- HH0053 -

Solution for Final (Thermal Physics II)

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• Effusion through a tiny hole

The flux through a tiny hole in the effusive regime is $\Phi = J_n A = \frac{1}{4}\bar{c}nA$. Due to effusion, the number of particles is reduced,

$$\frac{dN}{dt} = -\Phi = -\frac{1}{4}\bar{c}An \quad \rightarrow \quad \frac{dn}{dt} = -\left(\frac{\bar{c}A}{4V}\right)n. \tag{1}$$

The solution is $n(t) = n_0 \exp(-t/\tau_E)$ where $\tau_E = 4V/(\bar{c}A)$ is the effusion into the vacuum.

• Transport through a quantum dot

(a) The tunneling currents into the quantum dots are $I_a = q\gamma_a(f_a - N)$ with $a = 1, 2$. In the steady state, the particle number on the dot N is independent of time, indicating that the current from source to drain is $I = I_1 = -I_2$. It is straightforward to find that

$$I = q\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}(f_1 - f_2) \approx q\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}. \tag{2}$$

(b) Due to the uncertainty principle $\Delta E \Delta t \sim \hbar$, the above expression for tunneling current is not correct. Following the derivations in the notes, the conductance is

$$G = \left(\frac{dI}{dV_{sd}}\right)_0 = q^2\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}P(\epsilon) = \frac{q^2}{\sqrt{2\pi\sigma^2}}\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}. \tag{3}$$

In the symmetric setup, $\gamma_i = \gamma/2$ and $G = G_0 = q^2/h$, it is straightforward to find the variance of the broadened energy level $\sigma = h\gamma/\sqrt{32\pi}$.

• Boltzmann transport equation

(a) Within the relaxation time approximation, the incoming rate due to collision is $\gamma_{in} = f_0/\tau_c$ and the outgoing rate due to collisions is $\gamma_{out} = -f/\tau_c$.

Thus, the Boltzmann transport equation in one dimension takes the simple form,

$$\frac{\partial f}{\partial t} + \frac{p}{M}\frac{\partial f}{\partial x} + F_{ex}\frac{\partial f}{\partial p} = -\left(\frac{f - f_0}{\tau_c}\right). \tag{4}$$

(b) Because there is no current in or out of the wire, the steady state is the equilibrium state. The Boltzmann transport equation reads,

$$\frac{p}{M}\frac{\partial f}{\partial x} + qE\frac{\partial f}{\partial p} = 0 \quad \rightarrow \quad f(x,p) = C\exp\left(-\frac{p^2}{2M\tau} + \frac{qEx}{\tau}\right), \tag{5}$$

where C is a constant determined by $\int dxdp/(2\pi\hbar) f(x,p) = N$. The integrals involved are fundamental and the constant turns out to be

$$C = \frac{n}{n_Q}\frac{(qEL/2\tau)}{\sinh(qEL/2\tau)}, \tag{6}$$

where $n_Q = \sqrt{M\tau/(2\pi\hbar^2)}$ is the quantum concentration in one dimension.

• Einstein relation

The charge current due to electric field is $J_E = qnu_d = qn\mu E$. On the other hand, the charge current due to density gradient is $J_D = -qD(dn/dx) = -qD(qE/\tau)n$, where I have used the spatial relation $n(x) \sim e^{qEx/\tau}$. The detail balance in equilibrium implies that $J_E + J_D = 0$ everywhere in the wire, leading to the Einstein relation

$$qD = \mu\tau \quad \rightarrow \quad \text{(fluctuations)} = \text{(dissipations)}. \tag{7}$$

It is rather remarkable that fluctuations in equilibrium is related to dissipations off equilibrium.

• Fokker-Planck equation

Starting from the master equation in momentum space,

$$P(p, t + \Delta t) = \int_{-\infty}^{\infty} d\Delta p\, f(\Delta p, p')P(p', t), \tag{8}$$

where $p' = p - \Delta p$. Expanding the Taylor series to quadratic order, it leads to the Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial p}\Big[A(p)P\Big] + \frac{1}{2}\frac{\partial^2}{\partial p^2}\Big[B(p)P\Big], \tag{9}$$

The coefficient functions $A(p)$ and $B(p)$ can be evaluated from the Langevin equation $M dv/dt = -\gamma v + f(t)$,

$$A(p) \equiv \lim_{\Delta t \to 0} \frac{\langle \Delta p \rangle}{\Delta t} = -\gamma p, \qquad B(p) \equiv \lim_{\Delta t \to 0} \frac{\langle (\Delta p)^2 \rangle}{\Delta t} = \Lambda, \tag{10}$$

where Λ is the strength of the noise correlator, $\langle f(t) f(t') \rangle = \Lambda \delta(t - t')$. The above partial differential equation can be solved analytically and the probability density for momentum distribution is

$$P(p,t) = \frac{1}{\sqrt{2\pi \Delta(t)}} \exp\left[-\frac{\left(p - p_{int}\right)^2}{2\Delta(t)} \right], \tag{11}$$

where $p_{int}(t) = p_0\, e^{-\gamma t}$ and $\Delta(t) = (\Lambda/2\gamma)(1 - e^{-2\gamma t})$ As one can tell from the analytic solution, thermalization is rather similar to diffusion except the time dependences of the drift and variance are different.

• Sound wave propagation

Differentiating the continuity equation and dropping higher-order terms, one can eliminate the velocity derivative with the help of the Euler equation,

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 P + \rho \nabla \cdot \boldsymbol{g} = 0. \tag{12}$$

The second term has been evaluated in class, $\nabla^2 P = c^2 \nabla^2 \rho$, where $c = \sqrt{\gamma \tau / M}$. The last term is simple, $\rho \nabla \cdot \boldsymbol{g} = -4\pi G \rho^2$ but its consequence is non-trivial. The wave equation for the sound propagation is

$$\frac{\partial^2 \rho}{\partial t^2} = c^2 \nabla^2 \rho + 4\pi G \rho^2. \tag{13}$$

The above wave equation is no longer linear and the wave propagation is non-trivial when the wave length is long (or the wave number k is small).

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(1) Please do not flip the sheet until instructed.
(2) Please try to be as neat as possible so that I can understand your answers without ambiguity.
(3) While it is certainly your rights to make wild guesses or memorize irrelevant details, I would truly appreciate if you try to make your answers logical.
(4) Good luck for all hard-working students!

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1. Effusion through a tiny hole (20%) Consider an ideal gas in the effusive regime contained within volume V. The gas molecules effuse into vacuum through a tiny hole of area A. The initial particle density is n_0 and the average speed of gas molecules is $\bar{c}$. Derive the dynamical equation for $n(t)$ and find the solution .

2. Transport through a quantum dot (20%) Consider transport of charge q particles from the source into a quantum dot with tunneling rate γ_1, followed by tunneling from the dot to the drain with rate γ_2. The chemical potentials for the source and the drain are

$$\mu_1 = \epsilon + \frac{1}{2} q V_{sd},$$
$$\mu_2 = \epsilon - \frac{1}{2} q V_{sd},$$

where ϵ is the energy level for the isolated quantum dot and V_{sd} is the voltage difference. **(a)** Ignore the broadening of the energy level and compute the tunneling current in the steady state. **(b)** In reality, the energy level of the quantum dot is broadened due to uncertainty principle. Suppose the probability distribution of the energy level is the normal distribution,

$$P(E) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(E-\epsilon)^2}{2\sigma^2} \right],$$

where $\sigma = \sigma(\gamma)$ is a function of the total tunneling rate $\gamma = \gamma_1 + \gamma_2$. As derived in class, the maximum conductance is $G_0 = q^2/h$ in the symmetric setup $\gamma_1 = \gamma_2$. Find the variance of the broadened energy level $\sigma(\gamma)$.

3. Bolzmann transport equation (20%) A stochastic system in the semiclassical regime is described by a probability function $f(\boldsymbol{r}, \boldsymbol{p}, t)$ satisfying the Boltzmann transport equation,

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla_r f + \boldsymbol{F}_{ex} \cdot \nabla_p f = \left(\frac{\partial f}{\partial t} \right)_{\mathrm{c}},$$

where the subscript c denotes the changing rate due to collisions. **(a)** Estimate the scattering rates in relaxation time approximation and write down the Boltzmann transport equation in one dimension. **(b)** Consider a finite wire with ends at $x = \pm L/2$ under the influence of a constant electric field E. Find the steady-state solution for $f(x, p)$ in the classical regime.

4. Einstein relation (20%) Evaluate the current densities due to the electric field and the density gradient in the one dimensional wire considered in the previous problem. Making use of the detail balance, derive the Einstein relation for the system in equilibrium and explain the physical meaning behind the equality.

5. Fokker-Planck equation (20%) For a homogeneous statistical system, it can be described by a probability density $P(p, t)$ without spatial dependence. Starting from the master equation, expand the integral equation to appropriate orders and derive the Fokker-Planck equation in the momentum space. Explain in detail how to extract the coefficient functions $A(p)$ and $B(p)$ from the Langevin equation and show that Boltzmann distribution is one of the steady-state solutions.

6. Sound wave propagation (Bonus 20%) In the absence of viscosity, the fluid dynamics is well captured by the linearized Euler equations,

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \boldsymbol{v} = 0,$$
$$\frac{\partial}{\partial t}\left(P \rho^{-\gamma} \right) = 0,$$
$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \nabla P = \boldsymbol{f}_{ex},$$

where the density variation is assumed to be much smaller than the average density ρ_0 of the fluid. In class, we derive the wave equation in the absence of external forces. With the inclusion of gravity, $\boldsymbol{f}_{ex} = \rho \boldsymbol{g}$, derive the wave equation for sound propagation.