NTHU MATH 2810 Midterm Examination Oct 30, 2007

Note. There are 7 problems in total. **To ensure consideration for partial scores, write down intermediate steps where necessary.**

1. (a) (*3pts*) Two years ago, Santa Claus was distributing 10 *identical* candies to 4 children at the mall. Being a kind-hearted soul, he figured he would give at least one candy to each child, but other than that he could give any number of the candies to any of the children. How many ways were there for Santa Claus to have distributed his candies?

   (b) (*3pts*) Santa Claus was back at the mall last year, but that time he brought 10 *different* books for the 4 kids. But no more "mister nice guy," he was willing to give some kids *no* books! How many ways were there for him to distribute his books that time?

2. (*8pts*) Suppose that A and B are two events such that

$$\begin{aligned} P(A \cup B) &= 3 \times P(B), \\ P(A \cap B) &= 0.4 \times P(A \cap B^c), \\ P((A \cup B)^c) &= 0.1. \end{aligned}$$

   Find the value of $P(A)$.

   [**Hint.** Let p_1, p_2, p_3, and p_4 be the probabilities of the four mutually exclusive events $A \cap B$, $A \cap B^c$, $A^c \cap B$, and $A^c \cap B^c$, respectively. Use p_1, p_2, p_3, and p_4 to represent the above equations and get the solution. It would be helpful to draw a Venn Diagram.]

3. We have two boxes, A and B, with five balls in each. We know one box contains 3 black balls and 2 red ones, while the other contains 2 black ones and 3 red ones. But, we do not know which is which. We draw a ball from box A, record the color and *replace* it. Then, we draw another ball from box A.

   (a) (*4pts*) What is the probability that the first ball drawn is black?

   [**Hint.** Let U_1 be the event that "A is the urn with 3 black balls" and U_2 be the event that "A is the other urn", then we can assume $P(U_1) = P(U_2) = 1/2$.]

   (b) (*2pts*) If the first ball is black, what is the probability that A is the box with 3 black balls?

   (c) (*4pts*) What is the probability that the second ball drawn is black, given that the first one is black?

4. There are two *fair* dice, one red and the other blue. Roll the two dice. Define the following events:

$$\begin{aligned} A &= \text{the red die is less than 4}, \\ B &= \text{the red die is 3, 4, or 5}, \\ C &= \text{the sum of the two dice is 5}. \end{aligned}$$

   (a) (*5pts*) Show that $P(A \cap B \cap C) = P(A)P(B)P(C)$.

   (b) (*3pts*) Examine whether A, B, and C are mutually independent.

5. (a) (*8pts*) The random variable X has cumulative distribution function (cdf) $F_X(x)$. What is the cumulative distribution function of the random variable $Y = aX + b$, where a and b may each be any real-value number? Express your answer in terms of F_X, a, and b.

   [**Hint.** Consider the three conditions individually: (i) $a > 0$, (ii) $a < 0$, and (iii) $a = 0$.]

   (b) (*3pts*) Suppose that X is a discrete random variable with the probability mass function (pmf):

$$f_X(x) = cx^2, \quad x = 1, 2, 3, 4.$$

   Find the value of c.

6. Two tickets are drawn *without replacement* from a box containing 1 ticket labelled "one," 2 tickets labelled "two," 3 tickets labelled "three," and 4 tickets labelled "four." Let X be the *absolute* difference between the labels on the two tickets drawn.

   (a) (*10pts*) Find the probability mass function of X.

   (b) (*2pts*) Find $E(X)$.

   (c) (*3pts*) Find $Var(X)$.

   (**Note.** If you cannot answer (a), you can state how to calculate $E(X)$ and $Var(X)$ for partial scores.)

7. (a) (*6pts*) Suppose that X is a geometric random variable with parameter p, i.e., its probability mass function is

$$f_X(x) = \begin{cases} (1-p)^{x-1}p, & x = 1, 2, 3, \cdots, \\ 0, & \text{otherwise.} \end{cases}$$

   Show that for any positive integers n and k,

$$P(X = n + k | X > n) = P(X = k).$$

   This is called *memoryless* property.

   [**Hint. Show** first that $P(X > n) = (1 - p)^n$.]

   (b) (*6pts*) Suppose that X is a negative binomial random variable with parameters r and p, i.e., its probability mass function is

$$f_X(x) = \begin{cases} \binom{x-1}{r-1} p^r (1-p)^{x-r}, & x = r, r+1, \ldots, \\ 0, & \text{otherwise,} \end{cases}$$

   and that Y is a binomial random variable with parameters n and p, i.e., its probability mass function is

$$f_Y(y) = \begin{cases} \binom{n}{y} p^y (1-p)^{n-y}, & y = 0, 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

   Show that $P(X > n) = P(Y < r)$ by considering the underlying sequence of Bernoulli trials.

   [**Hint.** Note that you don't have to compute $P(X > n)$ nor $P(Y < r)$ to answer this question. It is enough to show that the two events, $X > n$ and $Y < r$, are identical.]