1. (a) (*3pts*) Let n_i be the number of candies the ith child got, where $i = 1, 2, 3, 4$, then it must be satisfied that

$$n_i \geq 1,\ i = 1, 2, 3, 4, \quad \text{and} \quad n_1 + n_2 + n_3 + n_4 = 10. \tag{1}$$

The number of integer solutions for Eqn. (1) is $\binom{10-1}{4-1} = 84$.

(b) (*3pts*) For each of the 10 different books, there are four distinct (and independent) choices of child it can be given to. So, the answer is $4 \times 4 \times \cdots \times 4 = 4^{10}$.

2. (*8pts*) The three equations can be represented in terms of p_1, p_2, p_3, and p_4 as follows:

$$\begin{aligned} P(A \cup B) = 3P(B) &\quad \Rightarrow \quad p_1 + p_2 + p_3 = 3(p_1 + p_3), & (2)\\ P(A \cap B) = 0.4P(A \cap B^c) &\quad \Rightarrow \quad p_1 = 0.4p_2, & (3)\\ P((A \cup B)^c) = 0.1 &\quad \Rightarrow \quad p_4 = 0.1. & (4) \end{aligned}$$

Furthermore, because $(A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B^c) = \Omega$ (the whole sample space),

$$p_1 + p_2 + p_3 + p_4 = 1. \tag{5}$$

By solving Eqns. (2), (3), (4), and (5), we get

$$p_1 = 0.24, \quad p_2 = 0.6, \quad p_3 = 0.06, \quad \text{and} \quad p_4 = 0.1.$$

Therefore, $P(A) = p_1 + p_2 = 0.84$.

3. (a) (*4pts*) Let E be the event of picking a black ball the first draw. Then,

$$P(E|U_1) = \frac{3}{5} \quad \text{and} \quad P(E|U_2) = \frac{2}{5}.$$

So, by the law of total probability,

$$\begin{aligned} P(E) &= P(E|U_1) \cdot P(U_1) + P(E|U_2) \cdot P(U_2) \\ &= \frac{3}{5} \cdot \frac{1}{2} + \frac{2}{5} \cdot \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

(b) (*2pts*) By the Bayes' Rule, we get

$$P(U_1|E) = \frac{P(U_1 \cap E)}{P(E)} = \frac{P(E|U_1) \cdot P(U_1)}{P(E)} = \frac{\frac{3}{5} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{3}{5}.$$

(c) (*4pts*) Let F be the event that the second ball is black, then by the law of total probability,

$$\begin{aligned} P(E \cap F) &= P(E \cap F|U_1) \cdot P(U_1) + P(E \cap F|U_2) \cdot P(U_2) \\ &= \left(\frac{3}{5}\right)^2 \cdot \frac{1}{2} + \left(\frac{2}{5}\right)^2 \cdot \frac{1}{2} = \frac{13}{50}. \end{aligned}$$

Therefore,

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{13/50}{1/2} = \frac{13}{25}.$$

4. (a) (*5pts*) For the events A, B, and C,

$$\begin{aligned} P(A) &= P(\{\text{red die is 1, 2, or 3}\}) = 3/6 = 1/2, \\ P(B) &= P(\{\text{red die is 3, 4, or 5}\}) = 3/6 = 1/2, \\ P(C) &= P(\{\text{(red die, blue die) is (1, 4), (2, 3), (3, 2), or (4, 1)}\}) = 4/36 = 1/9. \end{aligned}$$

The event $A \cap B \cap C$ is exactly the event that the red die is 3 and blue die is 2. So,

$$P(A \cap B \cap C) = \frac{1}{36} = P(A)P(B)P(C).$$

(b) (*3pts*) The answer is NO because

$$P(A \cap B) = P(\{\text{red die is 3}\}) = \frac{1}{6} \neq \frac{1}{4} = P(A)P(B).$$

5. (a) (*8pts*) Let $F_Y(y)$ be the cumulative distribution function of Y, then

$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P(aX \leq y - b). \tag{6}$$

(i) When $a > 0$, from (6) we get

$$P(aX \leq y - b) = P\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right).$$

(ii) When $a < 0$, from (6) we get

$$P(aX \leq y-b) = P\left(X \geq \frac{y-b}{a}\right) = 1-P\left(X < \frac{y-b}{a}\right) = 1-F_X\left(\left(\frac{y-b}{a}\right)-\right),$$

where $F_X\left(\left(\frac{y-b}{a}\right)-\right)$ is the left limit of F_X at $\frac{y-b}{a}$.

(iii) When $a = 0$, from (6) we get

$$P(aX \leq y - b) = P(0 \leq y - b) = \begin{cases} 1, & \text{if } y \geq b, \\ 0, & \text{if } y < b. \end{cases}$$

(b) (*3pts*) Because all values of a probability mass function must sum up to one, we get

$$1 = \sum_{x=1}^{4} f_X(x) = c \cdot (1 + 4 + 9 + 16) = 30 \cdot c \quad \Rightarrow \quad c = \frac{1}{30}.$$

6. (a) (*10pts*) Let Y_1 and Y_2 be the labels on the first and second tickets drawn, respectively. Then,

$$P(Y_1 = y_1, Y_2 = y_2) = \begin{cases} (1\times 2)/(10\times 9), & \text{if } (y_1, y_2) = (1,2) \text{ or } (2,1), \\ (1\times 3)/(10\times 9), & \text{if } (y_1, y_2) = (1,3) \text{ or } (3,1), \\ (1\times 4)/(10\times 9), & \text{if } (y_1, y_2) = (1,4) \text{ or } (4,1), \\ (2\times 1)/(10\times 9), & \text{if } (y_1, y_2) = (2,2), \\ (2\times 3)/(10\times 9), & \text{if } (y_1, y_2) = (2,3) \text{ or } (3,2), \\ (2\times 4)/(10\times 9), & \text{if } (y_1, y_2) = (2,4) \text{ or } (4,2), \\ (3\times 2)/(10\times 9), & \text{if } (y_1, y_2) = (3,3), \\ (3\times 4)/(10\times 9), & \text{if } (y_1, y_2) = (3,4) \text{ or } (4,3), \\ (4\times 3)/(10\times 9), & \text{if } (y_1, y_2) = (4,4), \end{cases}$$

and

$$X = |Y_1 - Y_2| = \begin{cases} 1, & \text{if } (Y_1, Y_2) = (1,2) \text{ or } (2,1), \\ 2, & \text{if } (Y_1, Y_2) = (1,3) \text{ or } (3,1), \\ 3, & \text{if } (Y_1, Y_2) = (1,4) \text{ or } (4,1), \\ 0, & \text{if } (Y_1, Y_2) = (2,2), \\ 1, & \text{if } (Y_1, Y_2) = (2,3) \text{ or } (3,2), \\ 2, & \text{if } (Y_1, Y_2) = (2,4) \text{ or } (4,2), \\ 0, & \text{if } (Y_1, Y_2) = (3,3), \\ 1, & \text{if } (Y_1, Y_2) = (3,4) \text{ or } (4,3), \\ 0, & \text{if } (Y_1, Y_2) = (4,4). \end{cases}$$

Therefore, the probability mass function of X is

$$f_X(x) = \begin{cases} P(X=0) = (2+6+12)/90 = 20/90, & \text{if } x = 0, \\ P(X=1) = (2\times 2 + 6\times 2 + 12\times 2)/90 = 40/90, & \text{if } x = 1, \\ P(X=2) = (3\times 2 + 8\times 2)/90 = 22/90, & \text{if } x = 2, \\ P(X=3) = (4\times 2)/90 = 8/90, & \text{if } x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

(b) (*2pts*)

$$E(X) = \frac{0\times 20 + 1\times 40 + 2\times 22 + 3\times 8}{90} = 108/90 = 1.2.$$

(c) (*3pts*)

$$E(X^2) = \frac{0\times 20 + 1\times 40 + 4\times 22 + 9\times 8}{90} = 20/9,$$

$$Var(X) = E(X^2) - (E(X))^2 = 20/9 - (1.2)^2 = 176/225 \approx 0.782.$$

7. (a) (*6pts*) We first show that

$$\begin{aligned} P(X > n) &= \sum_{x=n+1}^{\infty} f_X(x) = \sum_{x=n+1}^{\infty} p\cdot(1-p)^{x-1} \\ &= p\cdot[(1-p)^n + (1-p)^{n+1} + (1-p)^{n+2} + \cdots] = p\cdot\frac{(1-p)^n}{p} = (1-p)^n. \end{aligned}$$

Then, for positive integers n and k,

$$\begin{aligned} P(X = n+k | X > n) &= \frac{P((X = n+k)\cap(X > n))}{P(X > n)} = \frac{P(X = n+k)}{P(X > n)} \\ &= \frac{p\cdot(1-p)^{n+k-1}}{(1-p)^n} = p\cdot(1-p)^{k-1} = P(X = k). \end{aligned}$$

(b) (*6pts*) The random variable X is the number of independent Bernoulli trials required until we get r successes where each trial has probability p of success. Suppose that we run n such trials and have not yet got r successes in the n trials. We need to continue in order to find the value of X and so we know that it is the event $X > n$. The number of successes in the first n trials, however, follows a binomial distribution with parameters n and p. We need to continue if, and only if, we have not found r successes in n trials, which is the event $Y < r$. Thus the two events $X > n$ and $Y < r$ are identical and hence have the same probability, i.e.,

$$P(X > n) = P(Y < r).$$