Midterm ODE (2013/11/15) Total:100points

1. (10%) Write

$$\begin{aligned} y'' + 3\sin(zy) + z' &= \cos t \\ z''' + z'' + 3y' + z'y &= t \end{aligned}$$

as an equivalent first order system of differential equations.

2. (10%) Solve the following problems

(i) $\frac{dx}{dt} = x^3$, $x(0) = 1$.

(ii) $\frac{dx}{dt} = h(t)x + k(t)$, $x(t_0) = x_0$.

3. (15%)State contraction mapping principle, and show how to apply it to prove the existence and uniqueness of the solution of I.V.P.

$$\begin{aligned} \frac{dx}{dt} &= f(t, x) \\ x(t_0) &= x_0 \end{aligned}$$

4. **(i)** (5%)State Ascoli-Arzela Theorem.

(ii) (15%)Let $f : [t_0,\, t_0 + h] \times \mathbb{R}^N \to \mathbb{R}^N$ be continuous and bounded. Show that the I.V.P. $\frac{dx}{dt} = f(t, x)$, $x(t_0) = x_0$ has at least one solution $x(t)$ defined on $[t_0, t_0 + h]$. (Hint: Divide $[t_0, t_0 + h]$ into n equal parts. Construct $\{x_n\}_{n=1}^{\infty}$

$$x_n(t) = \begin{cases} x_0, & t_0 \le t \le t_0 + \frac{h}{n} \\ x_0 + \displaystyle\int_{t_0}^{t-\frac{h}{n}} f(s, x_n(s))ds & \end{cases}$$

Apply Ascoli-Arzela Theorem.)

5. **(i)** (5%)Explain continuous dependence on initial conditions and parameters for the first order system $\frac{dx}{dt} = f(t, x)$.

(ii) (10%)Let $x(t)$ be a scalar differentiable function such that

$$\begin{aligned} x'(t) &\le f(t, x(t)) \text{ on } [t_0, b] \\ x(t_0) &\le x_0. \end{aligned}$$

If $\varphi(t)$ is the unique solution of I.V.P.

$$\begin{aligned} \frac{dy}{dt} &= f(t, y) \\ y(t_0) &= x_0. \end{aligned}$$

Show that $x(t) \le \varphi(t)$ on $[t_0, b]$.

6. (15%)Show that any solution of

$$x'' + x + x^3 = 0$$

exists for all $t \in \mathbb{R}$. Can the same be said about the equation

$$x'' + x' + x + x^3 = 0.$$

(Hint: Consider energy function

$$E(t) = \text{kinetics } + \text{ potential}$$

and energy level curves.)

7. (15%)Show that the solutions $x_1(t), x_2(t)$ of the Lotka-Volterra two species competition model

$$\begin{aligned} \frac{dx_1}{dt} &= r_1 x_1 (1 - \frac{x_1}{K_1}) - \alpha_1 x_1 x_2 \\ \frac{dx_2}{dt} &= r_2 x_2 (1 - \frac{x_2}{K_2}) - \alpha_2 x_1 x_2 \\ x_1(0) &> 0, x_2(0) > 0 \end{aligned}$$

are positive and bounded for all $t \geq 0$.