1. (10%) Let A be a general square matrix. Prove that the matrix A^TA has only non-negative real eigenvalues. (A^T means the transpose of A.)

2. Let $C[-1,1]$ be the vector space of continuous real-valued functions on $[-1,1]$ with the following inner product:

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

Let A be the span of subspace generated by $1, x, x^2$.

(a) (10%) Find an orthonormal basis for A.

(b) (10%) Let $T : A \to A$ be the linear transformation defined by $T(a_0 + a_1x + a_2x^2) = a_2 + a_1x + a_0x^2$. Determine the linear transformation $T^* : A \to A$ such that $\langle T(f), g\rangle = \langle f, T^*(g)\rangle$ for all $f, g \in A$.

3. (10%) Consider the following quadratic form:

$$F(x,y,z) = x^2 + y^2 + z^2 + 2xy + 2yz.$$

Suppose that in addition $x^2+y^2+z^2 = 1$. What is the minimum value that F can take? What values of x, y, z achieve this minimum?

4. (10%) Consider a difference equation $x_{n+1} = Ax_n$ with the matrix $A = \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix}$. Find $\lim_{n\to\infty} x_n$ for each $x_0 \in \mathbb{R}^2$.

5. (10%) Let A be a $k \times k$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose that A has an $m \times n$ submatrix whose entries are all 1. Show that $mn \le k$.

6. For an $n \times n$ matrix A, define

$$\exp A := I_n + \sum_{k=1}^{\infty} \frac{1}{k!}A^k.$$

(a) (10%) Compute $\exp \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$.

(b) (5%) Prove or disprove (by giving a counterexample) that if A is nilpotent, then so is $\exp A - I_n$. (A matrix M is *nilpotent* if $M^k = 0$ for some positive integer k.)

(c) (5%) Prove or disprove (by giving a counterexample) that if $\exp A - I_n$ is nilpotent, then so is A.

7. Let $V = M_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices over $\mathbb{R}$. For a given matrix $A \in M_n(\mathbb{R})$, define a linear transformation $T_A : V \to V$ by

$$T_A(B) = AB - BA.$$

(a) (15%) Consider the case $n = 3$ and $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Determine the eigenvalues of T_A and their associated eigenspaces. Determine also the minimal polynomial of T_A.

(b) (5%) For general n, consider the family

$$\mathcal{F} = \{T_A : A \in M_n(\mathbb{R}) \text{ is a diagonal matrix}\}$$

of linear transformations. Prove that $\mathcal{F}$ is simultaneously diagonalizable.