Solution to Homework Assignment No. 4

1. (a) Since $\boldsymbol{A}$ is 4 by 4 and rank($\boldsymbol{A}$) = 1, it is singular. Therefore, $|\boldsymbol{A}| = 0$.

(b) $|\boldsymbol{U}| = 4 \cdot 1 \cdot 2 \cdot 2 = 16$.

(c) $|\boldsymbol{U}^T| = |\boldsymbol{U}| = 16$.

(d) Since $1 = |\boldsymbol{I}| = |\boldsymbol{U}\boldsymbol{U}^{-1}| = |\boldsymbol{U}||\boldsymbol{U}^{-1}| = 16|\boldsymbol{U}^{-1}|$, we have $|\boldsymbol{U}^{-1}| = 1/16$.

(e)

$$
\begin{aligned}
|\boldsymbol{M}| = \begin{vmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{vmatrix} &= (-1) \cdot \begin{vmatrix} 4 & 4 & 8 & 8 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix} \\
&= (-1) \cdot (-1) \cdot \begin{vmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{vmatrix} \\
&= |\boldsymbol{U}| = 16.
\end{aligned}
$$

(f)

$$
\begin{aligned}
|\boldsymbol{F}| = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 3/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 5/3 & 0 & 0 & 0 \\ 1 & 3/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} \\
&= (5/3) \cdot (3/2) \cdot 2 \cdot 1 = 5.
\end{aligned}
$$

2. (a)

$$
\begin{aligned}
\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \\
&= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & c^2-a^2-(b+a)(c-a) \end{vmatrix} \\
&= (b-a)(c-a)[(c+a)-(b+a)] \\
&= (b-a)(c-a)(c-b).
\end{aligned}
$$

(b) For a skew-symmetric matrix satisfying $\boldsymbol{A}^T = -\boldsymbol{A}$, we have $\left|\boldsymbol{A}^T\right| = |-\boldsymbol{A}|$. Since $\left|\boldsymbol{A}^T\right| = |\boldsymbol{A}|$ and $|-\boldsymbol{A}| = (-1)^n |\boldsymbol{A}|$, we can obtain $|\boldsymbol{A}| = (-1)^n |\boldsymbol{A}|$. Therefore, if n is odd, we have $|\boldsymbol{A}| = -|\boldsymbol{A}|$, which implies $|\boldsymbol{A}| = 0$.

3. (a) We have one-swap permutations as

$$(1,2,4,3),(1,3,2,4),(1,4,3,2),(2,1,3,4),(3,2,1,4),(4,2,3,1)$$

and three-swaps permutations as

$$(2,3,4,1),(2,4,1,3),(3,1,4,2),(3,4,2,1),(4,1,2,3),(4,3,1,2).$$

(b) We have

$$\boldsymbol{P}_\sigma = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and $|\boldsymbol{P}_\sigma| = (-1)^{n-1}|\boldsymbol{I}| = (-1)^{n-1}$ since $n-1$ row exchanges are needed to convert $\boldsymbol{P}_\sigma$ back to $\boldsymbol{I}$.

4. (a) We have

$$|\boldsymbol{A}_2| = -1$$

$$|\boldsymbol{A}_3| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

and

$$\begin{aligned} |\boldsymbol{A}_4| = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} &= -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \\ &= -1-1-1 \\ &= -3. \end{aligned}$$

(b)

$$
\begin{aligned}
|\boldsymbol{A}_n| &= \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ n-1 & n-1 & n-1 & n-1 & \cdots & n-1 \end{vmatrix} \text{[add all rows (except the last) to the last row]} \\
&= (n-1)\begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{vmatrix} \\
&= (n-1)\begin{vmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{vmatrix} \text{ [subtract the last row from each preceding row]} \\
&= (-1)^{n-1}(n-1). \quad \text{[all other terms in the big formula are zero]}
\end{aligned}
$$

5. (a) For $n \geq 4$, we have

$$
|\boldsymbol{B}_n| = \begin{vmatrix} & & & & 0 \\ & & \boldsymbol{B}_{n-1} & & \vdots \\ & & & & 0 \\ & & & & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} & & & & 0 & 0 \\ & & \boldsymbol{B}_{n-2} & & \vdots & \vdots \\ & & & & 0 & \vdots \\ & & & & -1 & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{vmatrix}.
$$

Applying the cofactor formula to the last row, we can have

$$|\boldsymbol{B}_n| = 2\cdot(-1)^{n+n}|\boldsymbol{B_{n-1}}| + (-1)\cdot(-1)^{n+n-1}\begin{vmatrix} & & & & 0 \\ & & \boldsymbol{B}_{n-2} & & \vdots \\ & & & & 0 \\ & & & & 0 \\ 0 & \cdots & 0 & -1 & -1 \end{vmatrix}$$

$$= 2|\boldsymbol{B}_{n-1}| - 1\cdot(-1)^{n+n}|\boldsymbol{B_{n-2}}| \text{ (apply the cofactor formula to the last column)}$$

$$= 2|\boldsymbol{B}_{n-1}| - |\boldsymbol{B}_{n-2}|.$$

Then we can obtain $a = 2$ and $b = -1$.

(b) We have

$$|\boldsymbol{B}_2| = 1$$
$$|\boldsymbol{B}_3| = 1$$
$$|\boldsymbol{B}_4| = 2|\boldsymbol{B}_3| - |\boldsymbol{B}_2| = 1.$$

Guess $|\boldsymbol{B}_n| = 1$ for all $n \geq 1$. We can check that

$$|\boldsymbol{B}_n| = 2|\boldsymbol{B}_{n-1}| - |\boldsymbol{B}_{n-2}| = 2 - 1 = 1.$$

(c)

$$\begin{aligned}
|\boldsymbol{B}_n| &= \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 2+(-1) & -1+0 & 0+0 & 0+0 & \cdots & 0+0 & 0+0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix} \\
&= |\boldsymbol{A}_n| + \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ -1 & & & & \\ 0 & & \boldsymbol{A}_{n-1} & & \\ \vdots & & & & \\ 0 & & & & \end{vmatrix} \\
&= |\boldsymbol{A}_n| - |\boldsymbol{A}_{n-1}|.
\end{aligned}$$

Furthermore, we have

$$|\boldsymbol{B}_n| = |\boldsymbol{A}_n| - |\boldsymbol{A}_{n-1}| = n + 1 - n = 1.$$

6. (a) For the system, we have

$$\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Using Cramer's rule, we can obtain

$$x = \frac{\begin{vmatrix} 1 & 4 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix}} = 3,\ y = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix}} = -1, \text{ and } z = \frac{\begin{vmatrix} 1 & 4 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix}} = -2.$$

(b) Let $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$, and we have $|\boldsymbol{A}| = 3$. By the cofactor formula, we can have

$$\left(\boldsymbol{A}^{-1}\right)_{11} = \frac{\boldsymbol{C}_{11}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}}{3} = -1$$

$$\left(\boldsymbol{A}^{-1}\right)_{21} = \frac{\boldsymbol{C}_{12}}{\det \boldsymbol{A}} = \frac{-\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix}}{3} = 0$$

$$\left(\boldsymbol{A}^{-1}\right)_{31} = \frac{\boldsymbol{C}_{13}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix}}{3} = 2$$

$$\left(\boldsymbol{A}^{-1}\right)_{12} = \frac{\boldsymbol{C}_{21}}{\det \boldsymbol{A}} = \frac{-\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix}}{3} = \frac{5}{3}$$

$$\left(\boldsymbol{A}^{-1}\right)_{22} = \frac{\boldsymbol{C}_{22}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix}}{3} = \frac{1}{3}$$

$$\left(\boldsymbol{A}^{-1}\right)_{32} = \frac{\boldsymbol{C}_{23}}{\det \boldsymbol{A}} = \frac{-\begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix}}{3} = -\frac{8}{3}$$

$$\left(\boldsymbol{A}^{-1}\right)_{13} = \frac{\boldsymbol{C}_{31}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}}{3} = \frac{2}{3}$$

$$\left(\boldsymbol{A}^{-1}\right)_{23} = \frac{\boldsymbol{C}_{32}}{\det \boldsymbol{A}} = \frac{-\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{3} = \frac{1}{3}$$

$$\left(\boldsymbol{A}^{-1}\right)_{33} = \frac{\boldsymbol{C}_{33}}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}{3} = -\frac{5}{3}.$$

Therefore, we can obtain the inverse of $\boldsymbol{A}$ as

$$\boldsymbol{A}^{-1} = \begin{bmatrix} -1 & 5/3 & 2/3 \\ 0 & 1/3 & 1/3 \\ 2 & -8/3 & -5/3 \end{bmatrix}.$$

7. (a)

$$\begin{aligned} |\boldsymbol{H}| &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \end{vmatrix} \\ &= \begin{vmatrix} -2 & 0 & -2 \\ 0 & -2 & -2 \\ -2 & -2 & 0 \end{vmatrix} = \begin{vmatrix} -2 & 0 & -2 \\ 0 & -2 & -2 \\ 0 & -2 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 0 & -2 \\ 0 & -2 & -2 \\ 0 & 0 & 4 \end{vmatrix} \\ &= 16. \end{aligned}$$

(b) The maximum volume is $L_1L_2L_3L_4$ reached when the edges are orthogonal in $\mathcal{R}^4$. If all the entries of the matrix are 1 or -1, the lengths L_1, L_2, L_3, L_4 are all equal to $\sqrt{1+1+1+1} = 2$. Therefore, the maximum determinant is $2^4 = 16$.