Chap 7. p1

9. (a) Let X_i denote the i-th urn is empty

$$X_i = \begin{cases} 1 & \text{if i-th urn is empty} \\ 0 & \text{o.w} \end{cases}$$

$$E(X_i) = 1 \times p(\text{i-th urn is empty}) + 0 \times p(\text{i-th urn is not empty})$$
$$= \left(\frac{i-1}{i} \cdot \frac{i}{i+1} \cdot \cdots \cdot \frac{1}{n}\right) = \frac{i-1}{n}$$
$$\therefore E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{i-1}{n} = \frac{n-1}{2}$$

(X: number of urns that are empty, $X = \sum_{i=1}^{n} X_i$)

(b) $p(\text{ball } n \text{ in urn } n) \cdot p(\text{ball } n-1 \text{ in urn } n-1) \cdots p(\text{ball } 1 \text{ in urn } 1)$
$$= \frac{1}{n} \cdot \frac{1}{n-1} \cdots \frac{1}{1} = \frac{1}{n!}$$

25. $$p(N=n) = p(X_1 \geq X_2 \geq \cdots \geq X_{N-1} < X_N)$$
$$= p(X_1 \geq X_2 \geq \cdots \geq X_{N-1}) - p(X_1 \geq X_2 \geq \cdots \geq X_N)$$
$$= \frac{1}{(N-1)!} - \frac{1}{N!} \qquad n = 2, 3 \cdots \infty$$
$$p(N \geq n) = \sum_{i=n}^{\infty} p(N=i) = \sum_{i=n}^{\infty} \left[\frac{1}{(i-1)!} - \frac{1}{i!}\right] = \frac{1}{(n-1)!} \qquad n = 2, 3 \cdots \infty$$
$$\therefore E(T) = \sum_{n=2}^{\infty} p(N \geq n) = \sum_{n=1}^{\infty} p(N \geq n) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = e$$

40. $$f_Y(y) = \int_0^\infty \frac{1}{y} e^{-(y+\frac{x}{y})} dx = \frac{1}{y} e^{-y} \int_0^\infty e^{-\frac{x}{y}} dx = e^{-y} \Rightarrow Y \sim Exp(1) \quad E(Y) = 1$$
$$E(X|Y) = \int_0^\infty x\, f_{X|Y}(x|y)\, dx = \int_0^\infty x \frac{f(x,y)}{f_Y(y)} dy = \int_0^\infty x \cdot \frac{1}{y} \cdot e^{-\frac{x}{y}} dx = y$$
$$\therefore E(X) = E\,E(X|Y) = E(Y) = 1$$
$$E(XY) = \int_0^\infty \int_0^\infty xy \cdot \frac{1}{y} e^{-(y+\frac{x}{y})} dx\, dy = \int_0^\infty e^{-y} \int_0^\infty x e^{-\frac{x}{y}} dx\, dy$$
$$= \int_0^\infty e^{-y} \left[-yxe^{-\frac{x}{y}}\Big|_0^\infty + y \int_0^\infty e^{-\frac{x}{y}} dx\right] dy$$
$$= \int_0^\infty y^2 e^{-y} dy = 2.$$
$$\therefore Cov(X, Y) = E(XY) - E(X)E(Y) = 2 - 1 \cdot 1 = 1$$

p2

45. ⓐ $\text{cov}(X_1+X_2, X_2+X_3)$

$= \text{cov}(X_1,X_2)+\text{cov}(X_1,X_3)+\text{cov}(X_2,X_2)+\text{cov}(X_2,X_3)$

$= 0+0+\text{var}(X_2)+0 = 1$

$\text{var}(X_1+X_2) = \text{var}(X_1)+\text{var}(X_2) = 2$ $\quad \text{var}(X_2+X_3)=2$

$\therefore \rho = \dfrac{\text{cov}(X_1+X_2, X_2+X_3)}{\sqrt{\text{var}(X_1+X_2)\text{var}(X_2+X_3)}} = \dfrac{1}{\sqrt{2\cdot 2}} = \dfrac{1}{2}$

ⓑ $\text{cov}(X_1+X_2, X_3+X_4)$

$= \text{cov}(X_1,X_3)+\text{cov}(X_1,X_4)+\text{cov}(X_2,X_3)+\text{cov}(X_2,X_4) = 0$

$\therefore \rho = 0$.

56. $I_i = \begin{cases} 1 & \text{the elevator stops at floor } i \\ 0 & \text{o.w.} \end{cases}$

$\sum_{i=1}^{N} I_i$ = the number of stops that the elevator will make before discharging all passengers

X : the number of people enter elevator on the ground floor

$\therefore E[\sum_{i=1}^{N} I_i \mid X=k] = \sum_{i=1}^{N} E[I_i \mid X=k]$

$= \sum_{i=1}^{N} P(\text{the elevator stops at floor } i \mid X=k)$

$= \sum_{i=1}^{N} \left(1-\left(\frac{N-1}{N}\right)^k\right)$

$\therefore E[\sum_{i=1}^{N} I_i] = E\, E[\sum_{i=1}^{N} I_i \mid X=k]$

$= E \sum_{i=1}^{N}\left(1-\left(\frac{N-1}{N}\right)^k\right)$

$= N E\left[1-\left(\frac{N-1}{N}\right)^k\right]$

$= N - N E\left[\left(\frac{N-1}{N}\right)^k\right]$

$= N - N \sum_{k=0}^{\infty} \left(\frac{N-1}{N}\right)^k \cdot \frac{e^{-10}10^k}{k!}$

$= N - N \sum_{k=0}^{\infty} \frac{e^{-10}[10(\frac{N-1}{N})]^k}{k!}$

$= N - N \underbrace{\left(\sum_{k=0}^{\infty} \frac{e^{-10\cdot(\frac{N-1}{N})}[10(\frac{N-1}{N})]^k}{k!}\right)}_{P_o(10(\frac{N-1}{N}))\text{的機率合}} \cdot e^{-\frac{10}{N}}$

$= N - Ne^{-\frac{10}{N}}$

P3

48 ⓐ $cov(X,Y|Z) = E[(X-E(X|Z))(Y-E(Y|Z))|Z]$

$= E[XY|Z] - E(Y|Z)E(X|Z) - E(X|Z)E(Y|Z) + E(X|Z)E(Y|Z)$

$= E(XY|Z) - E[X|Z]\,E[Y|Z]$

ⓑ $cov(X,Y) = E(XY) - E(X)E(Y)$

$= E[E(XY|Z)] - E(X)E(Y)$

$\overset{\text{by ⓐ}}{=} E[cov(X,Y|Z) + E[X|Z]E[Y|Z]] - E(X)E(Y)$

$= E(cov(X,Y|Z)) + E[E(X|Z)E(Y|Z)] - E(E(X|Z))E(E(Y|Z))$

$= E(cov(X,Y|Z)) + cov(E(X|Z), E(Y|Z))$

ⓒ $var(Y) = cov(Y,Y) = E(var(Y|Z)) + var(E(Y|Z))$

49. $X \sim LN(\mu,\sigma^2)$ $\quad Y = \log X \sim N(\mu,\sigma^2)$

$X = e^Y \quad M_Y(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} = E(e^{tY})$

$\therefore E(X) = E(e^Y) = E(e^{tY})\big|_{t=1} = e^{\mu + \frac{\sigma^2}{2}}$

$E(X^2) = E(e^{2Y}) = E(e^{tY})\big|_{t=2} = e^{2\mu + 2\sigma^2}$

$Var(X) = E(X^2) - [E(X)]^2 = e^{2\mu+2\sigma^2} - [e^{\mu + \frac{\sigma^2}{2}}]^2 = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$

50. $M_X(t) = E(e^{tX}) = E\left(\sum_{i=0}^{\infty} \frac{(tX)^i}{i!}\right) = E\left(1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots\right)$

$= 1 + \frac{tE(X)}{1!} + \frac{t^2E(X^2)}{2!} + \frac{t^3E(X^3)}{3!} + \cdots$

$= \sum_{i=0}^{\infty} \frac{t^i E(X^i)}{i!}$

$\psi(t) = \log M_X(t) = \log \sum_{i=0}^{\infty} \frac{t^i E(X^i)}{i!}$

$$\psi'(t) = \frac{\sum_{i=1}^{\infty} \frac{t^{i-1}E(X^i)}{(i-1)!}}{\sum_{i=0}^{\infty} \frac{t^i E(X^i)}{i!}}$$

$$\psi''(t) = \frac{\left(\sum_{i=2}^{\infty} \frac{t^{i-2}E(X^i)}{(i-2)!}\right)\left(\sum_{i=0}^{\infty} \frac{t^i E(X^i)}{i!}\right) - \left(\sum_{i=1}^{\infty} \frac{t^{i-1}E(X^i)}{(i-1)!}\right)^2}{\left(\sum_{i=0}^{\infty} \frac{t^i E(X^i)}{i!}\right)^2}$$

$$\psi''(t)\big|_{t=0} = \frac{E(X^2)E(X^0) - (E(X))^2}{E(X^0)^2} = E(X^2) - [E(X)]^2 = var(X)$$