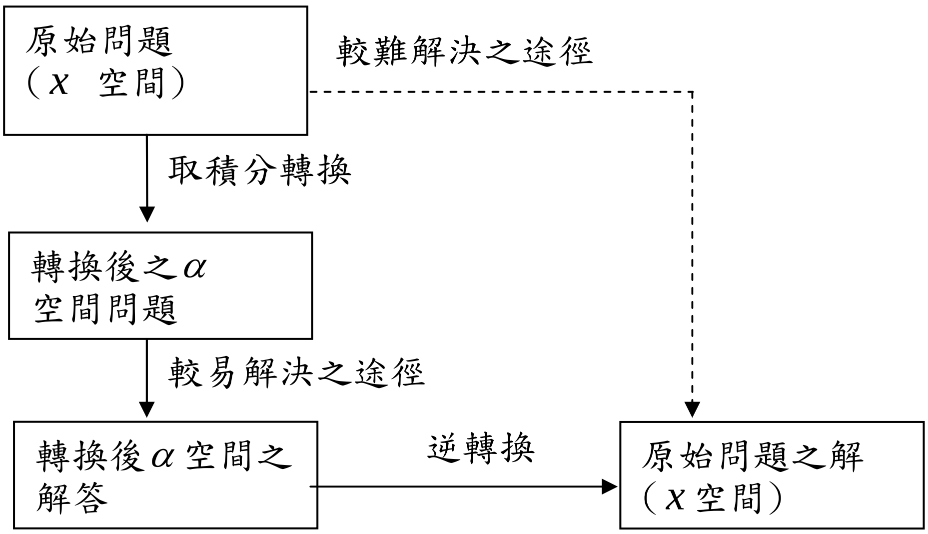
# Chapter 3 The Laplace Transform

◆ Preliminary Concepts

◆ 積分轉換(Integral transform)

積分轉換係將某函數 $f(x)$，乘上核函數(kernel function) $k(\alpha,x)$，並於指定變數區間 $x\in(a,b)$ 積分，而得一 $\alpha$ 之函數 $F(\alpha)$ 者。我們以下列通式來表達此一概念：

$$F(\alpha)\equiv\int_a^b f(x)k(\alpha,x)dx$$




◆ Note

(1) 在轉換後之 $\alpha$ 空間中的問題是否較原始問題簡單

(2) 逆轉換之求解是否容易

| 著名轉換 |
| --- |
| Laplace 轉換 <-> Laplace 逆轉換 |
| Fourier 指數轉換 <-> Fourier 指數逆轉換 |

(Def 3.2) 分段連續函數：(piecewise continuous function)

設函數 $f(t)$ 於其定義區間 $a \le t \le b$，滿足：

- 不連續點為有限個
- 於不連續點處左右極限均存在

則稱 $f(x)$ 為在區間 $a \le t \le b$ 之分段連續函數

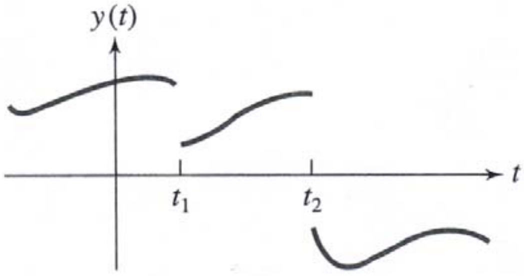



**FIGURE 3.1** *A function having jump discontinuities at $t_1$ and $t_2$.*

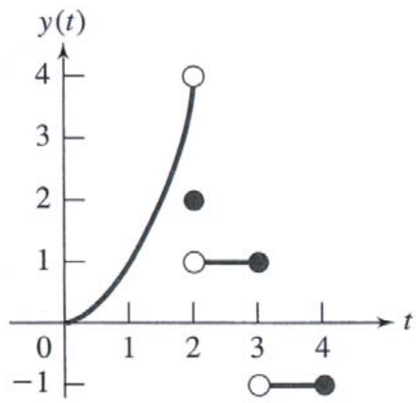



**FIGURE 3.2**

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 2 \\ 2 & \text{if } t = 2 \\ 1 & \text{if } 2 < t \le 3 \\ -1 & \text{if } 3 < t \le 4 \end{cases}.$$

Ex41: $f(t)=\sin\frac{1}{t}$在含$t=0$之區間並非分段連續，因為函數

在$t=0$之左右極限均不存在

◆ 指數階層函數(exponential order function)

若存在一正實數$M>0$，及實數$\alpha$，使得對所有$t>T$，恆有$\left|f(t)\right|<Me^{\alpha t}$ 或 $\left|f(t)e^{-\alpha t}\right|<M$，則稱當$t\to\infty$時，$f(t)$為$\alpha$指數階層函數，或簡稱為指數階層函數

Ex42: $f(t)=t^2$，則對所有$t>0$，恆有$\left|f(t)\right|<e^{3t}$

故$f(t)$為指數階層函數

Ex43: $f(t)=e^t$，因對所有$t>0$，恆有$\left|f(t)\right|<e^{1.5t}$

故$f(t)$為指數階層函數

Ex44: $f(t)=e^{t^2}$，因對所有$\alpha\in R$ $\lim_{t\to\infty}e^{t^2}\bullet e^{-\alpha t}\to\infty$，故$f(t)$不為指數階層函數

◆ 由定義及上述例子可知，指數階層函數之特性為當$t\to\infty$時，$f(t)$之發散速率(可能會收斂)比$Me^{\alpha t}$之發散速率小

## 3.1 Definition and Basic Properties

(Def 3.1) Laplace 轉換之定義:

設 $f(t)$ 為定義於 $t>0$ 之函數，則其 Laplace 轉換由下式所

定義： $\mathcal{L}\{f(t)\} \equiv F(s) = \int_0^\infty f(t)e^{-st}dt$

(Thm 3.2) Laplace 轉換存在(積分值收斂)之<u>充分</u>條件：

設函數 $f(t)$ 滿足

1. 對所有 $T>0^+$， $f(t)$ 於 $t\in[0^+,T]$ 上為分段連續
2. 存在常數 $M>0$，$\alpha\in R$，$T>0^+$，使得當 $t\geq T$ 時，

   恆有 $|f(t)|<Me^{\alpha t}$, 則 $\forall S>\alpha$，$\mathcal{L}\{f(t)\}$ 恆存在

◆ 幾種基本函數之 Laplace 轉換

1. $\mathcal{L}\left\{t^a\right\} = \int_0^\infty t^a e^{-st} dt \quad (a = 0,1,2,3,\ldots)$

$$= \int_0^\infty \left(\frac{\tau}{s}\right)^a e^{-\tau} \frac{d\tau}{s}$$

$$= \frac{1}{s^{a+1}} \int_0^\infty \tau^{(a+1)-1} e^{-\tau} d\tau = \frac{\Gamma(a+1)}{s^{a+1}} = \frac{a!}{s^{a+1}}$$

其中 $st = \tau$，$dt = \dfrac{d\tau}{s}$

且

| $t$ | 0 | $\infty$ |
|---|---|---|
| $\tau$ | 0 | $\infty$ |

$$\int_0^\infty \tau^{(a+1)-1} e^{-\tau} d\tau = \Gamma(a+1)$$

( $\Gamma(n)$ is the Gamma function，$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$

$\Gamma(n+1) = n\Gamma(n)$，for positive integers，$\Gamma(n+1) = n!$)

2. $\mathcal{L}\left\{e^{at}\right\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{(a-s)t} dt$

$$= \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty = \frac{0-1}{a-s} = \frac{1}{s-a} \quad (a-s<0, s>a)$$

3. $\mathcal{L}\{\cos at\} = ?$ $\mathcal{L}\{\sin at\} = ?$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2} \quad , \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$$

◆ 我們僅對$t \geq 0$時之$f(t)$作轉換。一般為了方便，均定義$t < 0$時之$f(t) = 0$

◆ 基本函數之逆轉換

1. $\mathcal{L}^{-1}\left\{\frac{1}{s^{a+1}}\right\} = \frac{t^a}{\Gamma(a+1)} = \frac{t^a}{a!} \qquad (a = 0, 1, 2, \cdots)$

2. $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

3. $\mathcal{L}^{-1}\left\{\frac{s}{s^2 + a^2}\right\} = \cos at$

4. $\mathcal{L}^{-1}\left\{\frac{1}{s^2 + a^2}\right\} = \frac{1}{a}\sin at$

## ◆ Laplace 轉換之基本定理

(Thm 3.1) 線性運算:

設 $\mathcal{L}\{f(t)\}=F(s)$ 且 $\mathcal{L}\{g(t)\}=G(s)$

則 $\mathcal{L}\{c_1 f(t)+c_2 g(t)\}=c_1 F(s)+c_2 G(s)$

[Proof]：由定義知

$$\begin{aligned}\mathcal{L}\{c_1 f(t)+c_2 g(t)\} &= \int_0^\infty (c_1 f(t)+c_2 g(t))e^{-st}dt \\ &= c_1\int_0^\infty f(t)e^{-st}dt + c_2\int_0^\infty g(t)e^{-st}dt \\ &= c_1 F(s)+c_2 G(s)\end{aligned}$$

## 3.3 Shifting Theorems and Heaviside Function

(Thm 3.7) 第一平移定理：

設 $\mathcal{L}\{f(t)\}=F(s)$，則 $\mathcal{L}\{f(t)e^{at}\}=F(s-a)$

[Proof]：由定義知

$$\begin{aligned}\mathcal{L}\{f(t)e^{at}\} &= \int_0^\infty f(t)e^{at}e^{-st}dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}dt = F(s-a)\end{aligned}$$

Ex45：$\mathcal{L}\{e^{at}\cos bt\}=\dfrac{(s-a)}{(s-a)^2+b^2}$

推理：設 $\mathcal{L}^{-1}\{F(s)\}=f(t)$

則 $\mathcal{L}^{-1}\{F(s-a)\}=e^{at}f(t)$

(Def 3.4) Heaviside Function:

The Heaviside function $H(t)$, (有時用 $u(t)$) is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

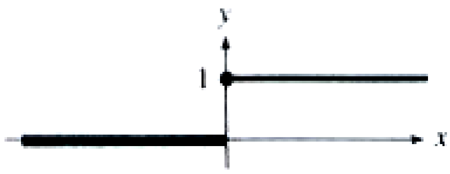



**FIGURE 3.10** *The Heaviside function $H(t)$.*

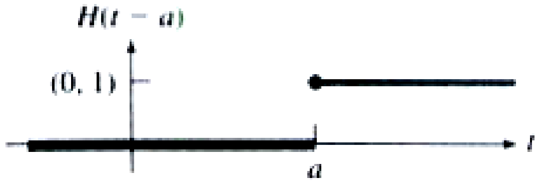



**FIGURE 3.11** *A shifted Heaviside function.*

$$H(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

$$H(t-a)g(t) = \begin{cases} 0 & \text{if } t < a \\ g(t) & \text{if } t \geq a \end{cases}$$

$$H(t-\pi)g(t) = H(t-\pi)\cos(t) = \begin{cases} 0 & \text{if } t < \pi \\ \cos(t) & \text{if } t \geq \pi \end{cases}$$

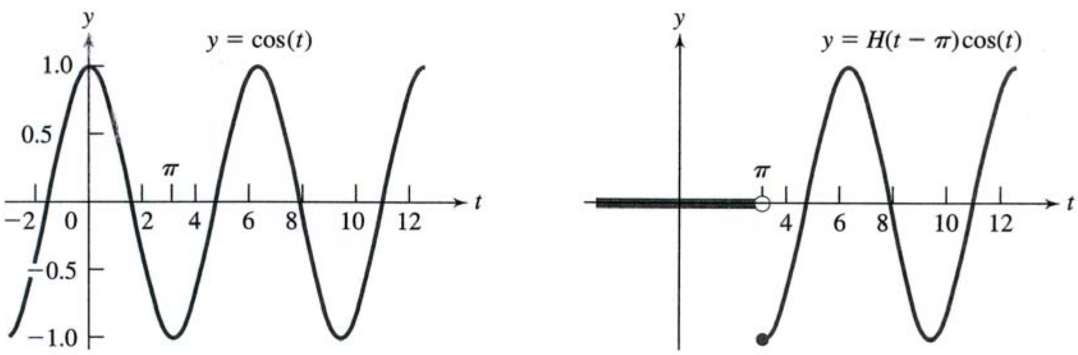



**FIGURE 3.12** *Comparison of $y = \cos(t)$ and $y = H(t - \pi)\cos(t)$.*

$$H(t-a)g(t-a)=\begin{cases} 0 & \text{if } t<a \\ g(t-a) & \text{if } t \ge a \end{cases}$$

$$H(t-\pi)\cos(t-\pi)=\begin{cases} 0 & \text{if } t<\pi \\ \cos(t-\pi) & \text{if } t \ge \pi \end{cases}$$

$$\mathcal{L}\{u(t)\}=\int_0^\infty u(t)e^{-st}dt=\int_0^\infty e^{-st}dt=\frac{-e^{-st}}{s}\Big|_0^\infty=\frac{1}{s}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=1$$

(Def 3.5) Pulse:

A pulse is a function of the form $H(t-a)-H(t-b)$, in which $a<b$

$$\left[H(t-1)-H(t-2)\right]e^t=\begin{cases} 0 & \text{if } t<1 \\ e^t & \text{if } 1\le t<2 \\ 0 & \text{if } t\ge 2 \end{cases}$$

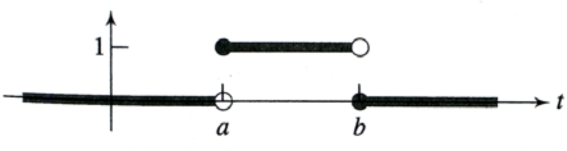



**FIGURE 3.13** *Pulse function* $H(t-a)-H(t-b)$.

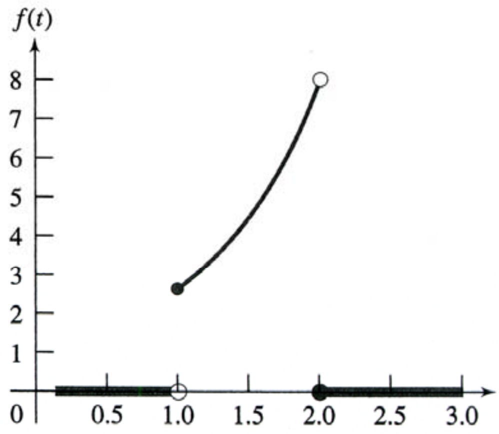



**FIGURE 3.14** *Graph of* $f(t)=[H(t-1)-H(t-2)]e^t$.

(Thm 3.8) 第二平移定理：

設 $\mathcal{L}\{f(t)\}=F(s)$，則 $\mathcal{L}\{f(t-a)u(t-a)\}=e^{-as}F(s)$

$$\text{其中 } f(t-a)u(t-a)\equiv\begin{cases}0 & t<a\\ f(t-a) & t\geq a\end{cases}$$

[Proof]：由定義知：

$$\begin{aligned}\mathcal{L}\{f(t-a)u(t-a)\}&=\int_0^\infty f(t-a)u(t-a)e^{-st}dt\\&=\int_a^\infty f(t-a)e^{-st}dt\\&=\int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau\\&=e^{-as}\int_0^\infty f(\tau)e^{-s\tau}d\tau=e^{-as}F(s)\end{aligned}$$

其中 $t-a=\tau$，$dt=d\tau$

且

| $t$ | $a$ | $\infty$ |
|---|---|---|
| $\tau$ | $0$ | $\infty$ |

Ex46：求 $\mathcal{L}\left\{\left(t^2+1\right)u(t-1)\right\}=?$

[解]：1.(誤解)

$$\begin{aligned}\mathcal{L}\left\{\left(t^2+1\right)u(t-1)\right\}&=e^{-s}\mathcal{L}\left\{(t^2+1)\right\}\\&=e^{-s}\left[\frac{2}{s^3}+\frac{1}{s}\right]\end{aligned}$$

2.（正解）

推理： $\mathcal{L}\{f(t)u(t-a)\}=\mathcal{L}\{g(t-a)u(t-a)\}$

$$=e^{-as}\mathcal{L}\{g(t)\}=e^{-as}\mathcal{L}\{f(t+a)\}$$

$$\mathcal{L}\left\{\left(t^2+1\right)u\left(t-1\right)\right\}=e^{-as}\mathcal{L}\left\{(t+1)^2+1\right\}$$

$$=e^{-as}\mathcal{L}\left\{t^2+2t+2\right\}$$

$$=e^{-s}\frac{2}{s^3}+e^{-s}\frac{2}{s^2}+e^{-s}\frac{2}{s}$$

Ex47: Find the Laplace transform of $f\left(x\right)=\begin{cases}1 & 0\le x<1\\ e^x & 1\le x<4\\ 0 & 4\le x\end{cases}$

[解]：

Ex48: Find the Laplace transform of $f\left(t\right)e^{-3t}$

Where $f\left(t\right)=\begin{cases}0 & t<8\\ t^2-4 & t\geq 8\end{cases}$

[解]：

Exercise Q:

1. $\mathcal{L}\left\{4e^{5t}+6t^3-3\sin 4t+2\cos 2t\right\}$

$\{\frac{4}{s-5}+\frac{36}{s^4}-\frac{12}{s^2+4^2}+\frac{2s}{s^2+2^2}\}$

2. $\mathcal{L}\left\{e^{at}t^n\right\}$ $\{\frac{n!}{(s-a)^{n+1}}\}$

3. $\mathcal{L}\left\{u(t-a)\right\}$ $\{\frac{e^{-as}}{s}\}$

4. $\mathcal{L}\left\{f(t)\right\}$ where $f(t)=\begin{cases}1 & 0\le t<7\\ \cos t & t\ge 7\end{cases}$

$\{\frac{1}{s}\left(1-e^{-7s}\right)+\frac{s}{s^2+1}\cos(7)e^{-7s}-\frac{1}{s^2+1}\sin(7)e^{-7s}\}$

5. $\mathcal{L}\ \left\{e^{-t}\left[1-t^2+\sin(t)\right]\right\}$ $\{\frac{1}{s+1}-\frac{2}{(s+1)^3}+\frac{1}{(s+1)^2+1}\}$

6. $\mathcal{L}^{-1}\left\{\frac{1}{(s-5)^3}e^{-s}\right\}$ $\{\frac{1}{2}e^{5(t-1)}(t-1)^2 H(t-1)\}$

7. $\mathcal{L}\ \left\{(t+2)^2e^t\right\}$ $\{\frac{\left(4s^2-4s+2\right)}{(s-1)^3}\}$

◆ 尺度變換：

設 $\mathcal{L}\{f(t)\}=F(s)$ 且 $a>0$，則 $\mathcal{L}\{f(at)\}=\frac{1}{a}F\left(\frac{s}{a}\right)$

[Proof]：由定義知：

$$\mathcal{L}\{f(at)\}=\int_0^\infty f(at)e^{-st}dt$$

$$=\int_0^\infty f(\tau)e^{-s\left(\frac{\tau}{a}\right)}\frac{d\tau}{a}=\frac{1}{a}\int_0^\infty f(\tau)e^{-\left(\frac{s}{a}\right)\tau}d\tau=\frac{1}{a}F\left(\frac{s}{a}\right)$$

其中 $at=\tau$，$dt=\frac{d\tau}{a}$

且

| $t$ | 0 | $\infty$ |
|---|---|---|
| $\tau$ | 0 | $\infty$ |

Ex49: $\mathcal{L}\{\sin 3t\}=?$

[解]：

◆ 積分的 Laplace 轉換：

設 $\mathcal{L}\{f(t)\}=F(s)$，則 $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}=\frac{1}{s}F(s)$

[Proof]：由定義知：

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}=\int_0^\infty\left[\int_0^t f(\tau)d\tau\right]e^{-st}dt$$

$$=\frac{-e^{-st}}{s}\left[\int_0^t f(\tau)d\tau\right]_0^\infty+\frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$=\frac{1}{s}F(s)$$

其中，令 $u(t)=\int_0^t f(\tau)d\tau$，$dv=e^{-st}dt$

$\frac{du}{dt}=f(t)$（by 第二微積分基本定理）

$du=f(t)dt$，$v=\frac{-e^{-st}}{s}$

Ex50: $\mathcal{L}\left\{\int_0^t \sin 2x dx\right\}=?$

[解]：

◆ Laplace 轉換的積分：

設 $\mathcal{L}\left\{f(t)\right\}=F(s)$，則 $\mathcal{L}\left\{\frac{f(t)}{t}\right\}=\int_s^\infty F(u)du$

[Proof]：由定義知：

$$
\begin{aligned}
\int_s^\infty F(u)du &= \int_s^\infty (\int_0^\infty f(t)e^{-ut}dt)du \\
&= \int_0^\infty f(t)(\int_s^\infty e^{-ut}du)dt \\
&= \int_0^\infty f(t)\frac{e^{-ut}}{-t}\Big|_{u=s}^{u=\infty} dt \\
&= \int_0^\infty f(t)(\frac{0-e^{-st}}{-t})dt \\
&= \int_0^\infty \frac{f(t)}{t}e^{-st}dt \\
&= \mathcal{L}\left\{\frac{f(t)}{t}\right\}
\end{aligned}
$$

Ex51: $\mathcal{L}\left\{\frac{\sin t}{t}\right\}=?$

[解]：

Exercise R:

1. $\mathcal{L}\left\{e^{-3t}\int_0^t \frac{\sin 2x}{x}dx\right\}$ $\{\frac{\pi}{2(s+3)}-\frac{1}{s+3}\tan^{-1}\left(\frac{s+3}{2}\right)\}$

2. Given $\mathcal{L}\left\{\frac{\sin t}{t}\right\}=\tan^{-1}(\frac{1}{s})$ , find $\mathcal{L}\left\{\frac{\sin at}{t}\right\}=?$

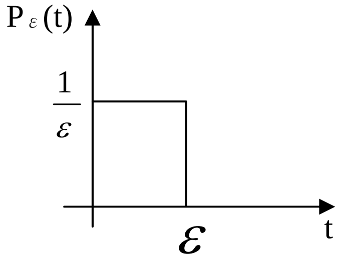
$\{\tan^{-1}\left(\frac{a}{s}\right)\}$

3.5 Unit Impulses and the Dirac Delta Function

◆ 單位脈衝函數(unit impulse function)之 Laplace 轉換

單位脈衝函數$\delta(t)$又叫做 Dirac delta 函數，其為一應用很廣之奇異函數(singular function)，在討論其定義以前，先考慮方塊波函數

$$P_\varepsilon(t)=\begin{cases}1/\varepsilon & 0\le t<\varepsilon\\ 0 & t\ge\varepsilon\end{cases}$$




由定義可知$P_\varepsilon(t)$之高度為$\dfrac{1}{\varepsilon}$，寬度為$\varepsilon$，故其面積等於 1，而且其可用單位階梯函數表示為：$P_\varepsilon(t)=\dfrac{1}{\varepsilon}\left[u(t)-u(t-\varepsilon)\right]$

當$\varepsilon\to 0$，$P_\varepsilon(t)$之高度增至無窮大，寬度減至無窮小，但面積仍維持等於 1。此一極限函數叫做單位脈衝函數。亦即：

$$\delta(t)\equiv\lim_{\varepsilon\to 0}P_\varepsilon(t)\equiv\lim_{\varepsilon\to 0}\frac{u(t)-u(t-\varepsilon)}{\varepsilon}$$

$$=\begin{cases}0 & t\neq 0^+\\ \infty & t=0^+\end{cases}$$

性質：由單位脈衝函數之定義，可得$\delta(t)$具下列性質：

1. $\int_0^\infty \delta(t)\,dt = 1$
2. 設$g(t)$為連續函數，則：
   a. (Thm 3.12)$\int_0^\infty g(t)\delta(t)\,dt = g(0^+)$
   b. $g(t)\delta(t) = g(0^+)\delta(t)$
3. $u'(t) = \delta(t)$

◆ $\delta(t)$之 Laplace 轉換：

[Proof]：由 $\mathcal{L}\{u(t)\} = \dfrac{1}{s}$ 且 $\mathcal{L}\{u(t-a)\} = e^{-as}\dfrac{1}{s}$

$$則\ \mathcal{L}\{\delta(t)\} = \lim_{\varepsilon\to 0}\left[\frac{\dfrac{1}{s} - \dfrac{e^{-s\varepsilon}}{s}}{\varepsilon}\right]$$

$$= \lim_{\varepsilon\to 0}\frac{1-e^{-\varepsilon s}}{s\varepsilon} = \lim_{\varepsilon\to 0}\frac{se^{-s\varepsilon}}{s} = 1$$

Ex52:$\mathcal{L}\{\delta(t-a)\} = ?$

$$\equiv \mathcal{L}[\lim_{\varepsilon\to 0} P_\varepsilon(t-a)] = \mathcal{L}[\lim_{\varepsilon\to 0}\frac{u(t-a)-u(t-a-\varepsilon)}{\varepsilon}]$$

$$= \lim_{\varepsilon\to 0}\left\{\frac{e^{-as}}{\varepsilon s} - \frac{e^{-(a+\varepsilon)s}}{\varepsilon s}\right\} = \lim_{\varepsilon\to 0}\left\{\frac{e^{-as} - e^{-as}e^{-\varepsilon s}}{\varepsilon s}\right\}$$

$$= \lim_{\varepsilon\to 0}\left\{\frac{e^{-as}(1-e^{-\varepsilon s})}{\varepsilon s}\right\} = \lim_{\varepsilon\to 0}\left\{\frac{e^{-as}(se^{-\varepsilon s})}{s}\right\} = e^{-as}$$

## 3.4 Convolution

(Def 3.6) 褶合積分(Convolution Integral):

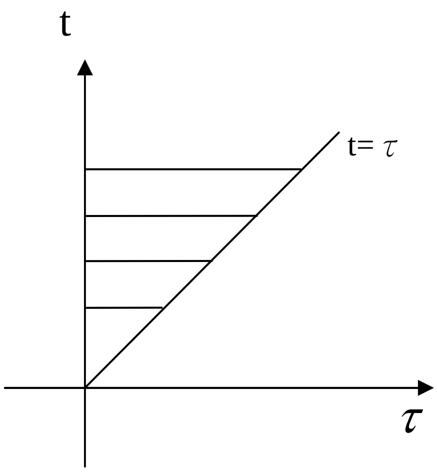
定義：設 $f(t)$，$g(t)$ 均有定義於 $t>0$，則積分 $\int_0^t f(\tau)g(t-\tau)d\tau$ 稱為 $f(t)$ 與 $g(t)$ 之褶合積分記為 $f(t)*g(t)$，即 $f(t)*g(t)\equiv\int_0^t f(\tau)g(t-\tau)d\tau$

(Thm 3.9) 褶合定理(Convolution Theorem):

若 $\mathcal{L}\left[f(t)\right]=F(s)$，$\mathcal{L}\left[g(t)\right]=G(s)$，且 $f(t)$ 與 $g(t)$ 之褶合

$$f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)d\tau=\int_0^t g(\tau)f(t-\tau)d\tau$$ , 則

$$\mathcal{L}\left[f(t)*g(t)\right]=F(s)G(s)$$




[Proof]：已知 $\mathcal{L}\left[f(t)\right]=F(s)$

$$\mathcal{L}\left[g(t)\right]=G(s)$$

$$\mathcal{L}\left[\int_0^t g(\tau)f(t-\tau)d\tau\right]$$

$$=\int_0^\infty e^{-st}(\int_0^t g(\tau)f(t-\tau)d\tau)dt$$

$$=\int_0^\infty g(\tau)(\int_\tau^\infty e^{-st}f(t-\tau)dt)d\tau$$

$$=\int_0^\infty g(\tau)\int_0^\infty e^{-s(x+\tau)}f(x)dxd\tau \quad （令 t=\tau+x）$$

$$=\int_0^\infty e^{-s\tau}g(\tau)(\int_0^\infty e^{-sx}f(x)dx)d\tau$$

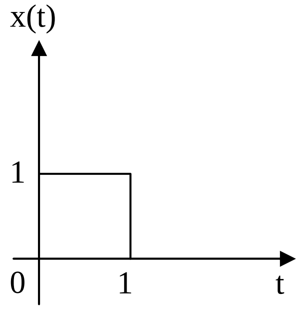
$$=\int_0^\infty e^{-s\tau}g(\tau)F(s)d\tau$$

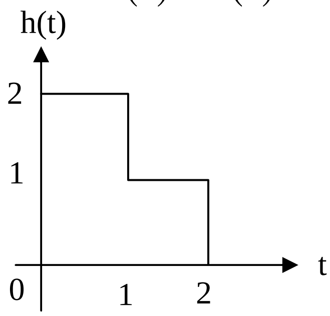
$$=F(s)\int_0^\infty e^{-s\tau}g(\tau)d\tau$$

$$=F(s)G(s)$$

同理可證 $\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right]=F(s)G(s)$

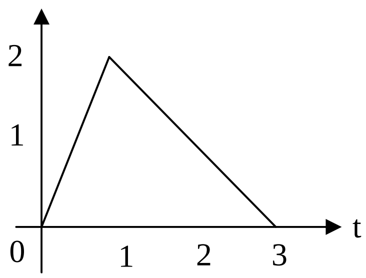
Ex53：已知 $x(t)$ 與 $h(t)$ 如下圖，試求 $x(t)*h(t)=?$




[解]：

即 $x(t)*h(t)$ 之圖形如下：



Ex54：By using the convolution theorem, find

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = ?$$

[解]：

Laplace 逆轉換

Laplace 逆轉換之求法：

部分分式展開：

設 $F(s)$ 為 $s$ 之有理分式函數，即 $F(s)=\dfrac{P(s)}{Q(s)}$，且 $P(s)$，$Q(s)$ 均為 $s$ 之多項式函數，則可根據 $Q(s)=0$ 時，根值情況將其化成較簡單之分式的和。

其中，我們均假定：

① $P(s)$，$Q(s)$ 中之係數為實數

② $P(s)$ 與 $Q(s)$ 無共同因式

③ $Q(s)$ 之次數比 $P(s)$ 之次數為高

(1) 情況 1：若 $Q(s)=0$ 具有 n 個不同的一次根 $a_k, k=1,2,...,n$

則 $\dfrac{P(s)}{Q(s)}=\dfrac{A_1}{s-a_1}+\dfrac{A_2}{s-a_2}+\cdots+\dfrac{A_k}{s-a_k}+\cdots+\dfrac{A_n}{s-a_n}$

故 $A_k=\dfrac{P(a_k)}{\dfrac{Q(a_k)}{(s-a_k)}}$

(2) 情況 2：若 $Q(s)=0$ 具有 $a_k$ 之 m 重根

則 $$\frac{P(s)}{Q(s)}=\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+\cdots+\frac{C_m}{(s-a_k)^m}$$

$$+\frac{C_{m-1}}{(s-a_k)^{m-1}}+\ldots+\frac{C_1}{s-a_k}+\ldots+\frac{A_n}{s-a_n}$$

$$C_m=\lim_{s\to a_k}\left[\frac{P(s)}{Q(s)}(s-a_k)^m\right]$$

$$C_{m-1}=\lim_{s\to a_k}\left\{\frac{d}{ds}\left[\frac{P(s)}{Q(s)}(s-a_k)^m\right]\right\}$$

$$C_{m-2}=\frac{1}{2!}\lim_{s\to a_k}\left\{\frac{d^2}{ds^2}\left[\frac{P(s)}{Q(s)}(s-a_k)^m\right]\right\}$$

(3) 情況 3：若 $Q(s)=0$ 具有共軛複數根

$$\lim_{s\to a+bi}\left\{\frac{P(s)}{Q(s)}\left[(s-a)^2+b^2\right]\right\}=A(a+bi)+B$$

Ex55：試求下列各函數之 Laplace 逆轉換

(I) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$ (II) $\mathcal{L}^{-1}\left\{\dfrac{(s-1)e^{-s}}{s(s+1)}\right\}$

[解]：

Ex56：Find the inverse Laplace transform of

$$\frac{(s+1)e^{-2s}}{(s^2+4)(s^2-2s+1)}$$

[解]：

Exercise S:

1. $\mathcal{L}^{-1}\left\{\dfrac{(s+1)}{(s^2+1)(s^2+4s+13)}\right\}$

$\{\dfrac{1}{20}\left(\cos t+2\sin t\right)-e^{-2t}\left[\dfrac{1}{20}\cos 3t+\dfrac{1}{15}\sin 3t\right]\}$

2. $\mathcal{L}^{-1}\left\{\dfrac{4s-8}{(s^2-16)}\right\}$ $\{3e^{-4t}+e^{4t}\}$

3. $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$ $\{\dfrac{1}{2}t\sin t\}$

4. $\mathcal{L}^{-1}\left\{\dfrac{1-e^{-s}}{s(s^2+1)}\right\}$ $\{1-\cos t-\left[1-\cos\left(t-1\right)\right]u\left(t-1\right)\}$

3.2 Solution of Initial Value Problems Using the Laplace Transform

◆ 用 Laplace 轉換求解常微分方程式之初值問題：

(Thm 3.5) 導函數之 Laplace 轉換：

設 $f(t)$ 於任意之 $0^+ \le t \le T$ 均為連續，且 $f'(t)$ 於 $0^+ \le t \le T$ 為分段連續，$f(t)$ 與 $f'(t)$ 於 $t \ge T$ 為指數階層函數，則

$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

[Proof]：$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt = \int_0^{t_1} f'(t)e^{-st}dt + \int_{t_1}^{t_2} f'(t)e^{-st}dt + \cdots + \int_{t_n}^{\infty} f'(t)e^{-st}dt$

$$= \left[ f(t)e^{-st}\Big|_0^{t_1} + s\int_0^{t_1} f(t)e^{-st}dt \right] + \cdots + \left[ f(t)e^{-st}\Big|_{t_n}^{\infty} + s\int_{t_n}^{\infty} f(t)e^{-st}dt \right]$$

$$= f(t)e^{-st}\Big|_0^{t_1} + \cdots + f(t)e^{-st}\Big|_{t_n}^{\infty} + s\int_0^{t_1} f(t)e^{-st}dt + \cdots + s\int_{t_n}^{\infty} f(t)e^{-st}dt$$

因為 $f(t)$ 為連續函數，所以

$$= f(t)e^{-st}\Big|_0^{\infty} + s\int_0^{\infty} f(t)e^{-st}dt$$

$$= 0 - f(0) + sF(s)$$

故 $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

(Thm 3.6) 設 $f(t)$，$f'(t)$， $\cdots$ ，$f^{(n-1)}(t)$於$t \geq 0^{+}$均為連續函數，而$f^{(n)}(t)$於$t \geq 0^{+}$為分段連續函數，且當$t > T$時，$f(t)$，$f'(t)$， $\cdots$ ，$f^{(n-1)}(t)$，$f^{(n)}(t)$皆為指數階層函數則：

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

[Proof]：（為了方便我們取$n = 2$時作一說明）

1.令$g(t) = f'(t)$

則 $\mathcal{L}\left\{f''(t)\right\} = \mathcal{L}\left\{g'(t)\right\}$

$\therefore$由導函數之 Laplace 轉換可得

$$\mathcal{L}\left\{f''(t)\right\} = \mathcal{L}\left\{g'(t)\right\} = s\mathcal{L}\left\{g(t)\right\} - g(0)$$

2.又 $\mathcal{L}\left\{g(t)\right\} = \mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$

代入上式可得：$\mathcal{L}\left\{f''(t)\right\} = s(sF(s) - f(0)) - g(0)$

$$= s^{2}F(s) - sf(0) - f'(0)$$

同理，$n \geq 3$時，可仿照上述的導證方式求證

Ex57：Please solve the initial value problem

$$\frac{d^2y}{dt^2}+7\frac{dy}{dt}+6y=6\sin 2t\,,\quad y(0)=0,\left.\frac{dy}{dt}\right|_{t=0}=0$$

[解]：

3.7 Differential Equations with Polynomial Coefficients

(Thm 3.13) Laplace 轉換之微分：

設 $\mathcal{L}\left\{f(t)\right\}=F(s)$，且 $F(s)$ 為可微分，則

$$\mathcal{L}\left\{tf(t)\right\}=(-1)\frac{d}{ds}F(s)$$

[Proof]：由定義知: $F(s)=\int_0^\infty f(t)e^{-st}dt$

$$\therefore \frac{d}{ds}F(s)=\frac{d}{ds}\int_0^\infty f(t)e^{-st}dt$$

$$=\int_0^\infty \left[-tf(t)\right]e^{-st}dt=\mathcal{L}\left\{-tf(t)\right\}=-\mathcal{L}\left\{tf(t)\right\}$$

(Cor 3.1) 設 $\mathcal{L}\left\{f(t)\right\}=F(s)$，且 $F(s)$ 至少可微分 n 次，則

$$\mathcal{L}\left\{t^n f(t)\right\}=(-1)^n\frac{d^n}{ds^n}F(s)$$

◆ 由 $\mathcal{L}\left\{tf(t)\right\}=(-1)\frac{d}{ds}F(s)$, $\left\{tf(t)\right\}=\mathcal{L}^{-1}\left[(-1)\frac{d}{ds}F(s)\right]$

$$f(t)=\frac{1}{-t}\mathcal{L}^{-1}\left[\frac{d}{ds}F(s)\right]$$

Ex58: (I) $\mathcal{L}\{t\sin at\} = ?$ (II) $\mathcal{L}^{-1}\left\{\ln\left(\dfrac{s+a}{s-a}\right)\right\} = ?$

［解］：

Ex59：試解 O.D.E. $ty''-ty'-y=0$， $y(0)=0, y'(0)=3$

［解］：

Exercise T :

1. $y''+2y'+2y=\delta(t-3)$, $\quad y(0)=0, y'(0)=0$

$\{ y=e^{-(t-3)}\sin(t-3)u(t-3) \}$

2. $\mathcal{L}^{-1}\left\{\ln(\frac{s-3}{s+1})\right\}$ $\quad \{\frac{e^{-t}-e^{3t}}{t}\}$

3. $y''+4y'+3y=0$, $\quad y(0)=3, y'(0)=1$ $\quad \{ y=-2e^{-3t}+5e^{-t} \}$

4. $y''-2y'+5y=8\sin t-4\cos t$, $\quad y(0)=1, y'(0)=3$

$\{ y=2\sin t+e^{t}\cos 2t \}$

5. $y''-y'-2y=4t^{2}$, $\quad y(0)=1, y'(0)=4$

$\{ y=-2t^{2}+2t-3+2e^{2t}+2e^{-t} \}$

3.6 Laplace Transform Solution of Systems

Ex60: $x''-2x'+3y'+2y=4$， $2y'-x'+3y=0$

$$x(0)=x'(0)=y(0)=0$$

[解]:

$$X(s)=\frac{4s+6}{s^2(s+2)(s-1)}$$

$$Y(s)=\frac{s\left(\frac{4}{s}\right)}{s(2s^2+2s-4)}=\frac{2}{s(s+2)(s-1)}$$

$$X(s)=-\frac{7}{2}\frac{1}{s}-3\frac{1}{s^2}+\frac{1}{6}\frac{1}{s+2}+\frac{10}{3}\frac{1}{s-1}$$

$$Y(s)=-\frac{1}{s}+\frac{1}{3}\frac{1}{s+2}+\frac{2}{3}\frac{1}{s-1}$$

$$x(t)=-\frac{7}{2}-3t+\frac{1}{6}e^{-2t}+\frac{10}{3}e^{t}$$

$$y(t)=-1+\frac{1}{3}e^{-2t}+\frac{2}{3}e^{t}$$

Ex61: Find the current in each loop

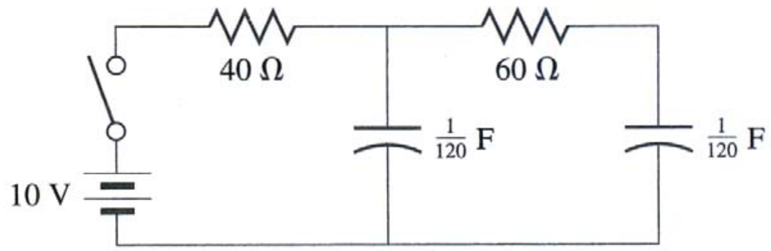



FIGURE 3.36

[解]: $40i_1+120(q_1-q_2)=10$

$$60i_2+120q_2=120(q_1-q_2)$$

$$40\frac{dq_1}{dt}+120(q_1-q_2)=10$$

$$60\frac{dq_2}{dt}+120q_2=120(q_1-q_2)$$

$$q_1(t)=\frac{1}{6}-\frac{1}{60}e^{-6t}-\frac{3}{20}e^{-t}$$

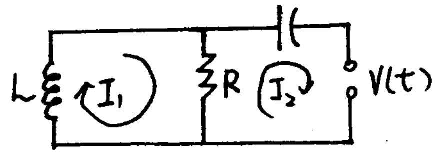
$$q_2(t)=\frac{1}{12}+\frac{1}{60}e^{-6t}-\frac{1}{10}e^{-t}$$

$$i_1(t)=\frac{3}{20}e^{-t}+\frac{1}{10}e^{-6t} \qquad i_2(t)=\frac{1}{10}e^{-t}-\frac{1}{10}e^{-6t}$$

Exercise U:

1. 求下圖所示電路中電流$I_1$，$R = 2.5\,\text{ohms}$，$C = 0.04\,\text{farad}$，

$L = 1\,\text{henry}$，$V(t) = 20\sin t$ ，$I_1(0) = I_2(0) = 0$




$$\{\, I_1(t) = \frac{-120}{169}\cos t - \frac{50}{169}\sin t + \frac{120}{169}e^{-5t} + \frac{50}{13}te^{-5t} \,\}$$

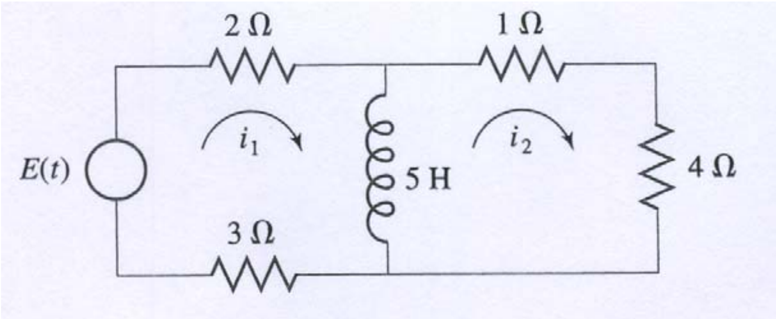
2. $3x' - y = 2t$ ，$x' + y' - y = 0; x(0) = y(0) = 0$

$$\{\, x(t) = \frac{t^2}{2} + \frac{1}{2}t + \frac{3}{4} - \frac{3}{4}e^{2t/3},\ \ y(t) = t + \frac{3}{2} - \frac{3}{2}e^{2t/3} \,\}$$

3. $x' + 2y' - y = 1$，$2x' + y = 0; x(0) = y(0) = 0$

$$\{\, x(t) = \frac{1}{3}t + \frac{4}{9} - \frac{4}{9}e^{3t/4},\ \ y(t) = -\frac{2}{3} + \frac{2}{3}e^{3t/4} \,\}$$

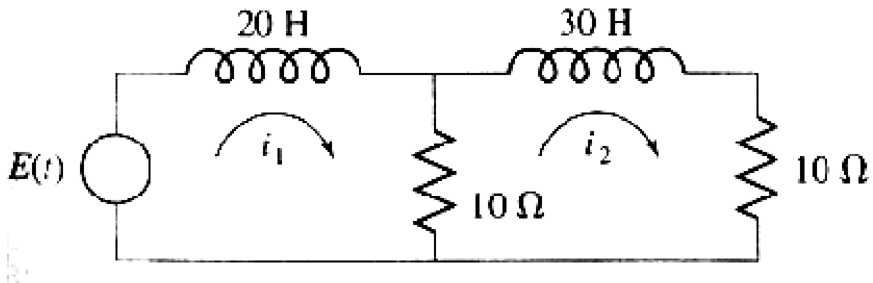
4. Solve for the currents in the circuit of following figure assuming that the currents and charges are initially zero and that $E(t)=2H(t-4)-H(t-5)$




$$\{i_1(t)=\frac{2}{5}\left[1-\frac{1}{2}e^{-\frac{1}{2}(t-4)}\right]H(t-4)-\frac{1}{5}\left[1-\frac{1}{2}e^{-\frac{1}{2}(t-5)}\right]H(t-5),$$

$$i_2(t)=\frac{1}{5}e^{-\frac{1}{2}(t-4)}H(t-4)-\frac{1}{10}e^{-\frac{1}{2}(t-5)}H(t-5)\}$$

5. Solve for the currents in the circuit of following figure if $E(t)=5H(t-5)$ and the initial currents are zero




$$\{i_1(t)=\left[1-\frac{1}{10}e^{-(t-5)}-\frac{9}{10}e^{-(t-5)/6}\right]H(t-5),$$

$$i_2(t)=\left[\frac{1}{2}+\frac{1}{10}e^{-(t-5)}-\frac{3}{10}e^{-(t-5)/6}\right]H(t-5)\}$$

| $f(t)$ | $F(s)=L\left[f(t)\right]$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t$ | $\frac{1}{s^2}$ |
| $t^n\ (n=1,2,3,\cdots)$ | $\frac{n!}{s^{n+1}}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $te^{at}$ | $\frac{1}{(s-a)^2}$ |
| $t^n e^{at}$ | $\frac{n!}{(s-a)^{n+1}}$ |
| $\sin(at)$ | $\frac{a}{s^2+a^2}$ |
| $\cos(at)$ | $\frac{s}{s^2+a^2}$ |
| $t\sin(at)$ | $\frac{2as}{(s^2+a^2)^2}$ |
| $t\cos(at)$ | $\frac{(s^2-a^2)}{(s^2+a^2)^2}$ |
| $e^{at}\sin(bt)$ | $\frac{b}{(s-a)^2+b^2}$ |

| | |
|---|---|
| $e^{at}\cos(bt)$ | $\dfrac{s-a}{(s-a)^2+b^2}$ |
| $\dfrac{1}{t}\sin(at)$ | $\tan^{-1}\left(\dfrac{a}{s}\right)$ |
| $\delta(t-a)$ | $e^{-as}$ |

| $f(t)$ | $F(s)$ |
|---|---|
| $af(t)+bg(t)$ | $aF(s)+bG(s)$ |
| $e^{at}f(t)$ | $F(s-a)$ |
| $f(t-a)H(t-a)$ | $e^{-as}F(s)$ |
| $f(at)$ | $\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$ |
| $\int_0^t f(\tau)d\tau$ | $\dfrac{1}{s}F(s)$ |
| $\dfrac{f(t)}{t}$ | $\int_s^\infty F(u)du$ |
| $f'(t)$ | $sF(s)-f(0+)$ |
| $f^{(n)}(t)$ | $s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$ |
| $tf(t)$ | $-F'(s)$ |
| $t^nf(t)$ | $(-1)^nF^{(n)}(s)$ |