ODE Final Exam (2014/1/15) Total:150points

1. (10%) Sketch the phase portrait of

$$\frac{dx_1}{dt} = x_1 + 5x_2$$
$$\frac{dx_2}{dt} = -x_1 - 2x_2.$$

2. (20%) By the definition of e^{At}, prove the following:

   (i) Let A, $B \in \mathbb{R}^{n\times n}$. If $AB = BA$ then $e^{(A+B)t} = e^{At}e^{Bt}$.

   (ii) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$. Prove that $e^{(A+B)t} \neq e^{At}e^{Bt}$.

3. (20%)Prove the following variation of constant formula and $x(t, x_0)$ be the solution of

$$\begin{cases} \frac{dx}{dt} = Ax + g(t),\ A \in \mathbb{R}^{n\times n} \\ x(0) = x_0 \end{cases}$$

   Then

$$x(t, x_0) = e^{At}x_0 + \int_0^t e^{A(t-s)}g(s)ds.$$

4. (20%)Consider the following Predator-prey system

$$\begin{cases} \frac{dx}{dt} = \gamma x(1 - \frac{x}{k}) - \alpha xy \\ \frac{dy}{dt} = (\beta x - d - \delta y)y \\ x(0) > 0,\ y(0) > 0 \text{ and } \gamma,k,\alpha,\beta,d,\delta > 0. \end{cases}$$

   (i) Find all equilibria with nonnegative components.

   (ii) Do stability analysis for each equilibrium.

   (iii) Do isocline analysis & plot the portrait of the flow.

   (iv) Predict the global behavior of the solution $(x(t),y(t))$.

5. (20%)

   (i) State stable manifold theorem.

   (ii) Show that if $y(t)$, $t \leq 0$ is a bounded solution of integral equation

$$y(t) = e^{At}Py(0) + \int_0^t e^{A(t-s)}Pg(y(s))ds + \int_{-\infty}^t e^{A(t-s)}Qg(y(s))ds,$$

   then $y(t)$ is a solution of $\frac{dx}{dt} = Ax + g(x)$, $g(x) = o(|x|)$ as $x \to 0$.

6. (20%)

(i) Consider linear inhomogeneous system

$$x' = Ax + f(t) \quad (*) \ ,$$

where $f(t)$ is a continuous 2π-periodic function. If there is a 2π-periodic solution $y(t)$ of adjoint equation $y' = -A^T y$ such that

$$\int_0^{2\pi} y^T(t) f(t) dt \neq 0.$$

Show that every solution $x(t)$ of $(*)$ is unbounded.(Hint: compute $\frac{d}{dt}(y^T(t)x(t))$ and integrate from 0 to ∞).

(ii) Show that the resonance occurs for the second order linear equation

$$x'' + \omega_0^2 x = F \cos \omega t$$

when $\omega = \omega_0$.

7. (20%)

(i) State Abel's formula.

(ii) Show that $\det e^A = e^{\text{trace} A}$ for any $A \in \mathbb{R}^{n \times n}$.

(iii) Let φ_t denote the flow of the autonomous system $\frac{dx}{dt} = f(x)$, $x \in \mathbb{R}^n$, and let Ω be a bounded region in $\mathbb{R}^n$. Define the volume of $\varphi_t(\Omega)$,

$$V(t) = \int_{\varphi_t(\Omega)} dx_1 ... dx_n.$$

Use Abel's formula and change of variables formula for multiple integral to prove

$$\frac{dV}{dt} = \int_{\varphi_t(\Omega)} \text{div} f(x) dx_1 dx_2 ... dx_n,$$

where $\text{div} f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$.

8. (20%)

(i) State Floque Theorem.

(ii) Let

$$A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\cos t \sin t \\ -1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix}.$$

Verify $(-e^{t/2}\cos t, \ e^{t/2}\sin t)$ is a solution of $x' = A(t)x$. Show that the characteristic multipliers are $-e^{\pi/2}$, $-e^{-\pi}$.