Spectral Theorem

Chi-chao Chao

Department of Electrical Engineering
National Tsing Hua University
ccc@ee.nthu.edu.tw

May 15, 2013

Spectral Theorem

Spectral Theorem

Suppose $\mathbf{A}$ *is an* n *by* n *real symmetric matrix. Then* $\mathbf{A}$ *has the factorization*

$$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{T}$$

where $\boldsymbol{\Lambda}$ *is a diagonal matrix with real eigenvalues on the diagonal and* $\mathbf{Q}$ *is an orthogonal matrix with columns formed by orthonormal eigenvectors.*

Schur's Theorem

- For an n by n complex matrix $\mathbf{Q}$, $\mathbf{Q}$ is called a *unitary* matrix if $\overline{\mathbf{Q}}^T = \mathbf{Q}^{-1}$.
- If a unitary matrix is real, then it is an orthogonal matrix.

Schur's Theorem

Every square matrix factors into

$$\mathbf{A} = \mathbf{QTQ}^{-1} = \mathbf{QT}\overline{\mathbf{Q}}^T$$

where $\mathbf{T}$ *is upper triangular and* $\mathbf{Q}$ *is unitary. If* $\mathbf{A}$ *has real eigenvalues, then* $\mathbf{Q}$ *and* $\mathbf{T}$ *can be chosen real:* $\mathbf{Q}^T = \mathbf{Q}^{-1}$, *i.e.,* $\mathbf{Q}$ *is orthogonal.*

Proof of Schur's Theorem

- We prove this by induction.
- The result is obvious if $n = 1$: $a = 1 \cdot a \cdot 1^{-1}$.
- Assume the hypothesis holds for k by k matrices and let $\mathbf{A}$ be a $k+1$ by $k+1$ matrix.
- Let λ_1 be an eigenvalue of $\mathbf{A}$ and $\mathbf{q}_1$ be a corresponding unit eigenvector.
- Using the Gram-Schmidt process, we can find $\mathbf{q}_2, \mathbf{q}_3, \ldots, \mathbf{q}_{k+1}$ such that $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{k+1}$ forms an orthonormal basis for $\mathcal{C}^{k+1}$, where $\mathcal{C}$ is the set of complex numbers.
- Let

$$\mathbf{Q}_1 = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_{k+1} \end{bmatrix}$$

and then $\mathbf{Q}_1$ is unitary.

- We can have

$$
\begin{aligned}
\overline{\mathbf{Q}}_1^T \mathbf{A}\mathbf{Q}_1 &= \begin{bmatrix} \overline{\mathbf{q}}_1^T \\ \overline{\mathbf{q}}_2^T \\ \vdots \\ \overline{\mathbf{q}}_{k+1}^T \end{bmatrix} \begin{bmatrix} \mathbf{A}\mathbf{q}_1 & \mathbf{A}\mathbf{q}_2 & \cdots & \mathbf{A}\mathbf{q}_{k+1} \end{bmatrix} \\
&= \begin{bmatrix} \overline{\mathbf{q}}_1^T \\ \overline{\mathbf{q}}_2^T \\ \vdots \\ \overline{\mathbf{q}}_{k+1}^T \end{bmatrix} \begin{bmatrix} \lambda_1\mathbf{q}_1 & \mathbf{A}\mathbf{q}_2 & \cdots & \mathbf{A}\mathbf{q}_{k+1} \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & \times & \cdots & \times \\ 0 & & & \\ \vdots & & \mathbf{A}_2 & \\ 0 & & & \end{bmatrix}
\end{aligned}
$$

where the last equality follows since $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{k+1}$ are orthonormal.

- By the induction hypothesis, since $\mathbf{A}_2$ is k by k,

$$\mathbf{A}_2 = \mathbf{Q}_2 \mathbf{T}_2 \overline{\mathbf{Q}}_2^T$$

where $\mathbf{Q}_2$ is unitary and $\mathbf{T}_2$ is upper triangular.

- Let

$$\mathbf{Q} = \mathbf{Q}_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{Q}_2 & \\ 0 & & & \end{bmatrix}.$$

- Then $\mathbf{Q}$ is unitary since

$$
\begin{aligned}
\mathbf{Q}\overline{\mathbf{Q}}^T &= \mathbf{Q}_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{Q}_2 & \\ 0 & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \overline{\mathbf{Q}}_2^T & \\ 0 & & & \end{bmatrix} \overline{\mathbf{Q}}_1^T \\
&= \mathbf{Q}_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{I}_k & \\ 0 & & & \end{bmatrix} \overline{\mathbf{Q}}_1^T = \mathbf{Q}_1 \mathbf{I}_{k+1} \overline{\mathbf{Q}}_1^T \\
&= \mathbf{Q}_1 \overline{\mathbf{Q}}_1^T = \mathbf{I}_{k+1}
\end{aligned}
$$

where $\mathbf{I}_n$ is the n by n identity matrix.

- We can have

$$
\begin{aligned}
\overline{\mathbf{Q}}^T\mathbf{A}\mathbf{Q} &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \overline{\mathbf{Q}}_2^T & \\ 0 & & & \end{bmatrix} \overline{\mathbf{Q}}_1^T\mathbf{A}\mathbf{Q}_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{Q}_2 & \\ 0 & & & \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \overline{\mathbf{Q}}_2^T & \\ 0 & & & \end{bmatrix} \begin{bmatrix} \lambda_1 & \times & \cdots & \times \\ 0 & & & \\ \vdots & & \mathbf{A}_2 & \\ 0 & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{Q}_2 & \\ 0 & & & \end{bmatrix}
\end{aligned}
$$

$$
= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \overline{\mathbf{Q}}_2^T & \\ 0 & & & \end{bmatrix} \begin{bmatrix} \lambda_1 & \times & \cdots & \times \\ 0 & & & \\ \vdots & & \mathbf{A}_2\mathbf{Q}_2 & \\ 0 & & & \end{bmatrix}
$$

$$
= \begin{bmatrix} \lambda_1 & \times & \cdots & \times \\ 0 & & & \\ \vdots & & \overline{\mathbf{Q}}_2^T\mathbf{A}_2\mathbf{Q}_2 & \\ 0 & & & \end{bmatrix} = \begin{bmatrix} \lambda_1 & \times & \cdots & \times \\ 0 & & & \\ \vdots & & \mathbf{T}_2 & \\ 0 & & & \end{bmatrix} = \mathbf{T}.
$$

- Then $\mathbf{T}$ is upper triangular since $\mathbf{T}_2$ is upper triangular.
- Therefore, $\mathbf{A} = \mathbf{Q}\mathbf{T}\overline{\mathbf{Q}}^T$.

- If λ_1 is a real eigenvalue, then $\mathbf{q}_1$ and $\mathbf{Q}_1$ can stay real.
- The induction step keeps everything real when $\mathbf{A}$ has real eigenvalues.
- Induction starts with the 1 by 1 case, and there is no problem.
- This ends the proof for Schur's Theorem.

Proof of Spectral Theorem

- In class we have shown that every symmetric $\mathbf{A}$ has real eigenvalues.
- By Schur's Theorem,
$$\mathbf{A} = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$$
where $\mathbf{Q}$ is orthogonal: $\mathbf{Q}^T = \mathbf{Q}^{-1}$ and $\mathbf{T}$ is upper triangular.
- Then $\mathbf{T} = \mathbf{Q}^T\mathbf{A}\mathbf{Q}$, which is a symmetric matrix since
$$\mathbf{T}^T = \mathbf{Q}^T\mathbf{A}\mathbf{Q} = \mathbf{T}.$$
- If $\mathbf{T}$ is triangular and also symmetric, it must be diagonal: $\mathbf{T} = \mathbf{\Lambda}$.
- Therefore,
$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T.$$