- Relation between $C(A)$ and $C(R)$

Case I. $r=m=n$ A: 可逆方陣 $R: [I]$ $\Rightarrow C(A)=C(R)$

Case II. $r=m<n$ A: 寬 $R: [I\ F]$ $\Rightarrow C(A)=C(R)$

Case III. $r=n<m$ A: 窄 $R: \begin{bmatrix} I \\ 0 \end{bmatrix}$ $\Rightarrow C(A)\neq C(R)$

Case IV. $r<m, r<n$ A: not full rank $R: \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ $\Rightarrow C(A)\neq C(R)$

$C(R)$? $C(A)$	r	$\rightarrow$	$r=m\ (=)$, $r<m\ (\neq)$
$C(R^T) = C(A^T)$	r		
$N(R) = N(A)$	$n-r$		
$N(R^T)$? $N(A^T)$	$m-r$		
dim 則R.A相等 $\uparrow$			

Def: The dimension of a vector space is the number of vectors in every basis

ex: $\dim(\mathbb{R}^2)=2$ $\quad \dim(\mathbb{R}^n)=n$ $\quad \dim(C(A))=\dim(C(R))$

$M_{2\times 2}$ real matrices $\Rightarrow \dim(M)=4$ $\quad \dim(A_{n\times n})=n^2$

$\dim(U_{n\times n})=\frac{n(n+1)}{2}$ $\quad \dim(D_{n\times n})=n$ $\quad \dim(S_{n\times n})=\frac{n(n+1)}{2}$

- Dimensions of the Four Subspaces

$A_{n\times n} \rightarrow$

Row space : $C(A^T)$

Column space : $C(A)$

Nullspace : $N(A)$

Left Nullspace : $N(A^T) \rightarrow N(A^T)=\{\underline{y} : A^T\underline{y}=\underline{0}\}=\{\underline{y} : \underline{y}^T A = 0^T\}$

- Row space

$$A=\begin{bmatrix}1&3&5&0&7\\0&0&0&1&2\\1&3&5&1&9\end{bmatrix}\quad R=\begin{bmatrix}1&3&5&0&7\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}$$

Basis for $C(R^T)$: $(1\ 3\ 5\ 0\ 7),(0\ 0\ 0\ 1\ 2)$

$\dim(C(R^T))=r=2$

pf: $EA=R \iff A=E^{-1}R$

Every row of A 為 rows of R 的線性組合，反之亦然。

$\Rightarrow$ We can have $C(A^T)=C(R^T)$ $\Rightarrow$ A與R的 row space 一樣 (column 則不一定)

$\therefore \dim(C(A^T))=\dim(C(R^T))$

- Column space

$C(A)$ 不一定 $=C(R)$, $\dim(C(A))=\dim(C(R))$

- Nullspace

$A\underline{x}=\underline{0} \iff R\underline{x}=\underline{0}$ pf: $EA\underline{x}=E\underline{0} \Rightarrow R\underline{x}=\underline{0}$; $E^{-1}R\underline{x}=E^{-1}\underline{0} \Rightarrow A\underline{x}=\underline{0}$

$\therefore N(A)=N(R)$

$\dim(N(A))=\dim(N(R))=n-r$

- Left Nullspace

$\underline{y}^T R=\underline{0}^T \iff [y_1\ y_2\ y_3]\begin{bmatrix}1&3&5&0&7\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}=[0\,0\,0\,0\,0]$

$y_1[1\,3\,5\,0\,7]+y_2[0\,0\,0\,1\,2]+y_3[0\,0\,0\,0\,0]=[0\,0\,0\,0\,0]$

$\Rightarrow y_1=0,\ y_2=0,\ y_3=$ free variable

$(y_1\ y_2\ y_3)=(0\ 0\ y_3)=y_3(0\ 0\ 1)$

Basis for $N(R^T)$: $[0\ 0\ 1]$

$\dim(N(R^T))=m-r$

For $N(A^T)$, A^T is $n\times m$

$\dim(N(A^T))=m-\text{rank}(A^T)=m-\text{rank}(A)=m-r$

$\dim(N(A^T))=\dim(N(R^T))$

$N(A^T)$ 不一定 $=N(R^T)$

pf: $\underline{y}^T A=\underline{0}^T \xrightarrow[A=E^{-1}R]{EA=R} (\underline{y}^T E^{-1})R=\underline{y}^T R=\underline{0}^T \Rightarrow \underline{y}^T E^{-1}=\underline{y}^T \Rightarrow (E^{-1})^T\underline{y}=\underline{y}$