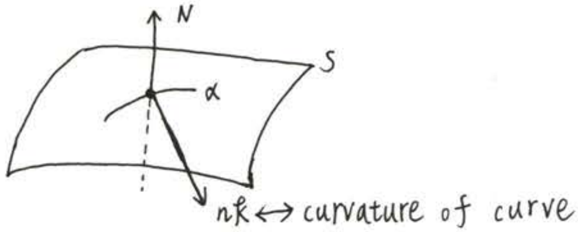
2018.2.22

<u>Recall</u>: The Sign of Gauss Curvature at $p \in S$ (regular surface)

k_1, k_2 are principle curvature of $S \subseteq \mathbb{R}^3$

e_1, e_2 are principal vectors of S if $v = \cos\theta e_1 + \sin\theta e_2$

the normal curvature of S in the v direction is $k_n = k\langle n, N\rangle = \mathrm{II}_p(v)$



$$k_n = \mathrm{II}_p(v) = -\langle dN_p(v), v\rangle = -\langle dN_p(\cos\theta e_1 + \sin\theta e_2), \cos\theta e_1 + \sin\theta e_2\rangle$$
$$= k_1\cos^2\theta + k_2\sin^2\theta \quad \text{Euler's Formula}$$

※ $-dN_p(e_1) = k_1 e_1$
$-dN_p(e_2) = k_2 e_2$

If $w = xe_1 + ye_2$, $\mathrm{II}_p(w) = k_1x^2 + k_2y^2$

$K = k_1k_2$, $H = \frac{k_1+k_2}{2}$ Gauss Curvature / Mean Curvature

- If the surface given by $z = h(x,y)$

$X(u,v) = (u, v, h(u,v))$

$$\rightarrow K = \frac{eg - f^2}{EG - F^2} = h_{uu}h_{vv} - h_{uv}^2$$

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{1}{2}(h_{uu} + h_{vv})$$

$$\text{Hessian of } h = \begin{pmatrix} h_{uu} & h_{uv} \\ h_{uv} & h_{vv} \end{pmatrix}$$

※ $N = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}$

$E = \langle X_u, X_u\rangle$, $F = \langle X_u, X_v\rangle$, $G = \langle X_v, X_v\rangle$

$e = \langle N, X_{uu}\rangle$, $f = \langle N, X_{uv}\rangle$, $g = \langle N, X_{vv}\rangle$

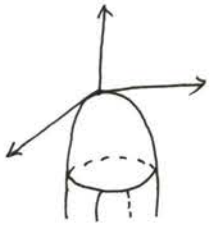
$\mathrm{II}_{p=(0,0,0)}(w) = h_{uu}u^2 + 2h_{uv}uv + h_{vv}v^2, \ \forall w=(u,v)$

① $K(p)>0$ $p\in S$ elliptic point

Gauss Curvature

$\to k_1(p)>0$ & $k_2(p)>0$ or $k_1(p)<0$ & $k_2(p)<0$

Thus S is tending away from its tangent plane $T_p(S)$ in all tangent directions at p.



$k_1(p)>0, k_2(p)>0$

If x, y are principal directions

$h_{xy}(0,0)=0, \ h_{xx}(0,0)=k_1(p), \ h_{yy}(0,0)=k_2(p)$

The quadratic approximation of surface S

$z=\frac{1}{2}(k_1x^2+k_2y^2)$ is paraboloid.

$X(u,v)=(u,v,h(u,v)) \to$ smooth

$\to$ Taylor's Expansion near p

$$h(x,y)=h(0,0)+h_x(0,0)x+h_y(0,0)y+\frac{1}{2}[h_{xx}(0,0)x^2+2h_{xy}(0,0)xy+h_{yy}(0,0)y^2]+R$$

when $(x,y)\to(0,0)$, $R\to 0$.

$h(0,0)=0, \ h_x(0,0)=0, \ h_y(0,0)=0$

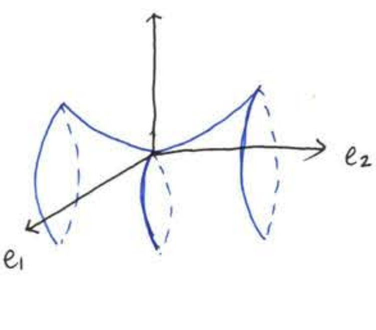
$\Rightarrow f=-\langle dN_p(X_u), X_u\rangle = 0.$

<u>Conclusion</u>: (k_1 and k_2 can control the shape of S)

$$\lim_{(x,y)\to(0,0)} \frac{R}{x^2+y^2} \to 0.$$

② $K(p)<0$, $p\in S$ hyperbolic point

$k_1(p)>0, k_2(p)<0$. $\left(k_1=\max_v \langle -dN_p(v), v\rangle, \ k_2=\min_v \langle -dN_p(v), v\rangle\right)$



$z=\frac{1}{2}(k_1x^2+k_2y^2)$

The quadratic approximation of S near p is hyperbolic paraboloid.

S is a saddle-shaped near p.

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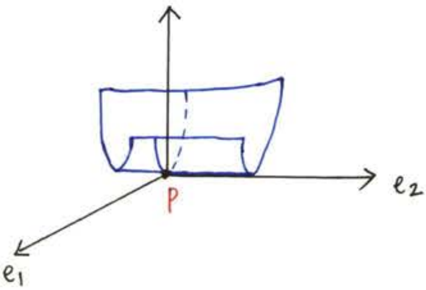
Ⅲ $K(p) = 0$

ⓐ if $k_1(p) \neq 0\ (>0)$, $k_2(p) = 0$ ($k_1(p)=0$, $k_2(p) \neq 0\ (<0)$)

The quadratic approximation of S is $z = \frac{1}{2} k_1 x^2$ (or $z = \frac{1}{2} k_2 y^2$)



∴ S is through - shaped near p
槽

→ p is called parabolic point

ⓑ $k_1(p)=0$, $k_2(p)=0$, $z=0$ → plane → p is a planner point

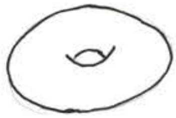
There is no information about the shaped of the surface near p.

$K = k_1 k_2$

- $k_1(p) = k_2(p)$ → umbilical point → <u>thm</u> :

 Torus

S = Torus



φ: outer half of S

f: inner half of S

ⓘ $K(p) > 0\ \forall p \in \varphi$

(The torus tends away from its tangent plane)

ⓘⓘ Near each point $p \in f$, S is a saddle - shaped and cut through $T_p(S)$

→ $K(p) < 0$ on f

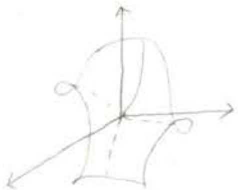
ⓘⓘⓘ Near each point on two circles (Top and bottom) which separate f and φ

∴ S is throogh - shaped, $K(p) = 0$

(There are no planner point on Torus

2018.3.1

- $z = xy$ (p.159)



$X(u,v) = (u, v, uv)$

three hills an valleys meet p must be a planner point

However, the shape of S near p is too complicated for all other three possibilities in above

<u>Def</u>: Let S and $\bar{S}$ be two oriented regular surface

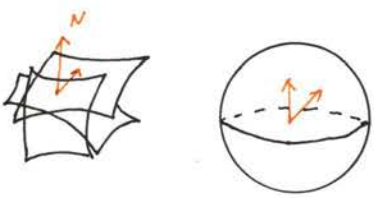
Let $\varphi: S \to \bar{S}$ be differentiable map and $p \in S$ such that $d\varphi$ is <u>non-singular</u>

$d\varphi: T_p(S) \to T_{\varphi(p)}(\bar{S})$

we say φ is orientation-Preserving at p if given a <u>positively oriented basis</u> <u>$\{w_1, w_2\}$ of $T_p(S)$</u>, the image under $d\varphi_p$, $\{d\varphi_p(w_1), d\varphi_p(w_2)\}$ form a positive basis for $T_\varphi(\bar{S})$, otherwise, we say φ is orientation-reversing at p.

<u>recall</u>: Gauss map

$N: S \longrightarrow S^2$ (= Unit sphere)



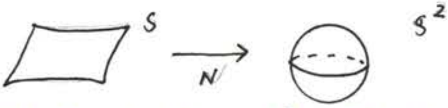
dN_p is non-singular

In particular, $\varphi = N: S \longrightarrow S^2$

Let N be an orientation S it induces an orientation on S^2 given by N

<u>prop</u>: The Gauss map $N: S \to S^2$ is orientation-preserving at elliptic points and orientation-reversing at hyperbolic points.

The differential of N is singular at points when $K = 0$.



$dN_p: T_p(S) \to T_{N(p)}(S^2) = T_p(S)$

Take $\{w_1, w_2\}$ to be a positive basis for $T_p(S)$

$(\det|w_1 \wedge w_2, N| > 0)$

We want to look at $\{dN_p(w_1), dN_p(w_2)\} \subseteq T_{N(p)}(S^2) = T_p(S)$

Let us write

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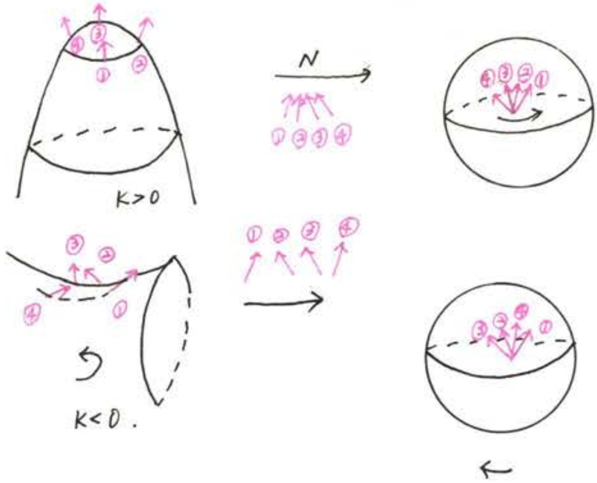
$$dN_p(w_1) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (a_{11}, a_{21}) = a_{11} w_1 + a_{21} w_2$$

$$dN_p(w_2) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a_{12} w_1 + a_{22} w_2$$

$$\begin{aligned} dN_p(w_1) \wedge dN_p(w_2) &= (a_{11}a_{22} - a_{12}a_{21})\, w_1 \wedge w_2 \\ &= \det |dN_p| \; w_1 \wedge w_2 \\ &= \underbrace{K}_{\text{Gauss Curvature}} \; w_1 \wedge w_2 \end{aligned}$$

$K > 0$, orientation - preserving

$K < 0$, orientation - reversing



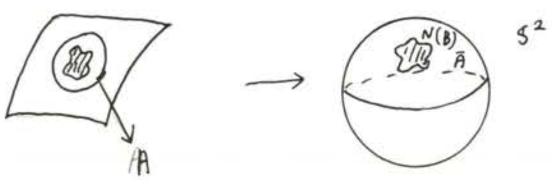
prop: Let $p \in S$ (where S is a regular, connected surface)

Let V be a connected Nbd of P where Gauss curvature K doesn't change sign. Then $K(p) = \lim\limits_{A \to 0} \dfrac{\bar{A}}{A}$

where A = area of a region $B \subseteq V$ containing P

$\bar{A}$ = Signed area of the Gauss image B.

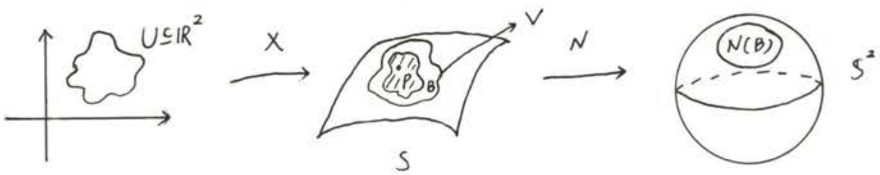
and limit taken over a sequence of regions B_K that converges to p.



$$\lim_{A \to 0} \frac{\bar{A}}{A} = K(p)$$

<u>Note</u> : Add sign orientation Preserving or Reversing.
"+" and "−" according to $N_u \wedge N_v$

<u>pf</u> :



Let $B \subseteq V$ is a region containing $p \in S$

Take $X : U \subseteq \mathbb{R}^2 \longrightarrow V \subset S$ to be a parametrization at p.

Let $X^{-1}(B) = R \subseteq U \subseteq (u, v)$ plane

$$A(B) = A = \iint_R |X_u \wedge X_v| \, du\, dv \quad \text{—} \; ①$$

refer 2-5

Let $\varphi = N \circ X : U \subseteq \mathbb{R}^2 \longrightarrow N(B) \subset S^2$

φ is differentiable (C^∞), where N is the Gauss map

Area of $N(B) = A(N(B)) = \bar{A} = \iint_R |\varphi_u \wedge \varphi_v| \, du\, dv$ where $\varphi_u = \frac{\partial}{\partial u}(N(X))$

$\varphi_v = \frac{\partial}{\partial v}(N(X))$

$$\therefore \bar{A} = \iint_R |dN_p(X_u) \wedge dN_p(X_v)| \, du\, dv = \iint_R |K| \, |X_u \wedge X_v| \, du\, dv \quad \text{—} \; ②$$

$$\frac{②}{①} = \frac{\iint_R |K| \, |X_u \wedge X_v| \, du\, dv \,/ A(R)}{\iint_R |X_u \wedge X_v| \, du\, dv \,/ A(R)} \qquad A(R) = \underline{\text{area of } R}$$

By the mean-value theorem for (connected hbhd) double integrals, $\exists (u_0, v_0) \in R$,

$$\underline{|K| \, |X_u \wedge X_v| (u_0, v_0)} = \frac{\iint_R |K| |X_u \wedge X_v| \, du\, dv}{A(R)}$$

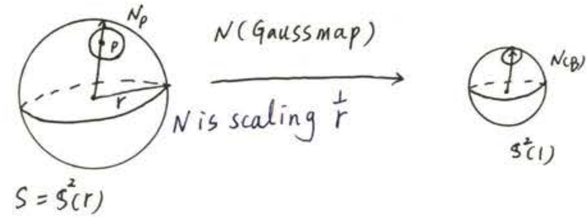
$$\exists (u_1, v_1) \in R, \text{ s.t } \underline{|X_u \wedge X_v| (u_1, v_1)} = \frac{\iint_R |X_u \wedge X_v| \, du\, dv}{A(R)}$$

As $R \to (0,0)$

$$\lim_{A \to 0} \frac{\bar{A}}{A} = \frac{|K| \, |X_u \wedge X_v|_p}{|X_u \wedge X_v|_p} = |K(p)|$$

Fact: Although the proof only gives the Absolute value of Gauss curvature, however, the sign can be recovered from the Gauss map if we define the signed area

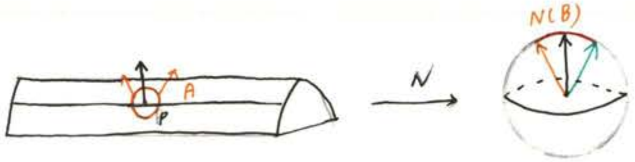
Example:



$$\Rightarrow K(p) = \lim_{A \to 0} \frac{\bar{A}}{A} = \frac{1}{r^2}$$

N collapses S onto a curve on S^2

$$K(p) = \lim_{A \to 0} \frac{\bar{A}}{A} = 0$$

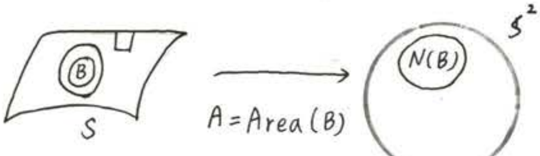


Area(⌒) = 0

$K(p) = 0$

↓

By Thm.



k doesn't sign in Nbd p, $p \in S$, $K(p) = \lim_{A \to 0} \frac{\bar{A}}{A}$, $\bar{A}$ = Area $(N(B))$

Curve:

$\alpha(t)$

$\alpha(s)$ parametrized by arc-length

$\alpha'(s)$

$|\alpha'(s)| = 1$

$t = \alpha'$, $t' = \underset{\text{curvature}}{\underset{=}{k}}\, n$

remark: Let α be a plane curve with nonzero curvature ($k \neq 0$). $p_1 \in \alpha$

s = length of the arc-length of a small segment of α containing p.

σ = the arc-length of its image in the indicatrix of tangent. T

$k = \lim_{s \to 0} \frac{\sigma}{s}$

The tangent indicatrix T of a regular curve $\alpha : I \to \mathbb{R}^3$ is the curve of oriented unit vector tangent to α.

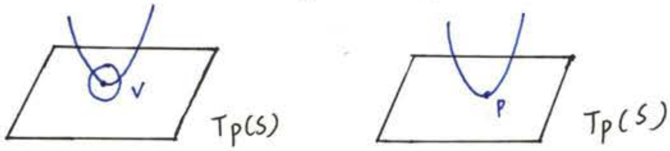
Moreprecisely, if $\alpha : I \to \mathbb{R}^3$ curve whose velocity vector $\alpha' = \frac{d\alpha}{dt} \neq 0$

Then $T = \frac{\alpha'(t)}{|\alpha'(t)|}$

local convex and curvature

Def: A regular surface $S \subseteq \mathbb{R}^3$ is a locally convex at $p \in S$ if there exist a Nbhd V of p s.t V is contained in one of the closed half-spaces determined by $T_p(S)$ in $\mathbb{R}^3$.

In addition, V has only one point (near p) in common with $T_p(S)$, then we call S strictly locally convex. ($V \cap T_p(S) = \{p\}$)



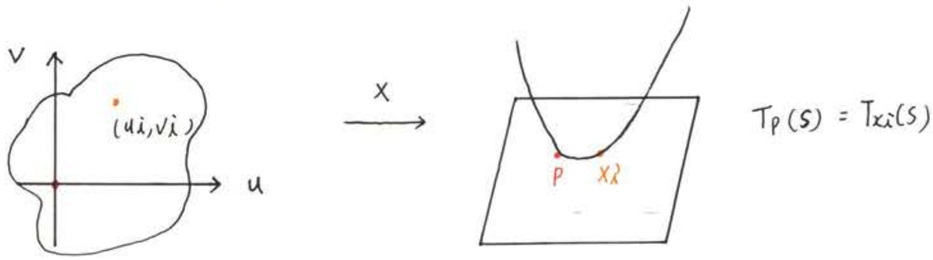
① $K > 0 \Rightarrow$ strictly locally convex at p.

pf: Suppose not

$\exists\, x_i \in S \setminus \{p\}$ s.t $x_i \to p$ and $x_i \in T_p(S)$

Note that V is a Nbhd of p lies on one side of $T_p(S)$ and $x_i \in T_p(S)$ → normal 相同

$\therefore T_p(S) = T_{x_i}(S)$



pick a parametrization at p

$X: U \subseteq \mathbb{R}^2 \longrightarrow S \subseteq \mathbb{R}^3$, $X(0,0) = p$, $X(u_i, v_i) = x_i$ (where $\{(u_i, v_i)\} \subseteq U$)

Normal vector at p, $N(0,0)$

$N(0,0) = \underbrace{N(u_i, v_i)}$ ↪ Normal vector at x_i

$$\frac{N(0,0) - N(u_i, v_i)}{u_i^2 + v_i^2} = 0.$$

$\exists\, \bar{u}_i \in (0, u_i)$, $\bar{v}_i \in (0, v_i)$

$N'(\bar{u}_i, \bar{v}_i) = 0$, where " ' " means diff in the direction e_i

$dN(\bar{u}_{ij}, \bar{v}_{ij})(e_{ij}) = 0 \Rightarrow \underbrace{dN(p)(e) = 0}_{K=0.}$ as $(\bar{u}_{ij}, \bar{v}_{ij}) \to (0,0)$

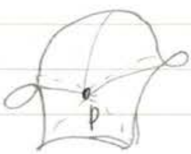
Ⅱ if S is locally convex at $p \Rightarrow K(p) \geq 0$.

HW If not, $T_p(S)$ two side $\Rightarrow K(p) < 0$

Ⅲ $K = 0 \not\Rightarrow$ local convex at p.

Example: $X(u,v) = (u, v, \underbrace{u^3(1+v^2)}_{h(u,v)})$

$X(0,0) = p$. $u(0) = 0$
$v(0) = 0$.



$$K = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1+h_u^2+h_v^2)^2}$$

$h_u = 3u^2(1+v^2)$

$h_{uu} = 6u(1+v^2)$

$h_v = 6u^2v$ at p

$h_{vv} = 6u^2$

$h_{uv} = 12uv$

$z = x^3(1+y^2)$

$(x_i, y_i) \to (0,0)$

$x_i \to 0^+$ $+$

0^- 0^-

$K(p) = 0$

Ⅳ $K(p) > 0$.

$K(q) \geq 0$ $q \in V$, V Nbhd of p.

$\Rightarrow$ local convex at p.

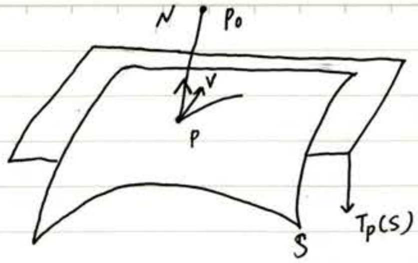
$\boxed{\text{yes}}$

HW

locally graph
$\to z = h(x,y)$

Hess$(h) > 0$ at p
≥ 0 $V \setminus \{0\}$

$\to$ hope $h(x,y) > 0$



$\alpha(0) = p$
$\alpha'(0) = v$

Let S be a regular surface and $f: S \longrightarrow \mathbb{R}$ be a differentiable map. given by $f(p) = |p - p_0|^2$, where $p \in S$ and p_0 is a fixed point of $\mathbb{R}^3$

① p is a critical point of f iff the line joining p and p_0 is normal to S at p.

pf: Let α be a regular curve $\alpha: I \to S$ s.t $\alpha(0) = p$ and $\alpha'(0) = v$

$$\frac{df}{dt}\Big|_{t=0} = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0} = \frac{d}{dt}(f(\alpha(t)))\Big|_{t=0} = \frac{d}{dt}\langle \alpha(t) - p_0, \alpha(t) - p_0 \rangle\Big|_{t=0}$$

$$= 2\langle \alpha'(0), \alpha(0) - p_0 \rangle$$

$$= 2\langle \underset{T_p(S)}{\underset{\uparrow}{v}}, p - p_0 \rangle$$

$$\frac{d}{dt} f\Big|_{t=0} \Leftrightarrow \langle v, \underset{/\!/\text{normal}}{p - p_0} \rangle = 0$$

$\therefore$ $p - p_0$ is a line normal to S at p $\therefore$ $p_0 - p = \lambda N(p)$ for some λ (*)

Ⅱ Find $\frac{d^2}{dt} f\Big|_{t=0} = ?$ if p is a critical point of f

sol: $$\frac{d^2}{dt^2} f\Big|_{t=0} = \frac{d^2}{dt^2}(f \circ \alpha)\Big|_{t=0} = \frac{d^2}{dt^2}(f(\alpha(t)))$$

$$= \frac{d}{dt} 2\langle \alpha'(t), \alpha(t) - p_0 \rangle\Big|_{t=0}.$$

$$= 2\left[\langle \alpha''(t), \alpha(t) - p_0 \rangle + |\alpha'(t)|^2\right)\Big|_{t=0}.$$

$$= 2\left[\langle \alpha''(0), p - p_0 \rangle + |v|^2\right]_{代(*)}$$

$$= 2\left[\langle \alpha''(0), -\lambda N(p) \rangle + |v|^2\right]$$

$$= 2\left[-\lambda \langle \alpha''(0), N \rangle + |v|^2\right]$$

$$\alpha' = u' X_u + v' X_v$$

$$= 2\left[-\lambda \mathrm{II}_p(\alpha'(0)) + |v^2|^2\right] \text{ —— (*)(*)}$$

$$\frac{df^2}{dt^2}\Big|_{t=0} = -2\left[-\lambda \mathrm{II}_p(\alpha'(0)) + |v|^2\right] \text{ —(*)(*)}$$

2018/03/08

Recall: local convex and curvature

① $K(p)>0 \Rightarrow$ strictly localy convex at p

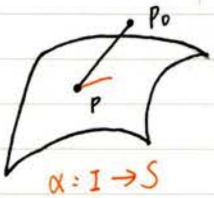
② S is locally convex $\Rightarrow K(p) \geq 0$ (H.W)

③ $K(p)=0 \not\Rightarrow$ locally convex
$z = x^3 + xy^2$

④ $\boxed{K(p)=0}$ $K(q) \geq 0 \ \forall q \in V \setminus \{p\} \Rightarrow S$ locally convex

$f : \underset{\text{surface}}{S} \longrightarrow \mathbb{R}$

$f(p) = |p - p_0|^2, \ p \in S, \ p_0 \in \mathbb{R}^3$



$\alpha : I \to S$
$\alpha(0) = p$
$\alpha'(0) = v$

(i) p is a critical point of f

$\Leftrightarrow p_0 - p = \underbrace{\lambda N(p)}_{\text{normal vector of } S \text{ at } p}$ —— ⊛ for some λ

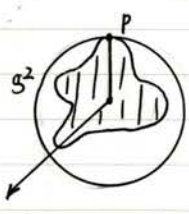
(ii) p is a critical point of f

$\frac{d^2}{dt^2} f \Big|_{t=0} = 2(-\lambda \mathrm{II}_p(v) + |v|^2)$ —— ⊛⊛

$\mathrm{II}_p(v) = \langle \alpha''(0), N \rangle$

Thm: A compact surface has an elliptic point
$K(p)>0$ for some $p \in S$

pf: $\because$ S is compact
S achieves its (global) maximum and (global) minimum
Namely, at the maximum point p.



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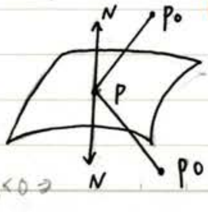
$\frac{d^2}{dt^2} f \Big|_{t=0} < 0 \Leftrightarrow$ ⊛⊛ <0.
($\because$ maximum $\Rightarrow$ 2次 diff <0) RHS

$-\lambda \mathrm{II}_p + |v|^2 < 0 \quad \Rightarrow \lambda \mathrm{II}_p > |v|^2 > 0$

If $\lambda > 0 \Rightarrow K(p) > 0$. o.K

reverse N s.t $\lambda > 0$



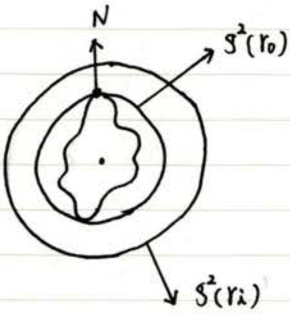
If $\lambda < 0 \Rightarrow$

$\Rightarrow \lambda_1' = -\lambda_1, \ \lambda_2' = -\lambda_2$

$K' = \lambda_1' \lambda_2' = \lambda_1 \lambda_2 = K \Rightarrow$ Gauss curvature 不變

Method II $\because$ S is compact, we find an large enough sphere s.t $S^2(r)$ containing S and decreasing r continuously

Assume that p is the 1st point in $S \cap S^2(r)$



$\exists$ a regular curve $\alpha: I \longrightarrow S \cap S^2(r)$

$\alpha(0) = p$, $\alpha'(0) = v$

Normal section = span $\{t, N\}$ 3-3, ⑯

$\because$ 越彎 k_n 越大 $\therefore$ S 的 k_n 會大於 $S^2(r_0)$ 的 k_n

$S^2(r_0)$ 的 $k_n = \frac{1}{r_0^2} > 0$ $\Downarrow > 0$.

3-3

The rigidity of the sphere

Thm2: Let S be a compact, connected, regular surface with constant Gauss Curvature. Then S is sphere.

Lemma: Let S be (oriented) regular surface and $p \in S$, k_1, k_2 are principle curvatures of S. ($k_1(t) \geq k_2(t)$)

S satisfying the following conditions:

① $K(p) > 0$, $p \in S$

② $k_1(p)$ has a local Maximum at p.

③ $k_2(p)$ has a local Minimum at p.

Then p is an umbical point of S (i.e $k_1(p) = k_2(p)$)

pf of Thm: 1899, H. Liebmann 1st proved

1909, D. Hilbert

✔ 1949, S.S Chern

Since S is compact and $K \equiv$ const., $k_1 k_2 = c$ for $c \neq 0$.

$\exists p \in S$, $K(p) > 0$ (By Thm 1) ①

By the compactness property, there is continuous function k_1 on S and its maximum occurs. at p

($\because$ compact, global Maximum) ②

$k_2 = \frac{c}{k_1}$ is decreasing function of k_1

$\therefore$ k_2 has global minimum at p.

By Lemma, p is an umbilical point $k_1(p) = k_2(p)$

Given any $q \in S$, $k_1(q) \geq k_2(q)$ $\begin{pmatrix} k_1 = \max \mathrm{II}_p(\alpha') \\ k_2 = \min \mathrm{II}_p(\alpha') \end{pmatrix}$

$\underbrace{k_1(p)}_{\text{maximumpoint}} \geq k_1(q) \geq k_2(q) \geq \underbrace{k_2(p)}_{\text{minimum point}} = k_1(p)$ (By Lemma)

$\Rightarrow$ $k_1(q) = k_2(q)$ $\forall q \in S$

$\therefore$ S is covered by umbilical points

recall: (3-2) prop.

S is contained in a sphere or a plane

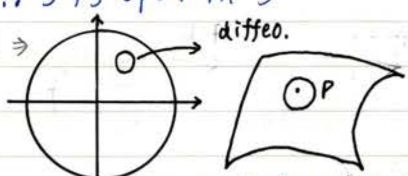
$\because$ $K(p) > 0$ $\therefore$ S is contained in a sphere

S is a compact surface

$\therefore$ S is closed in S^2

$\because$ S is a regular surface $\therefore$ S is open in S^2

By def of regular surface $\Rightarrow$



$\because$ S^2 connected, $S \subseteq S^2$, S both open and closed in S^2

By cor of Advanced calculus

$\therefore$ $S \equiv S^2$

Homework: (Jellet-Liebmann)

S is a compact, connected surface and $K > 0$ everywher

Suppose that mean curvature $H \equiv$ const. $= c$

then S is sphere $\qquad k_1 = c - k_2$

By the compactness property, there is cont. fun k_1 on S and has a max k_1 at p.
local max $\Rightarrow$ global max ($\because$ compact)

$\because$ Mean curvature $H = \frac{1}{2}(k_1 + k_2) \equiv c$ (const.) > 0 $\therefore$ $k_2 = 2c - k_1$

$\therefore$ has a global min k_2 at p

By Lemma, P is an umbilical point. $k_1(p) = k_2(p)$

Given any $q \in S$, $k_1(q) \geq k_2(q) \Rightarrow \underbrace{k_1(p)}_{\text{global max}} \geq k_1(q) \geq k_2(q) \geq k_2(p) = k_1(p)$

$\Rightarrow$ Everypoint of S is an umbilical point.

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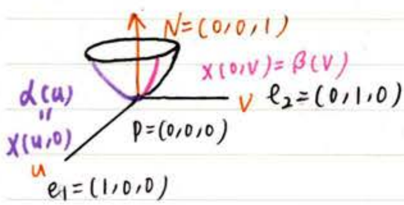
pf of Lemma: regular surface $\rightarrow$ locally graph



Let $p \in S$, $p = (0,0,0)$ and $N = (0,0,1)$
(normal vector at p)

$e_1 = (1,0,0)$ and $e_2 = (0,1,0)$ are the principle directions at p.

$\because$ S is a regular surface (By prop 3. 2-2)
Around p, S is locally the graph of a differentiable function h.
Let $X(u,v)$ be such a parametrization given by
$X(u,v) = (u, v, h(u,v))$

$X(0,0) = (0,0,0) = p$, $u(0) = 0$, $v(0) = 0$, $h(0,0) = 0$.

$X_u = (1, 0, h_u)$
$X_v = (0, 1, h_v)$

$X_u|_p = e_1 = (1,0,0) \Rightarrow h_u(0,0) = 0$
$X_v|_p = e_2 = (0,1,0) \Rightarrow h_v(0,0) = 0$.

$E = \langle X_u, X_u \rangle = 1 + h_u^2$ $\quad E|_0 = 1$ $\quad e = \langle N, X_{uu} \rangle = \frac{h_{uu}}{\sqrt{1+h_u^2+h_v^2}}$

$F = \langle X_u, X_v \rangle = h_u h_v$ $\quad F|_0 = 0$ $\quad f = \langle N, h_{uv} \rangle = \frac{h_{uv}}{\sqrt{1+h_u^2+h_v^2}}$

$G = \langle X_v, X_v \rangle = 1 + h_v^2$ $\quad G|_0 = 1$ $\quad g = \langle N, h_{vv} \rangle = \frac{h_{vv}}{\sqrt{1+h_u^2+h_v^2}}$

$$N = \frac{X_u \wedge X_v}{|X_u \wedge X_v|} = \frac{(-h_u, -h_v, 1)}{\sqrt{1+h_u^2+h_v^2}}, \quad dN_p = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \sim \begin{pmatrix} h_{uu} & 0 \\ 0 & h_{vv} \end{pmatrix}$$

$K = \frac{eg - f^2}{EG - F^2}$ $\quad \because u, v$ are principle directions.

$h_{uv}(0,0) = 0 \quad h_{vv}(0,0) = k_2(p)$

$h_{vu}(0,0) = 0 \quad h_{uu}(0,0) = k_1(p)$

Define $\overset{\text{curve}}{\alpha(u)} = X(u, 0)$ the image of 1^{st} coordinate function u.

$\overset{\text{curve}}{\beta(v)} = X(0, v)$ the image of 2^{nd} coordinate function v.

Let $G_1(v) = \frac{X_u(0,v)}{|X_u(0,v)|}$ u' where $X_u(0,v) \in T_{\beta(v)}(S)$

$G_2(u) = \frac{X_v(u,0)}{|X_v(u,0)|}$ v' where $X_v(u,0) \in T_{\alpha(u)}(S)$

Both are unit vectors

$$\mathrm{II}_{\beta(v)}(G_1(v)) = e(u')^2 = \frac{h_{uu}}{\sqrt{1+h_u^2+h_v^2}} \left(\frac{1}{|X_u(0,v)|^2} \right) = \left(\frac{h_{uu}}{\sqrt{1+h_u^2+h_v^2}} \right) \left(\frac{1}{1+h_u^2} \right)$$

$(\mathrm{II}_p(\alpha') = e(u')^2 + 2f(u'v') + g(v')^2$ if $\alpha' = X_u u' + X_v v')$

$$F_1(v) = \mathrm{II}_{\beta(v)}(G_1(v)) = \frac{h_{uu}}{\sqrt{1+h_u^2+h_v^2}} \left(\frac{1}{1+h_u^2} \right) (0,v)$$

$$F_2(u) = \mathrm{II}_{\alpha(u)}(G_2(u)) = g(v')^2 = \frac{h_{vv}}{\sqrt{1+h_u^2+h_v^2}} \left(\frac{1}{1+h_v^2} \right) (0,u)$$

$$F_1(v)\big|_{v=0} = \mathrm{II}_{\beta(v)}(G_1(v))\big|_{v=0} = \mathrm{II}_p(e_1) = k_1(p)$$

$$F_2(u)\big|_{u=0} = \mathrm{II}_{\alpha(u)}(G_2(u))\big|_{u=0} = \mathrm{II}_p(e_2) = k_2(p)$$

$\because F_1(v)\big|_{v=0} = \mathrm{II}_p(e_1) = k_1(p) = (\text{local Max at } p) \geq \mathrm{II}_{\beta(v)}(G_1(v)) = F_1(v)$

$F_1(v)|_{v=0} \geq F_1(v)$, 0 is local Max for F_1

$F_2(u)|_{u=0} = \mathrm{II}_p(e_2) = k_2(p)(\text{local min}) \leq \mathrm{II}_{\alpha(u)}(G_2(u)) = F_2(u)$

$\therefore F_2(u)|_{u=0} \leq F_2(u)$, 0 is local min for F_2

In particular, $\left.\begin{array}{l} F_2''(0) \geq 0 \\ F_1''(0) \leq 0 \end{array}\right\}$ ⊛

$$F_2(u) = \frac{h_{vv}}{\sqrt{1+h_u^2+h_v^2}} \frac{1}{(1+h_v^2)} \, (u,0)$$

$$= \left[\frac{-2h_v h_{vu}}{(1+h_v^2)^2} \cdot \frac{h_{vv}}{\sqrt{1+h_u^2+h_v^2}}\right] + \left[\frac{h_{vvu}}{\sqrt{1+h_u^2+h_v^2}} \cdot \frac{1}{1+h_v^2}\right] + \left[-\frac{1}{2} \cdot \frac{2h_u h_{uu} + 2h_v h_{vu}}{\sqrt{1+h_u^2+h_v^2}^3} \cdot \frac{h_{vv}}{1+h_v^2}\right]$$

① ② ③

$①'|_{u=0} = 0$

$②'|_{u=0} = -h_{vv} h_{uu}^2 \, (0,0)$ $\left(\begin{array}{c} h_u(0,0)=0 \\ h_v(0,0)=0 \\ h_{uv}(0,0)=0 \end{array}\right)$

$③'|_{u=0} = h_{vv} h_{uu}$

$\therefore F''(u)|_{u=0} = -h_{vv}(h_{uu}h_{uu}) + h_{vv}h_{uu}\,(u,0)\Big|_{u=0} \geq 0.$

$\therefore -h_{vv}(h_{uu}h_{uu}) + h_{vvuu}(u,0)\,|_{u=0} \geq 0.$

By the similar argument, we have $F_1(v)|_{v=0} = F_1''(0) = [-h_{uu}(h_{vv}h_{vv}) + h_{vvuu}]$
$(0,v)$

$F_2''(0) \geq 0 \geq F_1''(0)$

$F_2''(0) - F_1''(0) \geq 0$

$F_2''(0) - F_1''(0) = h_{uu}(h_{vv}h_{vv}) - h_{vv}(h_{uu}h_{uu}) = h_{uu}h_{vv}(h_{vv} - h_{uu})\,(0,0) \geq 0$

$\Rightarrow k_1(p)k_2(p)(k_2(p) - k_1(p)) \geq 0$

$K(p)(k_2(p) - k_1(p)) \geq 0.$

$\because K(p) > 0$ and $k_1(p) = \max II_p(\gamma') \geq k_2(p) = \min II_p(\gamma')$

$\therefore k_2 \leq k_1 \Rightarrow \therefore k_1(p) = k_2(p)$

$\therefore$ p is an umbilical point

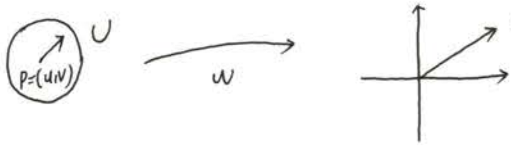
3/15 —

S is called ovaloids if S is compact, connected with positive Gauss curvature.

§ 3-4 Vector Fields (HW: 1, 11, 12)

<u>Def</u>: $U \subseteq \mathbb{R}^2$ open subset

A vector field is a map $w: U \to \mathbb{R}^2$ which can be thought of as an assignment of a vector to each $p \in U$



$w(t)$ is tangent to S .. $W(t)$ is called tangent vectorfield ①

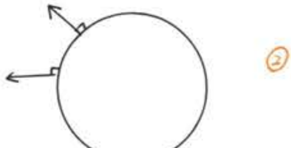
$w(t)$ is orthogonal to S, $w(t)$ is called normal vectorfield. ②

on circle.

If we write $(u,v) \in U$, and $w(u,v) = (a(u,v), b(u,v))$

then w is differentiable vectorfield if both $a(u,v)$ and $b(u,v)$ are differentiable. (C^∞)

From now on, all vector fields are assumed to be differentiable.



<u>Def</u>: A tragectory of a vector field w passing through a point (x_0, y_0) is differentiable curve $\alpha(t) = (x(t), y(t))$ $(\alpha: I \to U)$

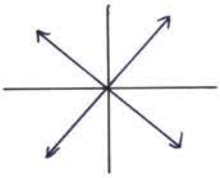
$\alpha(0) = (x(0), y(0)) = x_0, y_0$ and $\alpha'(t) = w(x(t), y(t)) = w(\alpha(t))$

Q: Given w, $?\exists\, \alpha$ s.t $\alpha' = w$

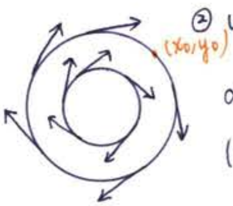
<u>Example</u>: ① Take $w = (x, y)$

$\alpha(t) = (x_0 e^t, y_0 e^t)$ is a tragectory. of w passing through (x_0, y_0), $\alpha'(t) = (x_0 e^t, y_0 e^t)$

$\alpha(t)$ is a straight line



② $w = (y, -x)$ $\;?\exists\, \alpha$ s.t $\alpha'(t) = w(\alpha(t))$



$\alpha(t) = (r\sin t, r\cos t)$, $t \in \mathbb{R}$ is a trajectory of w through (x_0, y_0), $\boxed{x_0^2 + y_0^2 = r^2}$, $\alpha'(t) = (r\cos t, -r\sin t)$

$= (y, -x)$

In general, if $w(x,y) = (a(x,y), b(x,y))$ is a vector field and if

$\alpha(t) = (x(t), y(t))$ is a tragectory to w through (x_0, y_0), then

$\alpha'(t) = (\underline{\underline{x'(t)}}, y'(t)) = w(t) = (\underline{\underline{a(x,y)}}, b(x,y))$

$$\Leftrightarrow \circledast \begin{cases} x'(t) = a(x(t), y(t)) \\ y'(t) = b(x(t), y(t)) \\ x(0) = x \\ y(0) = y \end{cases}$$

$\therefore$ By ODE, $\exists!$ solution in $\circledast$

For example ①,

$$\begin{cases} x' = x \\ y' = y \\ x(0) = x_0 \\ y(0) = y \end{cases}, \quad \frac{dx}{dt} = x$$

For example ②

$$\begin{cases} x' = y \\ y' = -x \\ x(0) = x_0 \\ y(0) = y \end{cases}, \quad x'' = y' = -x \quad \boxed{x'' + x = 0}$$

<u>Thm 1</u>: Let w be a vector field in an open domain $U \subseteq \mathbb{R}^2$.

Given $p \in U$, $\exists$ a trajectory $\alpha : I \to U$ of w with $\alpha(0) = p$.

This trajectory is unique in the following sense:

If $\beta : J \to U$ is another trajectory of w passing through p.

then $\underline{\underline{\alpha = \beta}}$ on $\underline{\underline{I \cap J}}$.

<u>Thm2</u>: Let w be a vector field on U. For each $p \in U$, $\exists$ a Nbhd V of p,

an interval I and $\alpha : I \times V \to U$ s.t

① For a fixed $q \in V$, the curve $\alpha(q,t)$ is a trajectory through q.

$$\begin{cases} \alpha(q,0) = q \\ \dfrac{\partial \alpha(q,t)}{\partial t} = w(\alpha(q,t)) \end{cases}$$

② α is differentiable map on $I \times V$.

<u>Fact</u>: such α is called a <u>local flow</u> of w at p.

(an integral curve of tangent vector field)

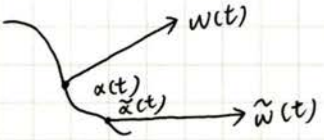
refer:

<u>Hurewicz</u> O.D.E

§3-5 Ruled Surface

(Surface is covered by all straight lines)

<u>Def</u>: A one parameter family of (straight) lines $\{\alpha(t), w(t)\}$ is correspondence that assigns to each $t \in I$, a point $\alpha(t) \in \mathbb{R}^3$ and a vector $w(t) \in \mathbb{R}^3$ $w(t) \neq 0$ s.t both $\alpha(t)$ and $w(t)$ are differentiable on t.



A one parameter family lines in $\mathbb{R}^3$ gives a parametrized surface $\boxed{X(t,v) = \underbrace{\alpha(t)}_{\text{pass point}} + \underbrace{v\,w(t)}_{\text{along direction}}}$

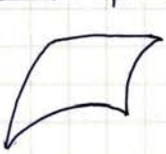
is called a ruled surface.

The lines $L_t = \alpha(t) + v\,w(t)$ in the ruled surface are called the ruling and $\alpha(t)$ is called directrix. (直紋曲面)

<u>Example1</u>: plane: $\alpha(t) = (t,0)$

$w(t) = (0,1)$

$X(t,v) = \alpha(t) + v\,w(t) = (t,0) + v(0,1) = (t,v)$, $t \in I$, $v \in \mathbb{R}$

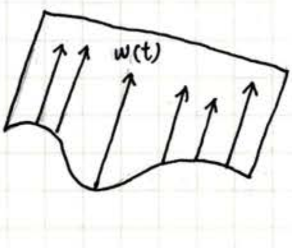


<u>Example 2</u>: cylinder: $\alpha(t) = (\cos(t), \sin(t), 0)$

$w(t) = (0,0,1)$

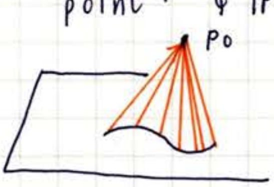
$X(t,v) = \alpha(t) + v\,W(t)$

$= (\cos t, \sin t, v)$



<u>Example 3</u>: <u>Cones</u>

$\alpha(t) \subseteq \mathbb{P}$ (plane) and the rulings all pass through a given point $P_0 \notin \mathbb{P}$



Example 4: Take a circle $= \{(x,y) \mid x^2+y^2=1\}$

parametrized $\alpha(t) = (\cos t, \sin t, 0)$

$w(t) = \alpha'(t) + e_3$, $e_3 = (0,0,1)$

The ruled surface, $X(t,v) = \alpha(t) + v(\alpha'(t)+e_3)$

$w(t) = \alpha'(t)+e_3$ or $w(t) = -\alpha'(t)+e_3$

$$X(t,v) = (\underbrace{\cos t - v\sin t}_{x(t,v)}, \underbrace{\sin t + v\cos t}_{y(t,v)}, \underbrace{v}_{z(t,v)})$$

hyperboloid $\quad x^2(t,v) + y^2(t,v) = 1 + z^2(t,v)$

of revolution surface $S = \{(x,y,z) \mid x^2+y^2-z^2=1\}$

HW: 沒有第3個 conponent.

Ruled Surface

$\{\alpha(t), w(t)\}$

point $\quad$ vector

$X(t,v) = \alpha(t) + v\,w(t)$, $v \in \mathbb{R}$, $t \in I$

Lines $L_t = \alpha(t) + v(w(t))$ rulings

$\alpha(t)$: directrix

① IP

② cylinder

③ Cone

④ $\alpha(t) = (\cos t, \sin t, 0)$

$w(t) = \alpha'(t) + e_3$, $e_3 = (0,0,1)$

$w(t) = -\alpha'(t) + e_3$

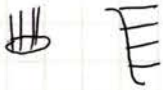
$X(t,v) = \alpha(t) + v\,w(t)$

hyperboloid of revolution has <u>two</u> sets of rules

$X(t,v) = \alpha(t) + (\alpha'(t))v$ if $w(t) = \alpha'(t)$

Not regular surface

$\mathbb{R}^2 \setminus$ disk



$\langle w, w' \rangle = 0 \leftarrow \langle w, w \rangle = 1$

$\uparrow$

$|w(t)|^2 = 1$

In order to develop the theory, we need assume that $w'(t) \neq 0$, $|w(t)| = 1$

noncylindrical

Find a good the diretrix $\beta(t) \ni \langle \beta', w' \rangle = 0$
on a ruled surface $X(t,u)$, $\beta \in X(t,u)$.

$\beta(t) = \alpha(t) + u\,w(t)$ for some real function u.

$\beta'(t) = \alpha'(t) + u' w(t) + u w'(t)$

$0 = \langle \beta'(t), w'(t) \rangle$

$= \langle \alpha'(t) + u' w(t) + w'(t) u, w'(t) \rangle$

$= \langle \alpha'(t), w'(t) \rangle + u |w'(t)|^2$

$\therefore u = \dfrac{-\langle \alpha'(t), w'(t) \rangle}{|w'(t)|^2}$ $\quad \langle \alpha', w' \rangle + u |w'|^2 = 0$

$\therefore \beta(t) = \alpha(t) - \dfrac{\langle \alpha'(t), w'(t) \rangle}{|w'(t)|^2} w$

$\beta(t)$ is called the line of striction. The points of β are control point.

- Show that β is independent of the choice of $\alpha(t)$

$X(t,v) = \overset{\text{diretrix}}{\alpha(t)} + u\,w(t)$ —①, $\beta(t) = \alpha(t) - \dfrac{\langle \alpha'(t), w'(t) \rangle}{|w'(t)|^2} w$

Let $\bar{\alpha}(t)$ be another diretrix of $X(t,v)$

$X(t,v) = \bar{\alpha}(t) + \bar{u}\,w(t)$ —②

$\bar{\beta}(t) = \bar{\alpha}(t) - \dfrac{\langle \bar{\alpha}', w' \rangle}{|w'|^2} w$

Hope $\bar{\beta}(t) - \beta(t) = 0$ (H.W)

If we write the rule's surface using the line of striction as a diretrix.

$$X(t,v) = \beta(t) + v\,w(t)$$

$$\begin{cases} X_t = \beta'(t) + v\,w'(t) \\ X_v = w(t) \end{cases}$$

$$X_t \wedge X_v = (\beta' + v w') \wedge w = \beta' \wedge w + v\, w' \wedge w$$

$$\boxed{\begin{array}{l} \because |w| = 1,\ w' \neq 0 \\ \langle \beta', w' \rangle = 0 \\ \langle w, w' \rangle = 0 \\ \therefore \beta' \wedge w = \lambda w' \end{array}}$$

$$\therefore = \lambda w' + v\, w' \wedge w$$

$$|X_t \wedge X_v|^2 = |\lambda w' + v\, w' \wedge w|^2$$

$$= \lambda^2 |w'|^2 + \underbrace{2\lambda \langle w', v\, w' \wedge w \rangle}_{0} + v^2 \overset{|w'|^2|w|^2}{\overset{\shortparallel}{|w' \wedge w|^2}}$$

$$= (\lambda^2 + v^2)|w'|^2$$

Hence the surface is regular iff $|X_t \wedge X_v| \neq 0$

either $\lambda \neq 0$ or $\lambda \neq 0$

$$|X_t \wedge X_v|^2 = (\lambda^2 + v^2)|w'|^2$$

$$\because \beta' \wedge w = \lambda w'$$

$$\langle \beta' \wedge w, w' \rangle = \lambda \langle w', w' \rangle = \lambda |w'|^2$$

$$\therefore \lambda = \frac{\langle \beta' \wedge w, w' \rangle}{|w'|^2} = \frac{(\beta', w, w')}{|w'|^2}$$

Next, compute Gauss Curvature of S at its regular points.

$X(t,v) = \beta(t) + v\,w(t)$ is a parametrization of S.

$$\begin{cases} X_t = \beta' + v w', & X_{tt} = \beta'' + v w'' \\ X_v = w, & X_{vv} = 0, \quad X_{tv} = w' \end{cases}$$

$$N = \frac{X_t \wedge X_v}{|X_t \wedge X_v|} = \frac{\lambda w' + v\, w' \wedge w}{|X_t \wedge X_v|}$$

$$K = \frac{eg - f^2}{EG - F^2}$$

$$\because X_{vv} = 0,\ g = \langle X_{vv}, N \rangle = 0$$

$$f = \langle X_{tv}, N \rangle = \frac{1}{|X_t \wedge X_v|} \langle w', \lambda w' + \underbrace{v\, w' \wedge w}_{0} \rangle = \frac{\lambda |w'|^2}{|X_t \wedge X_v|}$$

$$\therefore K = \frac{\dfrac{-\lambda^2 |w'|^2}{|X_t \wedge X_v|^2}}{(\lambda^2+v^2)|w'|^2} = \frac{-\lambda^2(t)|w'|^4}{(\lambda^2+v^2)^2|w'|^4}, \quad K = \frac{-\lambda^2(t)}{(\lambda^2+v^2)^2} \le 0.$$

also $|K|$ has maximum value at $v=0$ if we restricted on each ruling.

$K=0$ if $\lambda=0$ and $v\neq 0$, $\lambda = \dfrac{(\beta', w, w')}{|w'|^2} = \dfrac{\langle \beta' \wedge w, w' \rangle}{|w'|^2}$

- Def: A developable (可搭型曲面) Surface is a ruled surface $\ni (\alpha', w, w') = 0$

Claim: developable surface has Gauss Curvature 0 at regular points.

$K=0 \Rightarrow$ 至少有一個 priciple curvature $=0 \Rightarrow$ 在點上有直線

剪下去那邊是平面

$X(t,v) = \alpha(t) + v\,w(t)$

$X_t = \alpha' + v w'$, $X_v = w(t)$, $X_{tv} = w'$, $X_{tt} = \alpha'' + w''$, $X_{vv} = 0$

$K = \dfrac{-f^2}{EG-F^2}$ $\because f = \left\langle w', \dfrac{\alpha' \wedge w + v w' \wedge w}{|X_t \wedge X_v|} \right\rangle$

$f = \dfrac{-(\alpha', w, w')}{|X_t \wedge X_v| \neq 0}$ Numeration $=0$ by def of developable surface.

Case 1: α is a line of striction. (i.e. $\langle \alpha', w' \rangle = 0$)

$\alpha' \wedge w = \lambda(t)\, w'(t)$

$\therefore f = \lambda(t)|w'|^2$, $f=0$, $K=0$ only achieve at $\lambda(t)=0$

Case 2: α is any diretrix.

Numeration of $f = \langle w', \alpha' \wedge w + v w' \wedge w \rangle = \langle w', \alpha' \wedge w \rangle$

if S is developable $f=0 \Rightarrow K=0$.

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$X(t,v) = \alpha(t) + v\,w(t)$, $|w|=1$, $w' \neq 0$.

$\beta(t)$: line of striction $\langle \beta', w' \rangle = 0$

$$\beta(t) = \alpha(t) - \frac{\langle \alpha', w' \rangle}{|w'|^2} w$$

$\lambda(t) = \dfrac{(\beta', w, w')}{|w'|^2}$ is called the distribution parameter of ruled surface.

Developable surface $(\alpha', w, w') = 0$

$K_{ps} = 0$ at regular points.

Two subclasses of developable surface

<u>case1</u>: If $w \wedge w' = 0$ $\forall t$, $|w| = 1$, $\langle w, w' \rangle = 0 \Rightarrow w' = 0$

$\Rightarrow w$: const.

$\therefore$ S is <u>cylinder</u> over a curve obtained by intersecting S with the plane normal to $w(t)$.

<u>Case2</u>: If $w \wedge w' \neq 0$, $w' \neq 0$ $\forall t$

Hence <u>S is a noncylindrical</u>, $\exists\, \beta(t)$: line of striction.

$\langle \beta', w' \rangle = 0$.

By previousely argument.

$$\beta(t) = \alpha(t) - \frac{\langle \alpha', w' \rangle}{|w'|^2} w \ , \ \lambda(t) = \frac{(\beta', w, w')}{|w'|^2}$$

$$\beta'(t) = \alpha'(t) - \left(\frac{\langle \alpha', w' \rangle}{|w'|^2} \right)' w - \frac{\langle \alpha', w' \rangle}{|w'|^2} w'$$

$(\beta', w, w') \overset{?}{=} 0 = \langle \alpha', w, w' \rangle$ $\therefore$ $(\beta', w, w') = 0$

$\Rightarrow \lambda(t) = \dfrac{(\beta', w, w')}{|w'|^2} = 0$.

<u>recall</u>: $X(t,v) = \beta(t) + v\,w(t)$

$\boxed{X \text{ is singular iff } \lambda = 0 \text{ and } v = 0.}$

$\Leftrightarrow$ For developable surface, the whole line of striction are singular points.

under $w \wedge w' \neq 0$.

$\underline{\text{II}}$ⓐ If $\beta'(t) \neq 0 \ \forall t$ $\quad \begin{cases} \langle \beta', w' \rangle = 0 \\ \langle w, w' \rangle = 0 \end{cases}$ $\quad w \parallel \beta'$

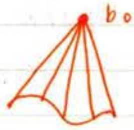
$$X(t,v) = \beta(t) + v\, c\, \beta'(t) = \beta(t) + \bar{v}\, \beta'(t)$$

$\therefore$ S is the tangent surface of β

IIⓑ If $\beta'(t) = 0 \ \forall t$

$X(t,v) = \alpha(t) + v\,w(t) \Rightarrow \beta = \text{const.} \equiv b_0$
namely

$$\beta(t) = \alpha(t) - \frac{\langle \alpha', w' \rangle}{|w'|^2} w(t)$$



$$X(t,v) = \underline{b_0} + \left(\frac{\langle \alpha', w' \rangle}{|w'|^2} + v \right) \underline{\underline{w}}$$

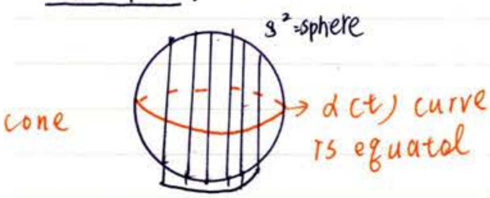
$\therefore$ S is a cone with vertex b_0

Hence developable surface is a piece of tangent surface, cylinder or cone #

Note: The cylinder and cone are ruled surface & developable surface. But the general hyperboliod of one sheet is not.
($\because$ it is locally ruled surface but $(\alpha', w, w') \neq 0$)

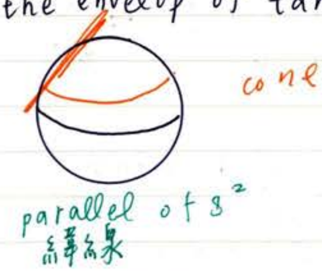
The envelopable of the family of tangent planes along a curve of surface.
i.e the surface is obtain by tangent plane along curve in the surface

Example: on S^2



If $\alpha(t)$ is a parallel of a sphere S^2, the envelop of tangent planes of S^2 along α

$$= \begin{cases} \text{cylinder} & \text{if } \alpha = \text{equator} \\ \text{cone} & \text{o.w} \end{cases}$$



Let S be a regular surface, and $\alpha(s)$ be a regular curve (by arc-length, i.e $|\alpha'(s)| = 1$)

Assume α' is never an asympotic direction (i.e $\mathrm{II}(\alpha') \neq 0$, normal curvature doesnot vanish)

Consider the ruled surface,

$$X(s,v) = \alpha(s) + v(w(s)) = \alpha(s) + v\,\frac{N(s) \wedge N'(s)}{|N'(s)|}$$

Where $N(s)$ = a normal vector along the curve α

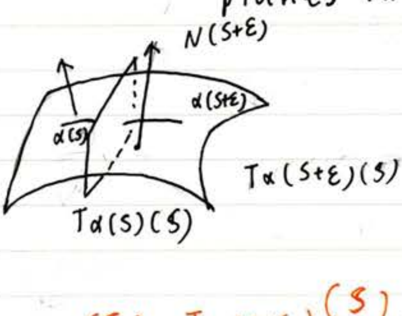
$\because$ α' is not asymptotic direction

$\mathrm{II}(\alpha) \neq 0$, $dN((\alpha'(s))) \Rightarrow dN(\alpha'(s)) \neq 0$.

$|N| = 1$, $\langle N, N' \rangle = 0$.

$\underline{N \wedge N'}$ $\perp$ the plane spanned by N and N'

Interpretation, $\{T_{\alpha(s)}(S^2)\}$ is a collection tangent planes in $\mathbb{R}^2$ to S along α



$T_{\alpha(s)}(S) \cap T_{\alpha(s+\varepsilon)}(S)$ is a line $/\!/$ $N(s) \wedge N(s+\varepsilon)$

$\varepsilon \to 0 \Rightarrow$ line $/\!/$ $\dfrac{N(s) \wedge N(\varepsilon+s)}{\varepsilon}$

$$\lim_{\varepsilon\to 0} \frac{N(s)\wedge N(s+\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{N(s)\wedge [N(s+\varepsilon)-N(s)]}{\varepsilon} = N(s)\wedge N'(s)$$

Consider the ruled surface

$$\boxed{X(\underset{=}{s},v) = \alpha(s)+vw(s) = \alpha(s) + v\,\frac{N(s)\wedge N'(s)}{|N'(s)|}}$$

where $N(s)$ = a normal vector along the curve α

$|N|=1$

Now we want to check this surface is developable surface

$\therefore$ Need $(\alpha', w, w')=0$

$$w(s) = \frac{N(s)\wedge N'(s)}{|N'(s)|}, \quad w'(s) = \Big(|N'(s)|\big[\overset{0}{N'(s)\wedge N'(s)} + N(s)\wedge N''(s)\big] - (N\wedge N')\,\big(\overset{0}{|N'(t)|}\big)'\Big) \Big/ |N'(s)|^2$$

$$(\langle N', N'\rangle^{\frac{1}{2}})' = -\langle N', N'\rangle^{-\frac{1}{2}} \langle N', N''\rangle^{0}$$

HW

- Show X is regular in Nbhd of 0 (v small) and it is tangent to S

$X(s,v) = \alpha(s) + v\,w(s)$

$X_s = \alpha' + v w' = \alpha' + v\left(\frac{N\wedge N''}{|N'|}\right)$

$X_v = w(s) = \frac{N\wedge N'}{|N'(s)|}$

$X_s\wedge X_v = \frac{\langle N', \alpha'\rangle N}{|N'|} - \frac{v(N, N', N'')N}{|N'|^2}$

$A\wedge B\wedge C = \langle A, C\rangle B - \langle B, C\rangle A$

check

$X_s\wedge X_v \,//\, N$ to S

$\therefore$ envelop surface is the tangent of S

normal curvature of α

$$\text{If } v=0,\ X_s\wedge X_v = \frac{\langle N', \alpha'\rangle N}{|N'|} = \frac{-\langle N, \alpha''\rangle N}{|N'|} = \frac{-\mathrm{II}(\alpha')N}{|N'|} = \frac{-k_n N}{|N'|} \neq 0$$

$\therefore$ regular

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Recall : Developable surface

Ⓘ ⓐ $w' \wedge w \neq 0$

If $\beta' \neq 0$, $\langle \beta', w' \rangle = 0$.

$(\beta', w, w') = 0 \Rightarrow (\beta', w', -w) = 0$

$\langle \beta' \wedge w', w \rangle = 0$, $w \in$ plane span by w' and β'

$|w| = 1 \Rightarrow \langle w, w' \rangle = 0 \Rightarrow w \parallel \beta'$

$|N|^2 = 1 \Rightarrow \langle \underset{\text{normal}}{N}, \underset{\text{tangent}}{N'} \rangle = 0$

$(|N'|^2)' = (\langle \underset{\text{tang}}{N'}, \underset{\text{tangent}}{N'} \rangle)' = \langle \underset{\text{Nors.}}{N''}, \underset{\text{tangent}}{N'} \rangle = 0$.

$X(t,v) = \alpha(t) + v \dfrac{N(s) \wedge N'(s)}{|N'(s)|}$ is developable surface

① X is regular

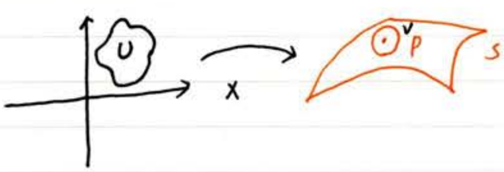
$\mathrm{II}(\alpha') \neq 0$, normal curvature

② $(\alpha', w, w') = 0$

H.W. check $K=0$ on regular points.

Minimal Surface

Def: A regular surface $S \subseteq \mathbb{R}^3$ is minimal if each parametrization its mean curvature is 0.



$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}$$

$N : S \to S^2$

$$H = \frac{k_1 + k_2}{2}$$

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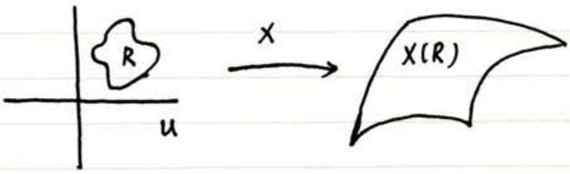
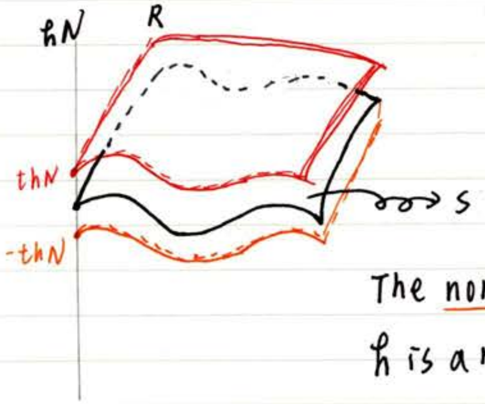
Interpretation of minimality:

$H=0 \Rightarrow k_1 = -k_2$



$A(X(R)) = \iint (EG-F^2)\,du\,dv$



N = normal vector

Let $S: U \to S$ be a parametrization of S. $D \subseteq U$, $\bar{D}$ = closure of D

$h: \bar{D} \to \mathbb{R}$ is differentiable

The normal variation of $X(\bar{D})$ determined by h is a map.

$\varphi: \bar{D} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$

$(u,v,t) \to X(u,v) + th(h,v)N(u,v)$

$\varphi(u,v,t) = X(u,v) + th(u,v)N(u,v)$

If $h=1$, move it to normal direction.

If $h \neq 1$, move it via h to Normal direction.

For each fix $t \in (-\varepsilon, \varepsilon)$

$$X^t(u,v) = \varphi(u,v,t)$$

$$X^t_u = X_u + th_uN + thN_u$$

$$X^t_v = X_v + th_vN + thN_v$$

$$E^t = \langle X^t_u, X^t_u \rangle = \overset{= E}{\langle X_u, X_u \rangle} + 2th_u \overset{0}{\langle N, \cancel{X}_u \rangle} + th\langle N_u, X_u\rangle$$

$$+ t^2h^2_u \underbrace{N \cdot N}_{=1} + t^2h^2 N_u^2 + t^2hh_u\langle N, N_u\rangle$$

$= \langle X_u, X_u \rangle + 2th \langle X_u, N_u \rangle + O(t^2)$

$O(t^2) = t^2 (h^2 \langle N_u, N_u \rangle + h_u^2)$

$\because \lim_{t \to 0} \frac{t^2}{t} = 0.$

$E^t = \langle X_u^t, X_u^t \rangle = E + 2th \langle \underbrace{X_u, N_u}_{= -\langle N, X_{uu} \rangle = -e} \rangle + O(t^2)$

$F^t = \langle X_u^t, X_v^t \rangle = F + ht \langle N_u, X_v \rangle + ht \langle N_v, X_u \rangle + O_2(t^2)$

where $O_2(t) = -t^2 (h^2 \langle N_u, N_v \rangle + th_u h_v)$

$G^t = \langle X_v^t, X_v^t \rangle = G + 2th \langle X_v, X_v \rangle + O_3(t^2)$

$F^t G^t - F^{t^2}$ $\qquad O_3(t^2) = t^2 (h^2 \langle X_v, X_v \rangle + h_v h_v)$

$E^t \cong E - 2hte$
$F^t \cong F - 2htf$
$G^t \cong G - 2htg$

$\therefore E^t G^t - F^{t^2} = (E - 2hte)(G - 2htg) - (F - 2htf)^2$

$= EG - F^2 - 2th(Eg - 2fF + Ge) + \underbrace{(-ht^2(2ge - 4hf))}_{= O_4(t^2)}$

$E^t G^t - F^{t^2} \cong EG - F^2 - 2ht(\underbrace{Eg - 2Ff + Ge}_{= 2H(EG - F^2)})$

$\because H = \frac{Eg - 2fF + Ge}{2EG - F^2} = (EG - F^2)(1 - 4thH)$

Hence the area of $X^t(D) = A(X^t(\bar{D}))$

$A(X^t(\bar{D})) = \iint_{\bar{D}} (E^t G^t - F^{t^2})^{\frac{1}{2}} du dv \cong \iint_{\bar{D}} (EG - F^2)^{\frac{1}{2}} (1 - 4thH)^{\frac{1}{2}} du dv$

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minimal → critical point

$$\frac{dA(X^t(D))}{dt}\Big|_{t=0} = \iint_{\bar{D}} \frac{1}{2}(EG-F^2)^{\frac{1}{2}}(1-4thH)^{-\frac{1}{2}}(-4hH)\Big|_{t=0}$$

$$= -\iint_{\bar{D}} 2hH(EG-F^2)^{\frac{1}{2}}(1-4thH)^{-\frac{1}{2}}\Big|_{t=0}$$

$$\frac{d}{dt}(A(X^t(\bar{D})))\Big|_{t=0} = -2\iint_{\bar{D}} hH(EG-F^2)^{\frac{1}{2}}\,du\,dv \quad — (*)$$

S is minimal if $H=0$.

$\frac{d}{dt}A(0)=0$.

"Fixed boundary"

So if X is minimal parametrization of S,

$H=0 \Rightarrow$ critical point of $(A(\bar{D}))$

i.e $\frac{d}{dt}(A(X^t(D)))\Big|_{t=0} = 0 \cdot \forall h$

<u>prop</u>. Let $X: U \to \mathbb{R}^3$ be a regular parametrization surface and let $D \subset U$ a b.dd (bounded) domain in U. Then X is a minimal iff $A'(0)=0$ for all such D and all normal variation h of $X(\bar{D})$.

"$\Rightarrow$"

"$\Leftarrow$"

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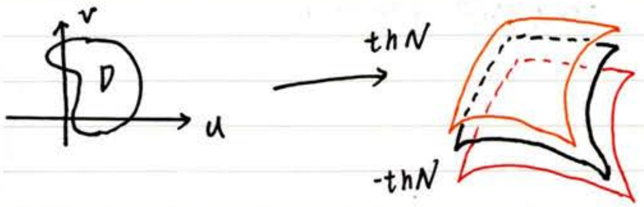
$X: D \subset U \to \mathbb{R}^3$

$h: \bar{D} \to \mathbb{R}$

$\varphi: \bar{D} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$

$\varphi(u,v,t) = X(u,v) + th\, N(u,v)$



on S

$A'(0) = 0$ iff $H=0$

"$\Leftarrow$") $\because$ $A'(t)\big|_{t=0} = -\iint_{\bar{D}} 2hH(EG-F^2)^{\frac{1}{2}}\, du\, dv$, $A'(0)=0$

"$\Rightarrow$" $\exists\, q$ s.t $H \neq 0$, choose $h = H_q$

$$A' = -\iint_{\bar{D}} 2H^2 (EG-F^2)^{\frac{1}{2}}\, du\, dv$$

$A'(0) < 0$.

$H=0$ iff $\frac{d}{dt}(A(X^t(D)))=0$ $\forall h$

critical point 不一定有極值

S minimal surface if $H=0$. 但 ∵ def, critical point ⇒ minimal surface

Def: A regular parametrization surface is isothermal if

$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle$ $E=G$

$\langle X_u, X_v \rangle = 0$. $F=0$

where $X: U \to \mathbb{R}^3$ parametrization

(recall p.78 X is diff.

dX_q is 1-1

Theorem: Let $X(u,v)$ be a regular parametrization surface. Assume X is isothermal.

Then $\Delta X = X_{uu} + X_{vv} = 2\lambda^2 \vec{H}$ where $\lambda^2 = \langle X_u, X_u \rangle = \langle X_v, X_v \rangle$ (Laplacian) $(= E = G)$

$\vec{H} = H N$ is called mean carvature vector.

H: mean curvature, N: normal vector

$\because E = G$

$E_u = G_u \Leftrightarrow \langle X_u, X_u \rangle_u = \langle X_v, X_v \rangle_u$

$2\langle X_{uu}, X_u \rangle \quad \langle X_v, X_{vu} \rangle$ — ①

$E_v = G_v \Leftrightarrow \langle X_u, X_u \rangle_v = \langle X_v, X_v \rangle_u$

$\langle X_{uv}, X_u \rangle = \langle X_{vv}, X_v \rangle$ — ②

$F = 0$, $F_u = \langle X_u, X_v \rangle_u \Rightarrow \langle X_{uu}, X_v \rangle = -\langle X_u, X_{vu} \rangle$ — ③

$F_v = \langle X_u, X_v \rangle_v \Rightarrow \langle X_{uv}, X_v \rangle = -\langle X_u, X_{vu} \rangle$

$\langle X_{uu}, X_u \rangle \underset{①}{=} \langle X_v, X_{vu} \rangle \underset{④}{=} -\langle X_u, X_{vv} \rangle = -\langle X_{vv}, X_u \rangle$

$\langle X_{uu} + X_{vv}, X_u \rangle = 0 \Rightarrow \langle \Delta X, X_u \rangle = 0$, $\Delta X \perp X_u$

$\langle X_{vv}, X_v \rangle \underset{②}{=} (X_{uv}, X_u) \underset{③}{=} -\langle X_{uu}, X_u \rangle$

$\Rightarrow \langle \Delta X, X_u \rangle = 0$, $\Delta X \perp X_u, X_v$

$\therefore \Delta X \parallel N$

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}$$

$\because X$ is isothermal $E = G$, $F = 0$

$$H = \frac{eE + Eg}{2E^2} = \frac{g+e}{2E} \quad \therefore g + e = 2EH$$

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On the other hand, $g+e = \langle X_{vv}, N\rangle + \langle X_{uu}, N\rangle$

$2EH = \langle \Delta X, N\rangle$

$E = G = \lambda^2$

$\langle \Delta X, N\rangle = 2\lambda^2 H$

$|\Delta X| = 2\lambda^2 H$, $\langle \Delta X, N\rangle N = 2\lambda^2 HN$

$\Delta X = 2\lambda^2 \vec{H}$

<u>Def</u>: $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is defined by $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ $\forall (x,y) \in U$

f is harmonic in U if $\Delta f = 0$.

<u>Cor</u>: Let $X(u,v) = (\underbrace{x(u,v), y(u,v), z(u,v)}_{\text{coordinate function}})$ be a parametrized surface and X be isothermal, iff its coordinate functions x, y, z are harmonic. then X is minimal

$\because \Delta X = 2\lambda^2 \vec{H} \Leftrightarrow \Delta X = 0$ iff $\Delta x, \Delta y, \Delta z$

1760 Lagrange

1850 Plateau: For each closed curve $\alpha \subseteq \mathbb{R}^3$

$\exists$ a regular surface S of minimal area with α as boundary

1930 Douglas & Radó

catenoid: $X(u,v) = (a\cosh v\cos u, a\cosh v\sin u, av)$

surface of revolution

$X(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$

1776 Meusnier < Catenoid. / helicoid

1835 Scherk (p.207)

Osserman

<u>Example</u>: Helicoid

$X(u,v) = (a\sinh v\cos u, a\sinh v\sin u, au)$

$0 < u < 2\pi$, $-\infty < v < \infty$

X: diff, dX_q: 1-1

check X is minimal surface

① X is isothermal $E = G,\ F = 0$.

② $\Delta x = 0$

$\Delta y = 0$

$\Delta z = 0$.

$X_u = (-a\sinh v \sin u,\ a\sinh v\cos u,\ a)$

$X_v = (a\cosh v\cos u,\ a\cosh v\sin u,\ 0)$

$X_{uu} = (-a\sinh v\cos u,\ -a\sinh v\sin u,\ 0)$

$X_{uv} = (-a\cosh v\sin u,\ a\cosh v\cos u,\ 0)$

$X_{vv} = (-a\sinh v\cos u,\ -a\sinh v\sin u,\ 0)$

$E = \langle X_u, X_u\rangle = a^2(\sin^2 hv + 1) = a^2\cos hv$

$G = \langle X_v, X_v\rangle = a^2\cos^2 hv = E$

$F = \langle X_u, X_v\rangle = 0$.

$\therefore$ X is isothermal

$$\left.\begin{array}{l} x_{uu} + x_{vv} = 0 \\ y_{uu} + y_{vv} = 0 \\ z_{uu} + z_{vv} = 0. \end{array}\right\} \text{ coordinate functions are harmonic}$$

By cor: X is minimal surface

Example: Helicoid

$X(u,v) = (u\cos v,\ u\sin v,\ bv)$ $0 < u < 2\pi$, $-\infty < v < \infty$

HW

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<u>Example</u>: $X(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$

Enneper's minial surface

$X_u = (1-u^2+v^2,\ 2uv,\ 2u)$

$X_v = (2uv,\ 1-v^2+u^2,\ -2v)$

$X_{uu} = (-2u,\ 2v,\ 2)$

$X_{uv} = (2v,\ 2u,\ 0)$

$X_{vv} = (2u,\ -2v,\ -2)$

$\begin{vmatrix} 1-u^2+v^2 & 2u \\ 2uv & -2v \end{vmatrix} = 2u^2v - 2v^3 - 2v - 4u^2v$

$E = (1-u^2+v^2)^2 + 4u^2v^2 + 4u^2$

$= u^4 + v^4 + 1 + 2u^2v^2 + 2u^2 + 2v^2 = (u^2+v^2+1)^2$

$G = E$, $F = \cancel{uv} - u^3v + uv^3 + \cancel{uv} + u^3v - uv^3 - \cancel{2uv} = 0$

$\Rightarrow$ isothermal

$x_{uu} + x_{vv} = 0$

$y_{uu} + y_{vv} = 0$

$z_{uu} + z_{vv} = 0$

<u>prop</u>: ① Any minimal surface of revolution S is an open subset of a plane or a catenoid.

$X(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$

② Any ruled minimal surface is an open subset of a plane or a helicoid. $X(t,v) = \alpha(t) + v\,w(t)$

<u>Theorem</u>: There is no minimal surface which can be a compact set.

(compact surface S, $\exists\, p \in S$, $K(p) > 0$)

$H = 0 = \frac{k_1 + k_2}{2} \Rightarrow k_1 = -k_2 \Rightarrow K = k_1 k_2 = -k_2^2 < 0$.

— 4/11

The intrinsic Geometry of surface

length of curve

<u>area</u>

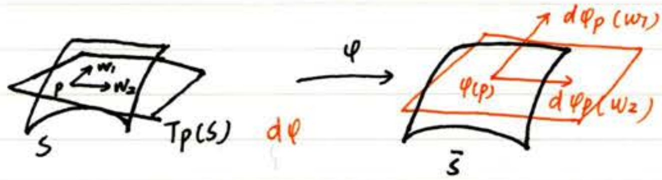
angle

S

$\bar{S}$

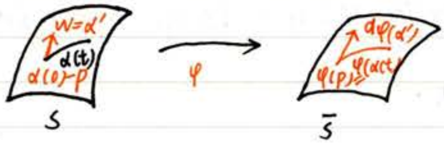
Isometries

Def : A diffeomorphism $\varphi: S \to \bar{S}$ is an isometry if $\forall\, p \in S$ and for all $w_1, w_2 \in T_p(S)$, we have $\langle w_1, w_2 \rangle_p \equiv \langle d\varphi_p(w_1), d\varphi_p(w_2) \rangle_{\varphi_p}$

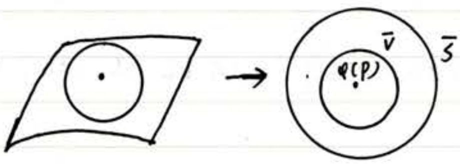


the surface S and $\bar{S}$ are said to be isometric i.e $I_p(w) = I_{\varphi(p)} d\varphi(w)$

$\forall\, w \in T_p(S)$, $w = w_1 = w_2$, $\langle w, w \rangle_p = \langle d\varphi_p(w), d\varphi_p(w) \rangle_{\varphi(p)}$



Def : A map $\varphi: V \to \bar{S}$ of a Nbhd V of $p \in S$ is a local isometry at p if a Nbhd $\bar{V}$ of $\varphi(p)$ of $\bar{S}$ $\ni$ $\varphi: V \to \bar{V}$ is isometry.



$\bar{S}$ and S are called local isometric.

Cylinder $\quad X(u,v) = (\cos u, \sin u, v)$, $u \in (0, 2\pi)$, $v \in \mathbb{R}$

$X_u = (-\sin u, \cos u, 0)$, $E = 1$, $F = 0$, $G = 1$

$X_v = (0, 0, 1)$

plane $\subseteq \mathbb{R}^2$ $\quad \bar{S} = \bar{X}(u,v) = P_0 + u w_1 + v w_2$, $(u,v) \in \mathbb{R}^2$

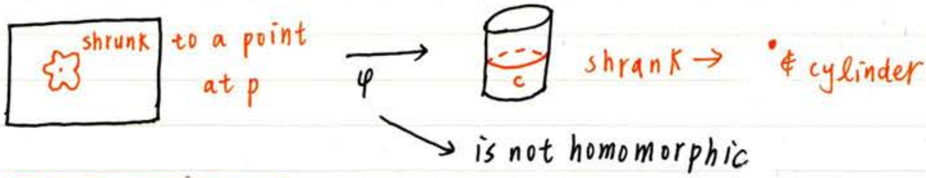
A plane is passing through P_0 & containing orthgonal vectors w_1, w_2, $|w_1| = 1$, $|w_2| = 1$

$\bar{X}_u = w_1 \qquad \bar{E} = \langle \bar{X}_u, \bar{X}_u \rangle = 1$

$\bar{X}_v = w_2 \qquad \bar{F} = 0$

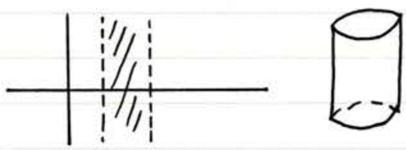
$\qquad\qquad \bar{G} = 1$

$\varphi: \underset{\bar{S}}{U} \to S$, $U = (0, 2\pi) \times \mathbb{R} \subseteq \mathbb{R}^2$



$\boxed{\underline{\text{locally}} \text{ isometric}}$

<u>Ideal</u>: Any simple closed curve in $\mathbb{P}$ (plane) can be shrank continuously into a point without leaving the plane.
But a parallel of cylinder does not have this property.
$\therefore$ the cylinder is locally isometric to $\mathbb{R}^2$.



<u>Example</u>: Every helicoid is locally isometric to catenoid.

<u>Rmk</u>: $\sinh v = \frac{e^v - e^{-v}}{2}$, $\cosh v = \frac{e^v + e^{-v}}{2}$

$X(u,v) = (a\sinh v\cos u,\ a\sinh v\sin u,\ au)$, $u \in (0, 2\pi)$, $v \in \mathbb{R}$

helicoid $E = a^2\cos^2 hv$, $F = 0$, $G = a^2\cos^2 v$

$\bar{X}(u,v) = (\underbrace{a\cosh v}_{f(v)}\cos u,\ \underbrace{a\cosh v}_{f(v)}\sin u,\ \underbrace{av}_{g(v)})$, $u \in (0, 2\pi)$, $v \in \mathbb{R}$

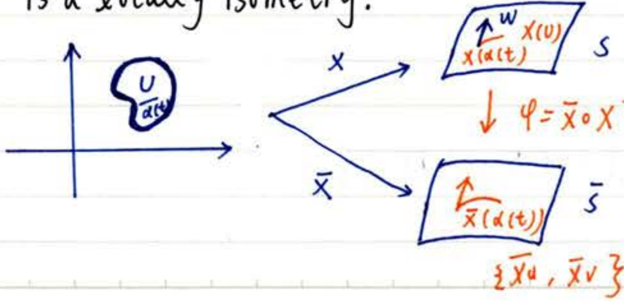
$\bar{X}_u = (-a\cosh v\sin u,\ a\cosh v\cos u,\ 0)$

$\bar{X}_v = (a\sinh v\cos u,\ a\sinh v\sin u,\ a)$

$\bar{E} = a^2\cos^2 hv$, $\bar{F} = 0$, $\bar{G} = a^2 + a^2\sin^2 hv = a^2\cos^2 hv$

<u>prop</u>: Let S and $\bar{S}$ be two regular surface. Assume the parametrization
$X: U \to S$ and
$\bar{X}: \to \bar{S}$ $\ni$ $\bar{E} = E$, $\bar{F} = F$ and $\bar{G} = G$ in U iff the map $\varphi: X(U) \to \bar{S}$
is a locally isometry.

"$\Rightarrow$"



$\langle d\varphi_p(w), d\varphi_p(w)\rangle$ $+\langle \bar{X}_v, \bar{X}_v\rangle v'^2$
$= \langle \bar{X}_u, \bar{X}_u\rangle u'^2 + 2\langle \bar{X}_u, \bar{X}_v\rangle u'v'$
$\overset{\shortparallel}{\bar{E}} = E$

Let α be a curve in U, $\alpha: I \to U$

$\alpha(t) = (u(t), v(t))$

$X(\alpha(0)) = p$, $\frac{d}{dt}\left(X(\alpha(t))\right)\Big|_{t=0} = w = X_u u' + X_v v'$

$\varphi(X(\alpha(t)))\Big|_{t=0} = \bar{X} \circ X^{-1}(X(\alpha(t)))\Big|_{t=0}$

$d\varphi_p(w) = \frac{d}{dt}(\varphi(X(\alpha(t)))_{t=0} = \frac{d}{dt}(\bar{X}(\alpha(t)))\Big|_{t=0} = \bar{X}_u u' + \bar{X}_v v'$

$\Leftarrow$ $\because$ φ is locally isometry...

$\therefore$ $L(X(\alpha))$ = length of curve in S
$\shortparallel$
$L(\bar{X}(\alpha))$ = length of curve in $\bar{S}$

$X: U \to S$

$\bar{X} = \varphi \circ X: U \to \bar{S}$

$\alpha: I \to U$, $I = [0, \varepsilon]$

$L(X(\alpha)) = \int_0^\varepsilon \left[E(u')^2 + 2F(u')(v') + G(v')^2\right]^{\frac{1}{2}} dt$

$= \int_0^\varepsilon \left[\bar{E}(u') + 2\bar{F}(u')(v') + \bar{G}(v')^2\right]^{\frac{1}{2}} dt$ — ⊛

$= L(\bar{X}(\alpha))$

given (u_0, v_0)

Look at $(\underbrace{u_0+t}_{u(t)}, \underbrace{v_0}_{v(t)})$ $(u') = 1$, $(v') = 0$

in ⊛ we have $\int_0^\varepsilon (E)^{\frac{1}{2}} dt = \int_0^\varepsilon (\bar{E})^{\frac{1}{2}} dt$

$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_0^\varepsilon (E)^{\frac{1}{2}} dt = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_0^\varepsilon (\bar{E})^{\frac{1}{2}} dt$, $E(u_0, v_0) = \bar{E}(u_0, v_0)$, $E = \bar{E}$

Now, look at $(\underbrace{u_0}_{u(t)}, \underbrace{v_0+t}_{v(t)})$ $u' = 0$, $v' = 1$

$\therefore \int_0^\varepsilon G\, dt = \int_0^\varepsilon \bar{G}\, dt \Rightarrow G = \bar{G}$

look at (u_0+t, v_0+t) $u' = 1$, $v' = 1$

$\int_0^\varepsilon (E + 2F + G)\, dt = \int_0^\varepsilon (\bar{E} + 2\bar{F} + \bar{G})\, dt \Rightarrow F = \bar{F}$

$E = \bar{E}$
$G = \bar{G}$ in $U \Leftrightarrow \varphi$ is locally isometry
$F = \bar{F}$

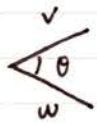
- **Conformal**

Def : A diffeomorphism $\varphi : S \to \bar{S}$ is called a comformal map if $\forall p \in S$ and $\forall v_1, v_2 \in T_p(S)$, we have
$<d\varphi_p(v_1), d\varphi_p(v_2)>_{\varphi(p)} = \lambda^2(p) <v_1, v_2>_p$ where $\lambda^2(p)$ is a nowhere zero differential function on S.

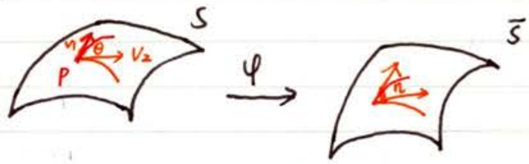
— 4/12

Recall : $\varphi : S \to \bar{S}$ diffeomorphism . $<v_1, v_2>_p = <d\varphi_p(v_1), d\varphi_p(v_2)>_{\varphi(p)}$
$\forall p \in S$, $\forall v_1, v_2 \in T_p(S)$)
φ is isometry
if $<d\varphi_p(v_1), d\varphi_p(v_2)>_{\varphi(p)} = \underline{\underline{\lambda}}^2 <v_1, v_2>_p$ $\lambda \neq 0, \lambda^2 > 0$ — ⊛
φ is conformal map.

The surface S and $\bar{S}$ are said to be conformal (conformal equivalent)
$S \sim \bar{S}$, $S_1 \sim S_2$ and $S_2 \sim S_3 \Rightarrow S_1 \sim S_3$
A map $\varphi : V \subseteq S \to \bar{S}$ of a Nbhd V of $p \in S$ into $\bar{V}$ (Nbhd of $\varphi(p)$) is called local conformal + $\varphi : V \to \bar{V}$ conformal — ⊛ holds



$\cos\theta = \dfrac{<v, w>}{|v||w|}$



Fact : A conformal map preserves the angle between two tangent vectors.

$|d\varphi_p(v)| = \sqrt{<d\varphi_p(v), d\varphi_p(v)>}$
$\because \varphi$ is conformal $= \sqrt{\lambda^2 <v, v>} = \lambda |v|$

$$\cos(\eta) = \frac{<d\varphi_p(v_1), d\varphi_p(v_2)>_{\varphi(p)}}{|d\varphi_p(v_1)|\,|d\varphi_p(v_2)|} = \frac{\lambda^2 <v_1, v_2>}{\lambda^2 |v_1||v_2|} = \frac{<v_1, v_2>}{|v_1||v_2|} = \cos\theta$$

prop : If $X : U \to S$ and $\bar{X} : U \to \bar{S}$

$$+\begin{cases} E = \lambda^2 \bar{E} \\ F = \lambda^2 \bar{F} \\ G = \lambda^2 \bar{G} \end{cases} \text{ for some function. } \lambda^2 : U \to \mathbb{R}^+ .$$

then φ is a local conformal map.

$$\langle d\varphi_p(v), d\varphi_p(v)\rangle = \lambda^2 \langle v, v\rangle_p = \lambda^2 I_p(v)$$

<u>Theorem</u>: Any two regular surface are conformal.

Ideal: Choose a special Parametrization. For a Nbhd of any point of S. By isothermal Parametrization, in which the first fundemental form. $E = G = \lambda^2$, $F = 0$.

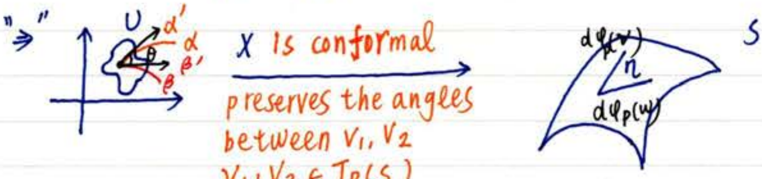
$\mathbb{P}$ = Plane, $E = G = 1$, $F = 1$

L. Bers : Riemann Surface, p 15-35

<u>Prop</u>: If $X : U \to S$ is conformal iff $E = G$, $F = 0$ (X is isothermal)

"$\Leftarrow$" $E_{\mathbb{R}^2} = 1 = G_{\mathbb{R}^2}$, $F_{\mathbb{R}^2} = 0$.

"$\Rightarrow$"



$\cos\theta = \cos\eta$.

$$\begin{matrix} \alpha' = v \\ \beta' = w \end{matrix}, \quad \cos\theta = \frac{\langle v, w\rangle}{|v|\,|w|}, \quad \cos\eta = \frac{\langle dX(v), dX(w)\rangle}{|dX(v)|\,|dX(w)|}$$

Namely, $\alpha' = v = (u_1'(t), v_1'(t))$
$\beta' = w = (u_2'(t), v_2'(t))$ $\Rightarrow$ $\cos\theta = \dfrac{\langle v, w\rangle}{|v|\,|w|} = \dfrac{u_1'u_2' + v_1'v_2'}{(u_1^2+v_1^2)^{\frac{1}{2}}(u_2^2+v_2^2)^{\frac{1}{2}}}$

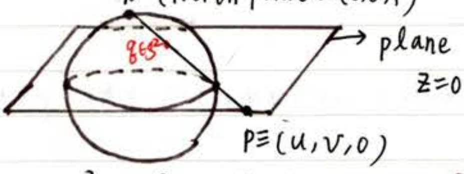
$$dX(v) = X_u(u_1') + X_v(v_1')$$
$$dX(w) = X_u(u_2') + X_v(v_2')$$

$$\cos\eta = \frac{\langle X_u u_1' + X_v v_1', X_u u_2' + X_v v_2'\rangle}{(E(u_1')^2 + 2F(u_1')(v_1') + G(v_1')^2)^{\frac{1}{2}}(E(u_2')^2 + 2F(u_2')(v_2') + G(v_2')^2)^{\frac{1}{2}}}$$

$$= \frac{E(u_1')(u_2') + G(v_1'v_2') + F(u_1'v_2' + v_1'u_2')}{(E(u_1')^2 + 2F(u_1')(v_1') + G(v_1')^2)^{\frac{1}{2}}(E(u_2')^2 + 2F(u_2')(v_2') + G(v_2')^2)^{\frac{1}{2}}}$$

$$At\ (u_1(t), v_1(t)) = (t+u_0, v_0) = (t,0)\ \text{ (if } (u_0,v_0)=(0,0))$$
$$(u_2(t), v_2(t)) = (u_0, v_0+t) = (0,t)$$
$$u_1'=1,\ v_1'=0,\ u_2'=0,\ v_2'=1$$
$$\cos\theta = \cos\left(\frac{\pi}{2}\right) = 0 = \cos\eta = \frac{F}{\bigcirc} \Rightarrow F=0.$$
$$At\ (u_1(t), v_1(t)) = (u_0+t, v_0+t) = (t,t) \qquad u_1'=1=v_1'$$
$$(u_2(t), v_2(t)) = (u_0+t, v_0-t) = (t,-t) \qquad v_2'=-1,\ u_2'=1$$
$$\Rightarrow 0=\cos\eta = \frac{E-G}{\bigcirc} \Rightarrow E=G$$

• Stereographic Projection



$S^2 \to \mathbb{R}^2 \qquad q=(x,y,z),\ x^2+y^2+z^2=1$

p, q and N ∈ straight line, $\exists t$, $q-N=t(p-N)$

$$q = N+t(p-N) = (0,0,1)+t((u,v,0)-(0,0,1))$$
$$= (tu, tv, 1-t) \in S^2$$

$$z=1-t,\ u=\frac{x}{t}=\frac{x}{1-z},\ v=\frac{y}{1-z}$$

Projection map. $f: S^2 \to \mathbb{P}$

$$f(x,y,z) = (u,v,0) = \left(\frac{x}{1-z}, \frac{y}{1-z}; 0\right)$$
$$f^{-1}: \mathbb{P} \to S^2$$
$$f^{-1}(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2}\right)$$
$$(tu)^2+(tv)^2+(1-t)^2=1$$
$$t^2(u^2+v^2+1)-(2t)=0.\ t=\frac{2}{1+u^2+v^2}$$

$$X(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2}\right) \Rightarrow X: \mathbb{P} \to S^2 \setminus \{(0,0,1)\}$$

∵ 一個 parametrization 不能包覆 sphere (至少要2個)

$$X_u = \frac{1}{(1+u^2+v^2)^2}\left[2(v^2-u^2+1),\ -4uv,\ 4u\right]$$

$$X_v = \frac{1}{(1+u^2+v^2)^2}\left[-4uv,\ 2(u^2-v^2+1),\ 4v\right]$$

$$\bar{E} = \frac{4(v^2-u^2+1)^2+16u^2v^2+16u^2}{(1+u^2+v^2)^4} = \frac{4(1+u^2+v^2)^2}{(1+u^2+v^2)^4} = \frac{4}{(1+u^2+v^2)^2} = \langle X_u, X_u\rangle$$

$$\bar{F} = \langle X_u, X_v\rangle = 0$$

$$\bar{G} = \langle X_v, X_v\rangle = \frac{16u^2v^2+16v^2+4(u^2-v^2+1)}{(1+u^2+v^2)^4} = \frac{4}{(1+u^2+v^2)^2}$$

$$\therefore \lambda^2 = \frac{4}{(1+u^2+v^2)^2}$$

<u>Recall</u>: ① Every regular surface is locally conformal to the plane (flat plane) — 4/18

* isothermal, $E=G,\ F=0$. $I_S(\alpha') = \lambda^2 I_{P}(\beta')$

② Any two regular surface are isothermal.

<u>4-3</u> Gauss Theorem and equations of compatibility

S is a regular surface in $\mathbb{R}^3$, $\{X_u, X_v, N\}$ forms a basis for $\mathbb{R}^3$ at each point $p \in S$

$$① \begin{cases} X_{uu} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + L_1 N \\ X_{uv} = \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + L_2 N \\ X_{vu} = \Gamma_{21}^1 X_u + \Gamma_{21}^2 X_v + L_3 N \\ X_{vv} = \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + L_4 N \end{cases}$$

$\langle N, X_{uu}\rangle = L_1$ ($\because \langle N, X_u\rangle = 0,\ \langle N, X_v\rangle = 0,\ \langle N, N\rangle = 1$) $= e$

$\langle N, X_{uv}\rangle = L_2 = f = L_3 = \langle N, X_{vu}\rangle$

$\langle N, X_{vv}\rangle = L_4 = g$