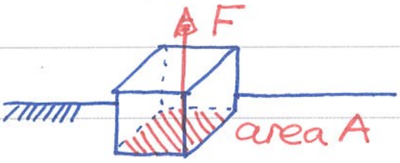
HH0097 Is Pressure in Liquids a Scalar, a Vector, or a Tensor?

Suppose we place a cube of copper on the desk,

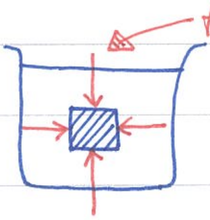


it is convenient to define the pressure in the area,

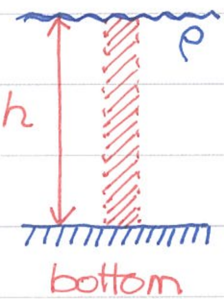
$$\boxed{P = \frac{F}{A}}$$

F is a vector. What about A?

Now we can turn to pressure in liquids. We know the



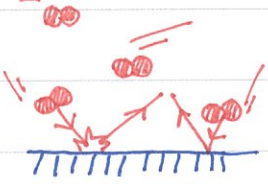
forces on the surface are always perpendicular! It seems that the pressure in liquids is very "smart" and know how to adjust its direction ☺



One can try to compute the pressure by the weight of liquid volume it supports as shown on the left $P \cdot \cancel{dA} = (\rho h \cdot \cancel{dA})\, g$

$$\rightarrow \boxed{P = \rho g h}$$

On the other hand, one can look into the atomic scale, the microscopic origin of pressure



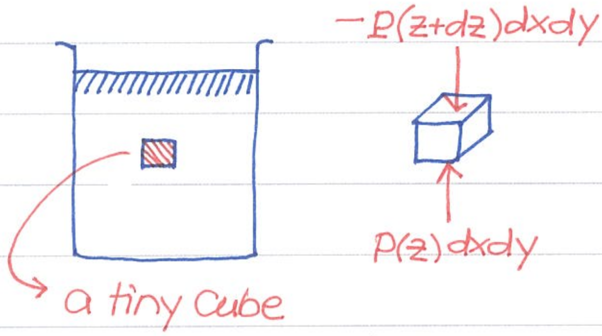
should arise from molecular collisions on the targeted surface. The pressure should only depend on the dynamics of molecules in the neighborhood. So, which picture is true to explain the origin of pressure?

(I) "Smart" pressure in liquids. Let us address this issue

first. An area element is actually a vector. Therefore, $\boxed{\vec{F} = -P\, d\vec{A}}$ Correct version ☺

Because pressure in liquids is a scalar, $\vec{F} \parallel d\vec{A}$, i.e. the force arisen from the pressure is always in the perpendicular direction to the surface. How can the pressure in liquids be so smart? Well, it arises from molecular collisions and thus results in a net force $\vec{F} \parallel d\vec{A}$ — the pressure in liquids is actually not smart, completely determined by the chosen $d\vec{A}$.

To compute P from microscopic origin  is tough, however, we can make use of Newton's 2nd law to help us.



① Pressure in liquids P is a scalar field $P = P(x, y, z)$.

② By translational symmetries, P is independent of x, y. Thus, it simplifies to $P = P(z)$.

③ Write down the EOM for the "tiny cube".

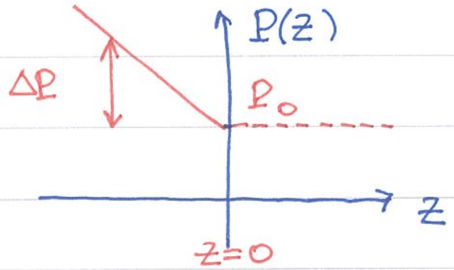
$$\left[-P(z+dz) + P(z)\right] dxdy + (-\rho g)\, dxdydz = dm \cdot a_z \quad (a_z \to 0)$$

By definition, $dP/dz = \frac{1}{dz}\left[P(z+dz) - P(z)\right]$, the EOM is

$$-\frac{dP}{dz} dz dx dy - \rho g\, dxdydz = 0 \rightarrow \boxed{\frac{dP}{dz} = -\rho g}$$

This is the "field equation" for P, quit simple ☺

$$\rightarrow \boxed{P(z) = P_0 - \rho g z} \quad \text{where } P_0 = P(z=0) \text{ is atmosphere } P.$$

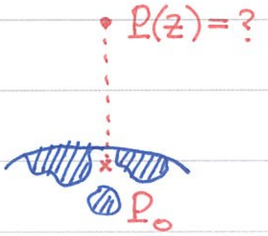


Plot the pressure $P(z)$. One notices

$\Delta P = \rho g |z|$ → the depth ☺

and $P(z) = P_0$ defines the surface.

We can apply the same field equation to the pressure in atmosphere $\boxed{\frac{dP}{dz} = -\rho g}$



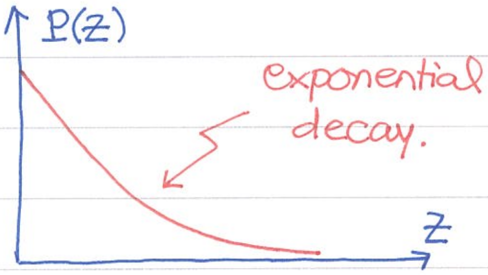
While it is reasonable to assume ρ = const in liquids, it is not true in gas phase. Assuming the ideal gas law works here $P = \left(\frac{N}{V}\right)kT \propto \rho \longrightarrow \boxed{\frac{P}{\rho} = \frac{P_0}{\rho_0}}$

Substitute the relation into the pressure field equation:

$$\boxed{\frac{dP}{dz} = -\rho g = -\left(\frac{\rho_0 g}{P_0}\right) P = -\frac{1}{a} P} \quad \text{where } a = \frac{P_0}{\rho_0 g}.$$

The solution is our old friend - the exponential function,

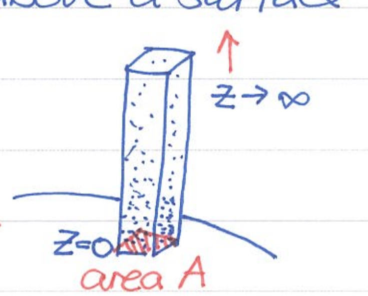
$$\boxed{P(z) = P_0 e^{-z/a}}$$



The constant $a = P_0/\rho_0 g \approx 8.5\,km$ for our Earth. Despite of the simple calculations, the exponential form agrees with the reallistic situation rather well - remarkable ☺

Let us try to compute the total weight W above a surface A on the ground. $dW = \rho g\, dV$

$$W = \int \rho g\, dV = \int_0^\infty \rho g A\, dz = gA\int_0^\infty \rho\, dz$$



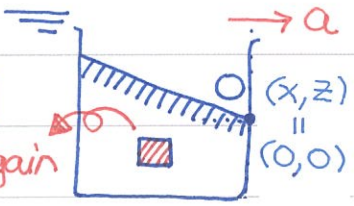
Make use of the relation between ρ and P,

$$\rho(z) = \frac{\rho_0}{P_0} P(z) = \rho_0 e^{-z/a} \longrightarrow W = A\cdot\rho_0 g \int_0^\infty e^{-z/a}\, dz$$

$$\longrightarrow W = A\rho_0 g\cdot(-a)\, e^{-z/a}\Big|_0^\infty = A\rho_0 g a = AP_0 \;!!$$

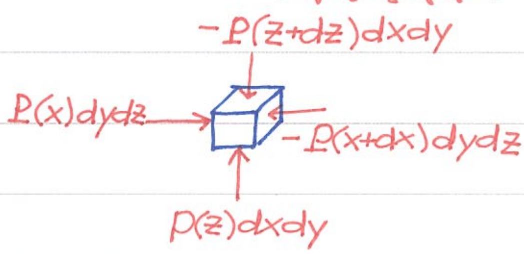
The pressure $P_0 = W/A$ is indeed the total weight/area ☺

⑦ Accelerating bucket. Consider the water level in an accelerating bucket — the pressure field equation is different. Make use of Newton's 2^{nd} law on the tiny cube to derive the field equation for the pressure.



① The pressure field $P = P(x, z)$

② EOM for the tiny cube reads,



In the x-direction,

$$\left[-P(x+dx)+P(x)\right]dydz = dm \cdot a$$

The mass $dm = \rho\, dxdydz$ and $\frac{\partial P}{\partial x} = \frac{1}{dx}\left[P(x+dx) - P(x)\right]$

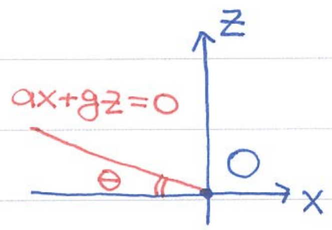
$$-\frac{\partial P}{\partial x}\cancel{dx}\cancel{dy}\cancel{dz} = \rho a \cancel{dx}\cancel{dy}\cancel{dz} \quad \Rightarrow \quad \boxed{\frac{\partial P}{\partial x} = -\rho a}$$

In the z-direction, the analysis is the same as in the static liquid and the field equation is $\boxed{\frac{\partial P}{\partial z} = -\rho g}$

The above equation can be solved easily and the soln is

$$\boxed{P = -\rho a x - \rho g z + P_0}$$

We have chosen the origin on the surface as shown in above



To find the surface of the accelerating bucket,

$$P(x,z) = P_0 \quad \Rightarrow \quad \boxed{gz + ax = 0}$$

The angle θ can be extracted from the slope,

$$\Rightarrow \boxed{\tan\theta = \frac{a}{g}}$$

the same as the tiltled angle of a pendulum in an accelerating frame.

We can think a bit deeper. Pressure in liquids is a scalar while the gravitational field $\vec{g}$ is a vector. Can we find their relation in general form?

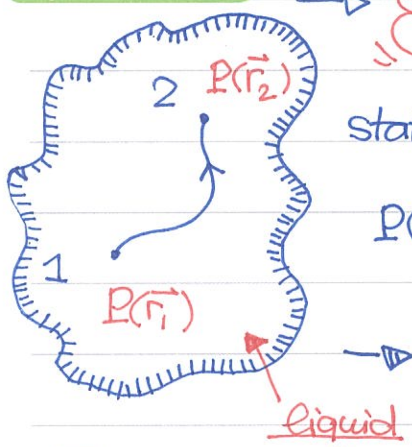
Recall the definition of gradient,

$$\boxed{\vec{\nabla} P \equiv \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial y} \right)}$$

Follow the same steps to analyze the "tiny cube", the pressure field equation for a static liquid is

$$\rightarrow \quad \vec{\nabla} P = \rho \vec{g}$$

The pressure difference between two points in a static liquid can be obtained by integration:



$$P(\vec{r}_2) - P(\vec{r}_1) = \int_1^2 \vec{\nabla} P \cdot d\vec{r} = \rho \int_1^2 \vec{g} \cdot d\vec{r} = -\rho \Delta \Phi$$

$$\rightarrow \quad \boxed{P(\vec{r}_2) = P(\vec{r}_1) - \rho \Delta \Phi}$$

pressure is closely related Φ

For an accelerating bucket, the effective gravitational field is

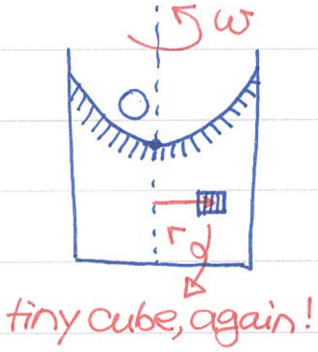
$$\vec{g}_{eff} = \vec{g} - \vec{a} = (-a, 0, -g) \Rightarrow \boxed{\Phi = -\vec{g}_{eff} \cdot \vec{r} + \text{const.}}$$

Thus, the pressure can be written down directly,

$$P(x,z) = P_0 - \rho \Delta \Phi = P_0 + \rho \vec{g}_{eff} \cdot \vec{r} = P_0 - \rho a x - \rho g z$$

Yes

Ⓓ Rotating bucket. Now consider a rotating bucket. We know the water level would become a curved surface. Follow the same logic flow as before:

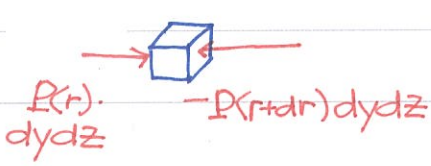


① The pressure field $P = P(r, z)$

② EOM in the r-direction is

$$\left[-P(r+dr) + P(r) \right] dy\,dz = -dm \cdot r\omega^2$$

$$-\frac{\partial P}{\partial r}\, dr\,\cancel{dy}\,dz = -\rho r \omega^2\, dr\,\cancel{dy}\,dz$$



the volume $dV = dr\,dy\,dz$ here!

The EOM in the z-direction is the same. Write both equations down together:

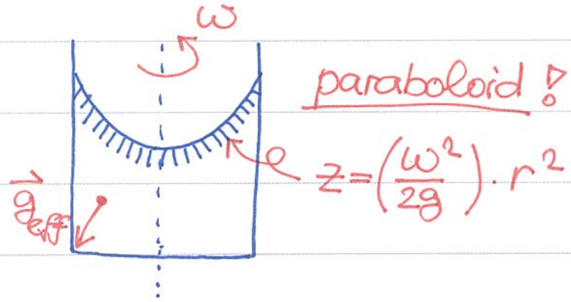
$$\boxed{\begin{aligned} \frac{\partial P}{\partial r} &= \rho\omega^2 r \\ \frac{\partial P}{\partial z} &= -\rho g \end{aligned}} \longrightarrow P(r,z) = P_0 + \frac{1}{2}\rho\omega^2 r^2 - \rho g z$$

We choose O on the surface.

The surface is determined by the equation $P(r,z) = P_0$, i.e. $\boxed{\frac{1}{2}\omega^2 r^2 - g z = 0}$

$z = \left(\frac{\omega^2}{2g}\right) r^2$

It is quite cute that the surface is a paraboloid ☺ Now we can ask whether one is able to understand $P(r,z)$ in terms of $\vec{g}_{eff}$ and its potential Φ. That is to say, $P(r,z) + \rho\,\Phi(r,z) = C$



This is a fun problem for students who want to bridge the scientific knowledge between liquid pressure and gravity.

清大東院
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