Solution to Final Examination

1. (a) True. Since

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = \det(\boldsymbol{A}-\lambda\boldsymbol{I})^T = \det(\boldsymbol{A}^T-\lambda\boldsymbol{I})$$

the eigenvalues of $\boldsymbol{A}$ are the same as the eigenvalues of $\boldsymbol{A}^T$.

(b) False. Both $\boldsymbol{A}$ and $\boldsymbol{B}$ are actually in Jordan forms:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since they have different Jordan forms, they are not similar.

(c) False. Since we know that $T(0,0,0) = (0,0,0) = T(1,-2,1)$ and hence the kernel of T does not contain the zero vector only, T has no inverse.

(d) True. In the standard basis, the matrix representing T is

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

which is a symmetric matrix. By the Spectral Theorem, there exist three orthonormal eigenvectors and this symmetric matrix is diagonalizable. Therefore, in this orthonormal basis (formed by the three orthonormal eigenvectors), the matrix representation for T is a diagonal matrix.

2. (a) Consider

$$\begin{aligned} \det(\boldsymbol{A}-\lambda\boldsymbol{I}) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} \\ &= (4-\lambda)(2-\lambda)^2 = 0. \end{aligned}$$

Thus, we have $\lambda = 4, 2, 2$. For $\lambda_1 = 4$, the AM of λ_1 equals 1. Besides,

$$\boldsymbol{A}-\lambda_1\boldsymbol{I} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

and the corresponding eigenvector is

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The GM of λ_1 is 1, which is equal to the AM of λ_1. For $\lambda_2 = 2$, the AM of λ_2 equals 2. Besides,

$$\boldsymbol{A} - \lambda_2 \boldsymbol{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and the corresponding eigenvector is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The GM of λ_2 is 1, which is smaller than the AM of λ_2. Therefore, $\boldsymbol{A}$ is not diagonalizable, and its Jordan form is

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(b) Consider

$$\begin{aligned} \det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= \begin{vmatrix} 3-\lambda & -1 & -2 \\ 2 & -\lambda & -2 \\ 2 & -1 & -1-\lambda \end{vmatrix} \\ &= -\lambda(1-\lambda)^2 = 0. \end{aligned}$$

Thus, we have $\lambda = 0, 1, 1$. For $\lambda_1 = 0$, the AM of λ_1 equals 1. Besides,

$$\boldsymbol{A} - \lambda_1 \boldsymbol{I} = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 3 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

and the corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The GM of λ_1 is 1, which is equal to the AM of λ_1. For $\lambda_2 = 1$, the AM of λ_2 equals 2. Besides,

$$\boldsymbol{A} - \lambda_2 \boldsymbol{I} = \begin{bmatrix} 2 & -1 & -2 \\ 2 & -1 & -2 \\ 2 & -1 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The GM of λ_2 is 2, which is equal to the AM of λ_2. Therefore, $\boldsymbol{A}$ is diagonalizable with

$$\boldsymbol{S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{\Lambda} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Let $\boldsymbol{u}_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$. We can have, for $k \geq 0$,

$$\boldsymbol{u}_{k+1} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = \boldsymbol{A}\boldsymbol{u}_k$$

where

$$\boldsymbol{A} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}.$$

Then $\boldsymbol{u}_k = \boldsymbol{A}\boldsymbol{u}_{k-1} = \boldsymbol{A}^2\boldsymbol{u}_{k-2} = \cdots = \boldsymbol{A}^k\boldsymbol{u}_0$. Let

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} 1/2 - \lambda & 1/2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$$

and we can obtain $\lambda = 1$ or $-1/2$. For $\lambda = 1$,

$$\boldsymbol{A} - 1 \cdot \boldsymbol{I} = \begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix} \Longrightarrow \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda = -1/2$,

$$\boldsymbol{A} - \left(\frac{-1}{2}\right)\boldsymbol{I} = \begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix} \Longrightarrow \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Expressing $\boldsymbol{u}_0$ as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we can have

$$\begin{aligned} & \boldsymbol{u}_0 = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ \Longrightarrow \quad & \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\right)^{-1} \boldsymbol{u}_0 = \frac{-1}{3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/6 \end{bmatrix} \\ \Longrightarrow \quad & \boldsymbol{u}_0 = \frac{1}{3}\boldsymbol{v}_1 + \frac{1}{6}\boldsymbol{v}_2. \end{aligned}$$

Hence we can obtain

$$\begin{aligned} \boldsymbol{u}_k &= \boldsymbol{A}^k\boldsymbol{u}_0 = \boldsymbol{A}^k \left(\frac{1}{3}\boldsymbol{v}_1 + \frac{1}{6}\boldsymbol{v}_2\right) \\ &= \frac{1}{3}\boldsymbol{A}^k\boldsymbol{v}_1 + \frac{1}{6}\boldsymbol{A}^k\boldsymbol{v}_2 \\ &= \frac{1}{3} \cdot 1^k\boldsymbol{v}_1 + \frac{1}{6}\left(\frac{-1}{2}\right)^k \boldsymbol{v}_2 \\ &= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{6}\left(\frac{-1}{2}\right)^k \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}. \end{aligned}$$

Therefore,

$$G_k = \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^k.$$

As $k \to \infty$, we have

$$\lim_{k\to\infty} G_k = \frac{1}{3}.$$

4. (a) Let

$$\boldsymbol{B} = \frac{1}{2}\left(\boldsymbol{A} + \boldsymbol{A}^T\right) = \frac{1}{2}\left\{\begin{bmatrix} 4 & -1 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ -1 & 4 \end{bmatrix}\right\} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

and we have $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x}$. The eigenvalues of $\boldsymbol{B}$ can be found as:

$$\begin{aligned} \det\left(\boldsymbol{B} - \lambda\boldsymbol{I}\right) &= \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 12 = 0 \\ \Longrightarrow \quad & \lambda = 2 \text{ or } 6. \end{aligned}$$

Since both the eigenvalues of $\boldsymbol{B}$ are positive and thus $\boldsymbol{B}$ is positive definite, we have $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x}$ is always positive for every nonzero vector $\boldsymbol{x}$.

(b) From class, we can have

$$\lambda_{\min} \le R(\boldsymbol{x}) = \frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}} = \frac{\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}} \le \lambda_{\max}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of $\boldsymbol{B}$, respectively. From (a), we have $\lambda_{\max} = 6$. Therefore,

$$\max_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{x}) = 6.$$

(c) A vector $\boldsymbol{x}$ achieves $\min_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{x})$ if $\boldsymbol{x}$ belongs to the eigenspace corresponding to $\lambda_{\min}$. From (a), we have $\lambda_{\min} = 2$. Since

$$\boldsymbol{B} - \lambda_{\min}\boldsymbol{I} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

we can choose $\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or any nonzero scalar multiplication of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

5. (a) For any nonzero singular value σ of $\boldsymbol{A}$, we can find a nonzero eigenvector $\boldsymbol{x}$ of $\boldsymbol{A}^T\boldsymbol{A}$ such that $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \sigma^2\boldsymbol{x}$. Therefore, we can have

$$\frac{\|\boldsymbol{A}\boldsymbol{x}\|}{\|\boldsymbol{x}\|} = \frac{(\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x})^{1/2}}{(\boldsymbol{x}^T\boldsymbol{x})^{1/2}} = \frac{(\sigma^2\boldsymbol{x}^T\boldsymbol{x})^{1/2}}{(\boldsymbol{x}^T\boldsymbol{x})^{1/2}} = \frac{\sigma\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|} = \sigma.$$

(b) Let $\boldsymbol{x}_i$ be an eigenvector of $\boldsymbol{A}$ corresponding to λ_i, for $i = 1, 2, \cdots, n$. Since $\boldsymbol{A}$ is an n by n symmetry matrix, we have $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}_i = \boldsymbol{A}\boldsymbol{A}\boldsymbol{x}_i = \lambda_i\boldsymbol{A}\boldsymbol{x}_i = \lambda_i^2\boldsymbol{x}_i$, for $i = 1, 2, \cdots, n$. Therefore, the singular values of $\boldsymbol{A}$ are given by $\sqrt{\lambda_i^2} = |\lambda_i|$, for $i = 1, 2, \cdots, n$.

(c) Let $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ be the singular value decomposition of $\boldsymbol{A}$. Since $\boldsymbol{A}$ is an n by n matrix, we have $\boldsymbol{\Sigma}^T\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\boldsymbol{\Sigma}^T = \boldsymbol{\Sigma}^2$. Besides, we know that $\boldsymbol{U}^{-1} = \boldsymbol{U}^T$ and $\boldsymbol{V}^{-1} = \boldsymbol{V}^T$. Hence we can obtain

$$\boldsymbol{A}^T\boldsymbol{A} = (\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T)^T\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{U}^T\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{\Sigma}^2\boldsymbol{V}^T$$

and

$$\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T(\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T)^T = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{U}^T = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\boldsymbol{U}^T = \boldsymbol{U}\boldsymbol{\Sigma}^2\boldsymbol{U}^T.$$

Choosing $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{V}^T$ which is invertible with $\boldsymbol{M}^{-1} = \boldsymbol{V}\boldsymbol{U}^T$, we can obtain

$$\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{M} = \boldsymbol{V}\boldsymbol{U}^T\boldsymbol{U}\boldsymbol{\Sigma}^2\boldsymbol{U}^T\boldsymbol{U}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{\Sigma}^2\boldsymbol{V}^T = \boldsymbol{A}^T\boldsymbol{A}.$$

As a result, $\boldsymbol{A}^T\boldsymbol{A}$ is similar to $\boldsymbol{A}\boldsymbol{A}^T$.

6. (a) Let $\boldsymbol{C}, \boldsymbol{D} \in M$, and we can check the following two conditions:

- $T(\boldsymbol{C}+\boldsymbol{D}) = \boldsymbol{A}(\boldsymbol{C}+\boldsymbol{D}) = \boldsymbol{A}\boldsymbol{C}+\boldsymbol{A}\boldsymbol{D} = T(\boldsymbol{C})+T(\boldsymbol{D})$.
- $T(c\boldsymbol{C}) = \boldsymbol{A}(c\boldsymbol{C}) = c(\boldsymbol{A}\boldsymbol{C}) = cT(\boldsymbol{C})$ for all c.

Therefore, T is linear.

(b) We can have

$$T(\boldsymbol{V}_1) = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = a\boldsymbol{V}_1 + 0\boldsymbol{V}_2 + c\boldsymbol{V}_3 + 0\boldsymbol{V}_4$$

$$T(\boldsymbol{V}_2) = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = 0\boldsymbol{V}_1 + a\boldsymbol{V}_2 + 0\boldsymbol{V}_3 + c\boldsymbol{V}_4$$

$$T(\boldsymbol{V}_3) = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} = b\boldsymbol{V}_1 + 0\boldsymbol{V}_2 + d\boldsymbol{V}_3 + 0\boldsymbol{V}_4$$

$$T(\boldsymbol{V}_4) = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = 0\boldsymbol{V}_1 + b\boldsymbol{V}_2 + 0\boldsymbol{V}_3 + d\boldsymbol{V}_4.$$

Therefore, the matrix representation for T in this basis β is

$$\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

7. (a) Perform singular value decomposition, and we can have

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the pseudoinverse of $\boldsymbol{A}$ is given by

$$\begin{aligned} \boldsymbol{A}^+ &= \boldsymbol{V}\boldsymbol{\Sigma}^+\boldsymbol{U}^T \\ &= \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \end{bmatrix}. \end{aligned}$$

(b) Since $\boldsymbol{A}$ has full column rank, there is a left inverse for $\boldsymbol{A}$. We can have

$$\boldsymbol{A}^{+}\boldsymbol{A} = \boldsymbol{I}$$

and hence the pseudoinverse $\boldsymbol{A}^{+}$ obtained in (a) is a left inverse for $\boldsymbol{A}$.

(c) The projection matrix onto the column space of $\boldsymbol{A}$ is given by

$$\boldsymbol{P}_c = \boldsymbol{A}\boldsymbol{A}^{+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(d) We can have

$$\boldsymbol{x}_r = \boldsymbol{A}^{+}\boldsymbol{b} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$