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Spectral Theorem

In this note we will prove the *Spectral Theorem*, which is stated below.

Spectral Theorem *Suppose $\boldsymbol{A}$ is an n by n real symmetric matrix. Then $\boldsymbol{A}$ has the factorization*

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^{-1} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix with real eigenvalues on the diagonal and $\boldsymbol{Q}$ is an orthogonal matrix with columns formed by orthonormal eigenvectors.

For an n by n complex matrix $\boldsymbol{Q}$, $\boldsymbol{Q}$ is called a *unitary* matrix if $\overline{\boldsymbol{Q}}^T = \boldsymbol{Q}^{-1}$. If a unitary matrix is real, then it is an orthogonal matrix. For the proof of the Spectral Theorem, we need the following *Schur's Theorem*:

Schur's Theorem *Every square matrix factors into*

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^{-1} = \boldsymbol{Q}\boldsymbol{T}\overline{\boldsymbol{Q}}^T$$

where $\boldsymbol{T}$ is upper triangular and $\boldsymbol{Q}$ is unitary. If $\boldsymbol{A}$ has real eigenvalues, then $\boldsymbol{Q}$ and $\boldsymbol{T}$ can be chosen real: $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$, i.e., $\boldsymbol{Q}$ is orthogonal.

Proof. We prove this by induction. The result is obvious if $n = 1$: $a = 1 \cdot a \cdot 1^{-1}$. Assume the hypothesis holds for k by k matrices and let $\boldsymbol{A}$ be a $k+1$ by $k+1$ matrix. Let λ_1 be an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{q}_1$ be a corresponding unit eigenvector. Using the Gram-Schmidt process, we can find $\boldsymbol{q}_2, \boldsymbol{q}_3, \ldots, \boldsymbol{q}_{k+1}$ such that $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_{k+1}$ forms an orthonormal basis for $\mathcal{C}^{k+1}$, where $\mathcal{C}$ is the set of complex numbers.

Let $\boldsymbol{Q}_1 = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \cdots & \boldsymbol{q}_{k+1} \end{bmatrix}$. Then $\boldsymbol{Q}_1$ is unitary and

$$\begin{aligned}
\overline{\boldsymbol{Q}}_1^T \boldsymbol{A}\boldsymbol{Q}_1 &= \begin{bmatrix} \overline{\boldsymbol{q}}_1^T \\ \overline{\boldsymbol{q}}_2^T \\ \vdots \\ \overline{\boldsymbol{q}}_{k+1}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{A}\boldsymbol{q}_1 & \boldsymbol{A}\boldsymbol{q}_2 & \cdots & \boldsymbol{A}\boldsymbol{q}_{k+1} \end{bmatrix} \\
&= \begin{bmatrix} \overline{\boldsymbol{q}}_1^T \\ \overline{\boldsymbol{q}}_2^T \\ \vdots \\ \overline{\boldsymbol{q}}_{k+1}^T \end{bmatrix} \begin{bmatrix} \lambda_1\boldsymbol{q}_1 & \boldsymbol{A}\boldsymbol{q}_2 & \cdots & \boldsymbol{A}\boldsymbol{q}_{k+1} \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & \times & \cdots & \times \\ 0 & & & \\ \vdots & & \boldsymbol{A}_2 & \\ 0 & & & \end{bmatrix}
\end{aligned}$$

where the last equality follows since $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_{k+1}$ are orthonormal. By the induction hypothesis, since $\boldsymbol{A}_2$ is k by k,

$$\boldsymbol{A}_2 = \boldsymbol{Q}_2\boldsymbol{T}_2\overline{\boldsymbol{Q}}_2^T$$

where $\boldsymbol{Q}_2$ is unitary and $\boldsymbol{T}_2$ is upper triangular. Let

$$\boldsymbol{Q} = \boldsymbol{Q}_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boldsymbol{Q}_2 & \\ 0 & & & \end{bmatrix}.$$

Then $\boldsymbol{Q}$ is unitary since

$$\begin{aligned} \boldsymbol{Q}\overline{\boldsymbol{Q}}^T &= \boldsymbol{Q}_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boldsymbol{Q}_2 & \\ 0 & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \overline{\boldsymbol{Q}}_2^T & \\ 0 & & & \end{bmatrix} \overline{\boldsymbol{Q}}_1^T \\ &= \boldsymbol{Q}_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boldsymbol{I}_k & \\ 0 & & & \end{bmatrix} \overline{\boldsymbol{Q}}_1^T = \boldsymbol{Q}_1 \boldsymbol{I}_{k+1} \overline{\boldsymbol{Q}}_1^T = \boldsymbol{Q}_1 \overline{\boldsymbol{Q}}_1^T = \boldsymbol{I}_{k+1} \end{aligned}$$

where $\boldsymbol{I}_n$ is the n by n identity matrix. We can have

$$\begin{aligned} \overline{\boldsymbol{Q}}^T \boldsymbol{A}\boldsymbol{Q} &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \overline{\boldsymbol{Q}}_2^T & \\ 0 & & & \end{bmatrix} \overline{\boldsymbol{Q}}_1^T \boldsymbol{A}\boldsymbol{Q}_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boldsymbol{Q}_2 & \\ 0 & & & \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \overline{\boldsymbol{Q}}_2^T & \\ 0 & & & \end{bmatrix} \begin{bmatrix} \lambda_1 & \times & \cdots & \times \\ 0 & & & \\ \vdots & & \boldsymbol{A}_2 & \\ 0 & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boldsymbol{Q}_2 & \\ 0 & & & \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \overline{\boldsymbol{Q}}_2^T & \\ 0 & & & \end{bmatrix} \begin{bmatrix} \lambda_1 & \times & \cdots & \times \\ 0 & & & \\ \vdots & & \boldsymbol{A}_2\boldsymbol{Q}_2 & \\ 0 & & & \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \times & \cdots & \times \\ 0 & & & \\ \vdots & & \overline{\boldsymbol{Q}}_2^T \boldsymbol{A}_2\boldsymbol{Q}_2 & \\ 0 & & & \end{bmatrix} = \begin{bmatrix} \lambda_1 & \times & \cdots & \times \\ 0 & & & \\ \vdots & & \boldsymbol{T}_2 & \\ 0 & & & \end{bmatrix} = \boldsymbol{T} \end{aligned}$$

where $\boldsymbol{T}$ is upper triangular since $\boldsymbol{T}_2$ is upper triangular. Therefore, $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{T}\overline{\boldsymbol{Q}}^T$.

If λ_1 is a real eigenvalue, then $\boldsymbol{q}_1$ and $\boldsymbol{Q}_1$ can stay real. The induction step keeps everything real when $\boldsymbol{A}$ has real eigenvalues. Induction starts with the 1 by 1 case, and there is no problem. ∎

We can now use Schur's Theorem to prove the Spectral Theorem.

Proof of the Spectral Theorem. In class we have shown that every symmetric $\boldsymbol{A}$ has real eigenvalues. By Schur's Theorem,

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T$$

where $\boldsymbol{Q}$ is orthogonal: $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$ and $\boldsymbol{T}$ is upper triangular. Then $\boldsymbol{T} = \boldsymbol{Q}^T\boldsymbol{A}\boldsymbol{Q}$, which is a symmetric matrix since $\boldsymbol{T}^T = \boldsymbol{Q}^T\boldsymbol{A}\boldsymbol{Q} = \boldsymbol{T}$. If $\boldsymbol{T}$ is triangular and also symmetric, it must be diagonal: $\boldsymbol{T} = \boldsymbol{\Lambda}$. Therefore, $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$. ∎