Expectation

- Recall. Expectation for univariate random variable.
- Theorem. For random variables $\mathbf{X}=(X_1, \ldots, X_n)$ with joint pmf $p_{\mathbf{X}}$/pdf $f_{\mathbf{X}}$, the *expectation* of a univariate random variable Y, where

$$Y=g(X_1, \ldots, X_n),\ g:\mathbb{R}^n\rightarrow\mathbb{R}^1,$$

is
$$\begin{aligned} E(Y) &\equiv \textstyle\sum_{y\in\mathcal{Y}} y\, p_Y(y) && (1)\\ &= \textstyle\sum_{\mathbf{x}=(x_1,\ldots,x_n)\in\mathcal{X}} g(x_1,\ldots,x_n)\, p_{\mathbf{X}}(x_1,\ldots,x_n) && (2)\\ &\equiv E[g(X_1,\ldots,X_n)] \end{aligned}$$

if $X_1, \ldots, X_n$ are discrete and the sum converges absolutely, or

$$\begin{aligned} E(Y) &\equiv \textstyle\int_{-\infty}^{\infty} y f_Y(y)\, dy && (3)\\ &= \textstyle\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\ldots,x_n) f_{\mathbf{X}}(x_1,\ldots,x_n)\, dx_1\cdots dx_n && (4)\\ &\equiv E[g(X_1,\ldots,X_n)] \end{aligned}$$

if Y and $X_1, \ldots, X_n$ are continuous and the integrals converges absolutely.

Proof. Like the univariate case.

➢ **Q**: What if Y is discrete and $X_1, \ldots, X_n$ are continuous?

➢ Notation.

- Shorthand notation. Combine (1) and (3), by writing

$$E(Y)=\int_{-\infty}^{\infty} y\, dF_Y(y)=\begin{cases} \sum_{y\in\mathcal{Y}} y\, p_Y(y), & \text{for discrete case,}\\ \int_{-\infty}^{\infty} y f_Y(y)\, dy, & \text{for continuous case,}\end{cases}$$

  and combine (2) and (4) by writing

$$\begin{aligned} E[g(\mathbf{X})] &= \int_{\mathbb{R}^n} g(\mathbf{x})\, dF_{\mathbf{X}}(\mathbf{x})\\ &= \begin{cases} \sum_{\mathbf{x}\in\mathcal{X}} g(\mathbf{x})\, p_{\mathbf{X}}(\mathbf{x}), & \text{for discrete case.}\\ \int_{\mathbb{R}^n} g(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}, & \text{for continuous case.}\end{cases} \end{aligned}$$

- Riemann-Stieltjes Integral. For example, for non-negative g,

$$\textstyle\int_a^b g(x)\, dF(x) = \lim \sum_{i=1}^{n} g(x_i)[F(x_i)-F(x_{i-1})].$$

  where the limit is taken over all $a=x_0<x_1<\cdots<x_n=b$ as $n\rightarrow 0$ and $\max_{i=1,\ldots,n}(x_i-x_{i-1})\rightarrow 0$.

  [Recall. The integral of g over (a, b] is defined as

$$\textstyle\int_a^b g(x)\, dx = \lim \sum_{i=1}^{n} g(x_i)(x_i-x_{i-1}).]$$

➢ Note.

- $g(X_1, \ldots, X_n)=X_i \Rightarrow E[g(X_1, \ldots, X_n)]=E(X_i)\equiv \mu_{X_i}$.
- $g(X_1, \ldots, X_n)=(X_i-\mu_{X_i})^2 \Rightarrow E[g(X_1, \ldots, X_n)]=Var(X_i)\equiv\sigma^2_{X_i}$.

➢Example (Distance between two points). Suppose that X, Y are i.i.d. ~ Uniform(0, 1).

Let $D=|X-Y|$. Find $E(D)$.

- The joint pdf of (X, Y) is
$$f(x,y)=\begin{cases}1, & 0\le x\le 1,\ 0\le y\le 1,\\ 0, & \text{otherwise.}\end{cases}$$
- $E(D)=\int_0^1\int_0^1|x-y|\,dydx=\int_0^1\left[\int_0^x(x-y)\,dy+\int_x^1(y-x)\,dy\right]dx$
$$=\int_0^1\left[-\tfrac{1}{2}(y-x)^2\Big|_{y=0}^{x}+\tfrac{1}{2}(y-x)^2\Big|_{y=x}^{1}\right]dx$$
$$=\int_0^1\tfrac{1}{2}\left[x^2+(1-x)^2\right]dx=\tfrac{1}{6}\left[x^3-(1-x)^3\right]\Big|_{x=0}^{1}=\tfrac{1}{3}.$$

- Theorem (Mean of Sum). For r.v.'s $X_1, \ldots, X_n$ and constants $-\infty< a_0, a_1, \ldots, a_n<\infty$,
$$E(a_0+a_1X_1+\cdots+a_nX_n)=a_0+a_1E(X_1)+\cdots+a_nE(X_n).$$

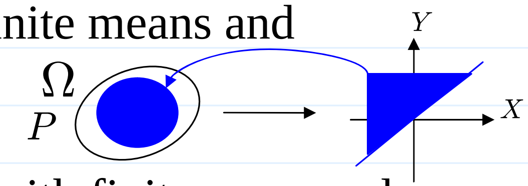
Proof. $E(a_0+a_1X_1+\cdots+a_nX_n)$
$$\begin{aligned}&=\int_{\mathbb{R}^n}(a_0+a_1X_1+\cdots+a_nX_n)\,dF_{\mathbf{X}}(\mathbf{x})\\&=\int_{\mathbb{R}^n}a_0\,dF_{\mathbf{X}}(\mathbf{x})+a_1\int_{\mathbb{R}^n}x_1\,dF_{\mathbf{X}}(\mathbf{x})\\&\qquad+\cdots+a_n\int_{\mathbb{R}^n}x_n\,dF_{\mathbf{X}}(\mathbf{x})\\&=a_0+a_1E(X_1)+\cdots+a_nE(X_n).\end{aligned}$$

➢Corollary. Suppose that $\mu=E(X_1)=\cdots=E(X_n)$. Let $\overline{X}_n=\frac{X_1+\cdots+X_n}{n}$,

then, $E(\overline{X}_n)=\mu$.

➢Corollary. If X and Y are r.v.'s with finite means and $P(X\le Y)=1$,

then $E(X)\le E(Y)$.



Proof. First, if Z is a random variable with finite mean and $P(Z\ge 0)=1$,

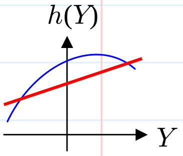
then $E(Z)=\int_0^\infty z\,dF_Z(z)\ge 0$.

For the general case, let $Z=Y-X$, then $Z\ge 0$ with probability one, and therefore, $0\le E(Z)=E(Y-X)=E(Y)-E(X)$.

➢Corollary. If $P(a\le X\le b)=1$ for some constants a, b, then
$$a\le E(X)\le b.$$

- Theorem. If two random vectors $\mathbf{X}$ $(\in\mathbb{R}^m)$ and $\mathbf{Y}$ $(\in\mathbb{R}^n)$ are independent (i.e., $F_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})=F_{\mathbf{X}}(\mathbf{x})\times F_{\mathbf{Y}}(\mathbf{y})$, or
$$f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})=f_{\mathbf{X}}(\mathbf{x})\times f_{\mathbf{Y}}(\mathbf{y}),\ \text{ or }\ p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})=p_{\mathbf{X}}(\mathbf{x})\times p_{\mathbf{Y}}(\mathbf{y})\ ),$$
then for $g:\mathbb{R}^m\to\mathbb{R}$ and $h:\mathbb{R}^n\to\mathbb{R}$,
$$E[g(\mathbf{X})\times h(\mathbf{Y})]=E[g(\mathbf{X})]\times \mathrm{E}[h(\mathbf{Y})].$$

Proof. We only prove it for the continuous case:



$$
\begin{aligned}
E[g(\mathbf{X})h(\mathbf{Y})] &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} g(\mathbf{x})h(\mathbf{y})f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})\, d\mathbf{y}d\mathbf{x} \\
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} g(\mathbf{x})h(\mathbf{y})f_{\mathbf{X}}(\mathbf{x})f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y}d\mathbf{x} \\
&= \int_{\mathbb{R}^m} g(\mathbf{x})f_{\mathbf{X}}(\mathbf{x}) \left[\int_{\mathbb{R}^n} h(\mathbf{y})f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y}\right] d\mathbf{x} \\
&= \left[\int_{\mathbb{R}^m} g(\mathbf{x})f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{x}\right]\left[\int_{\mathbb{R}^n} h(\mathbf{y})f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y}\right] \\
&= E[g(\mathbf{X})]E[h(\mathbf{Y})].
\end{aligned}
$$

- Corollary. For 2 independent r.v.'s X and Y, $E(XY)=E(X)E(Y)$.

- **Q**: For independent r.v.'s X and Y, $E(X/Y)=E(X)/E(Y)$?

- Note. $E[h(Y)]\neq h(E(Y))$ in general, e.g., $E(1/Y) \neq 1/E(Y)$.

• Covariance and Correlation between 2 random variables

- Definition. Suppose that X and Y are two random variables with finite means μ_X, μ_Y and variances σ_X^2, σ_Y^2, respectively.

  1.Let $g(x, y)=(x-\mu_X)(y-\mu_Y)$, then

  $$Cov(X,Y) \equiv E[g(X,Y)] = E[(X-\mu_X)(Y-\mu_Y)]$$

  is called the *covariance* between X and Y, denoted by σ_{XY}.

  2.The *correlation* (coefficient) between X and Y is defined as

  $$Cor(X,Y) = \sigma_{XY}/(\sigma_X\sigma_Y)$$

  and denoted by ρ_{XY}.

  3. X and Y are called *uncorrelated* if ρ_{XY}=0.

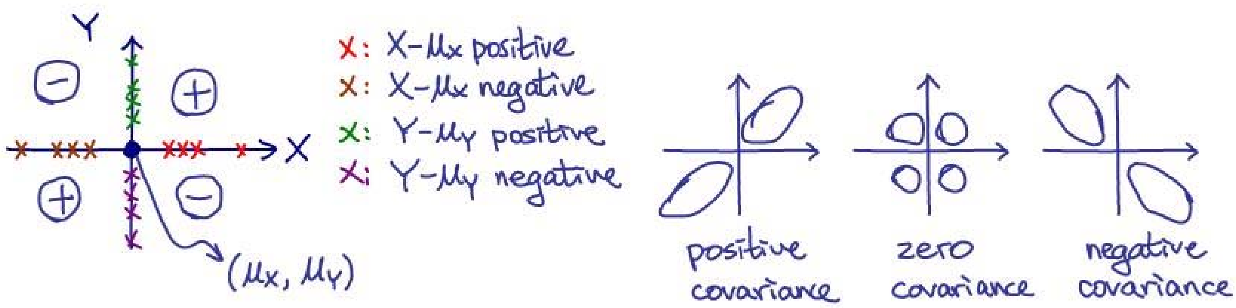
  - A special case of covariance: $Cov(X, X) = Var(X)$.

- Intuitive explanation of covariance and correlation

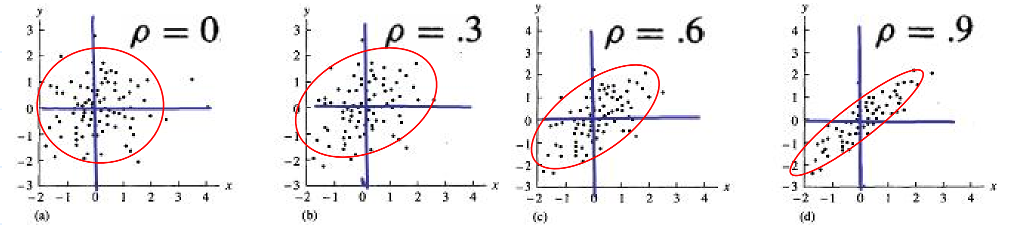
  - Covariance is a measure of the joint variability of X and Y, or their degree of association.

  - Covariance is the average value of the product of the deviation of X from its mean and the deviation of Y from its mean.

  - Positive Covariance and Negative Covariance



- Correlation Coefficient is unit free.
- Correlation coefficient measures the strength of the *linear* relationship between X and Y.

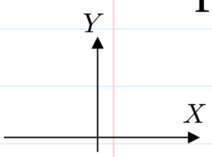


➢Theorem. $Cov(X, Y) = E(XY) - \mu_X\mu_Y$.

Proof.
$$\begin{aligned} Cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\ &= E(XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y) \\ &= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X\mu_Y \\ &= E(XY) - \mu_X\mu_Y - \mu_Y\mu_X + \mu_X\mu_Y. \end{aligned}$$

- Corollary. If X and Y are independent, then $Cov(X, Y)=0$, i.e. X and Y are uncorrelated.

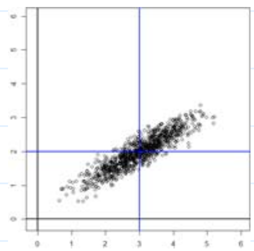
Proof. When X, Y are independent, $E(XY)=E(X)E(Y)=\mu_X\mu_Y$.

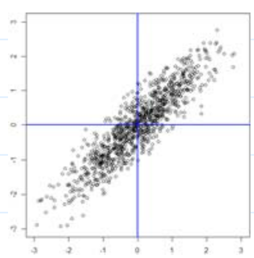


  - However, the converse statement is not necessarily true. (e.g., let X~Uniform(–1, 1) and $Y=X^2$, then $Cov(X, Y)=0$, but X and Y are not independent).

- Corollary.
$$\rho_{XY} = E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right].$$

Proof. By definition.





➢Example. If $(X_1, \ldots, X_m)$ ~ Multinomial(n, m, $p_1, \ldots, p_m$), then
$$Cov(X_i, X_j) = -np_ip_j, \quad \text{for } 1 \le i \ne j \le m.$$

- Because $(X_1, X_2, X_3+\cdots+X_m)$ ~ Multinomial(n, 3, p_1, p_2, $p_3+\cdots+p_m$), and

$X_3+\cdots+X_m = n - X_1 - X_2$,

$p_3+\cdots+p_m = 1 - p_1 - p_2$,

we have
$$\begin{aligned} E(X_1X_2) &= \sum x_1x_2\binom{n}{x_1, x_2, n-x_1-x_2} p_1^{x_1}p_2^{x_2}(1-p_1-p_2)^{n-x_1-x_2} \\ &= \sum x_1x_2\frac{n!}{x_1!x_2!(n-x_1-x_2)!}p_1^{x_1}p_2^{x_2}(1-p_1-p_2)^{n-x_1-x_2} \\ &= n(n-1)p_1p_2\left[\sum\frac{(n-2)!}{(x_1-1)!(x_2-1)!(n-x_1-x_2)!}\right. \\ &\qquad \left.\times p_1^{x_1-1}p_2^{x_2-1}(1-p_1-p_2)^{n-x_1-x_2}\right] \\ &= n(n-1)p_1p_2. \end{aligned}$$

- WLOG, we can get $E(X_iX_j) = n(n-1)p_ip_j$, for $i \neq j$.
  Therefore, $Cov(X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j)$
  $= n(n-1)p_ip_j - (np_i)(np_j) = -np_ip_j.$
- And, for $i \neq j$,

$$Cor(X_i, X_j) = \frac{-np_ip_j}{\sqrt{np_i(1-p_i)}\sqrt{np_j(1-p_j)}} = -\sqrt{\frac{p_ip_j}{(1-p_i)(1-p_j)}}.$$

• Expectations for Sums of Random Variables

➢Notation. In the following, let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be r.v.'s and $-\infty < a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m < \infty$ are constants.

➢Recall. $E(a_0+a_1X_1+\cdots+a_nX_n) = a_0+a_1E(X_1)+\cdots+a_nE(X_n)$.

➢Theorem (covariance of two sums).

$$\begin{aligned} & Cov(a_0 + a_1X_1 + \cdots + a_nX_n, b_0 + b_1Y_1 + \cdots + b_mY_m) \\ = & \textstyle\sum_{i=1}^n \sum_{j=1}^m a_ib_jCov(X_i, Y_j). \end{aligned}$$

Proof. Let $S = a_0+a_1X_1+\cdots+a_nX_n$ and $T = b_0+b_1Y_1+\cdots+b_mY_m$, then

$$\begin{aligned} S - E(S) &= \textstyle\sum_{i=1}^n a_i(X_i - \mu_{X_i}), \\ T - E(T) &= \textstyle\sum_{j=1}^m b_j(Y_j - \mu_{Y_j}), \\ [S - E(S)][T - E(T)] &= \textstyle\sum_{i=1}^n \sum_{j=1}^m a_ib_j(X_i - \mu_{X_i})(Y_j - \mu_{Y_j}). \end{aligned}$$

Therefore,
$$\begin{aligned} Cov(S, T) &= E\{[S - E(S)][T - E(T)]\} \\ &= \textstyle\sum_{i=1}^n \sum_{j=1}^m a_ib_jE[(X_i - \mu_{X_i})(Y - \mu_{Y_j})] \\ &= \textstyle\sum_{i=1}^n \sum_{j=1}^m a_ib_jCov(X_i, Y_j). \end{aligned}$$

➢Theorem (variance of sum).

$$\begin{aligned} Var(a_0 + a_1X_1 + \cdots + a_nX_n) &= \textstyle\sum_{i=1}^n \sum_{j=1}^n a_ia_jCov(X_i, X_j) \\ &= \textstyle\sum_{i=1}^n a_i^2Var(X_i) + 2\sum_{1\le i<j\le n} a_ia_jCov(X_i, X_j). \end{aligned}$$

Proof. $Cov(X_i, X_i)=Var(X_i)$.

- Corollary. If $X_1, \ldots, X_n$ are uncorrelated, then
$$Var(a_0 + a_1X_1 + \cdots + a_nX_n) = \textstyle\sum_{i=1}^n a_i^2Var(X_i).$$
- Corollary. If $X_1, \ldots, X_n$ are uncorrelated and
$$Var(X_1)=\cdots = Var(X_n) \equiv \sigma^2 < \infty,$$
then $Var(\overline{X}_n) = \sigma^2/n$.
- Corollary. Suppose that $X_1, \ldots, X_n$ are uncorrelated and have same mean μ and variance σ^2. Let
$$S^2 = \frac{\sum_{i=1}^n (X_i - \overline{X}_n)^2}{n-1},$$
then $E(S^2)=\sigma^2$.

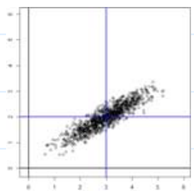
Proof.

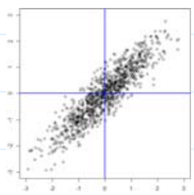
$$
\begin{aligned}
(n-1)S^2 &= \textstyle\sum_{i=1}^{n}[(X_i-\mu)-(\overline{X}_n-\mu)]^2 \\
&= \left[\textstyle\sum_{i=1}^{n}(X_i-\mu)^2\right]+\left[\textstyle\sum_{i=1}^{n}(\overline{X}_n-\mu)^2\right] \\
&\qquad -2(\overline{X}_n-\mu)\left[\textstyle\sum_{i=1}^{n}(X_i-\mu)\right] \\
&= \left[\textstyle\sum_{i=1}^{n}(X_i-\mu)^2\right]+n(\overline{X}_n-\mu)^2-2n(\overline{X}_n-\mu)^2 \\
&= \left[\textstyle\sum_{i=1}^{n}(X_i-\mu)^2\right]-n(\overline{X}_n-\mu)^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(n-1)E(S^2) &= \left\{\sum_{i=1}^{n}E[(X_i-\mu)^2]\right\}-nE\left[(\overline{X}_n-\mu)^2\right] \\
&= n\sigma^2-nVar(\overline{X}_n)=(n-1)\sigma^2.
\end{aligned}
$$

- Note. The previous three corollaries also hold if $X_1, \ldots, X_n$ are independent.





➢Theorem (ρ of linear transformation).

$Cor(a_0+a_1X_1, b_0+b_1Y_1)=\text{sign}(a_1b_1)\times Cor(X_1, Y_1)$,

and $|Cor(a_0+a_1X, b_0+b_1Y)|=|Cor(X, Y)|$,

i.e., $|\rho_{XY}|$ is invariant under location and scale changes.

Proof. Let $S=a_0+a_1X_1$ and $T=b_0+b_1Y_1$, then

$Cov(S, T)=Cov(a_0+a_1X_1, b_0+b_1Y_1)=a_1b_1Cov(X_1, Y_1)$,

$Var(S)= a_1^2\, Var(X_1)$, and $Var(T)= b_1^2\, Var(Y_1)$.

Therefore,

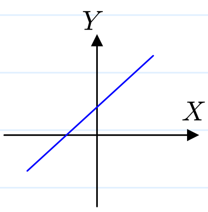
$$\rho_{ST}=\frac{Cov(S,T)}{\sigma_S\sigma_T}=\frac{a_1b_1Cov(X_1,Y_1)}{|a_1||b_1|\sigma_X\sigma_Y}=\frac{a_1b_1}{|a_1b_1|}\rho_{XY}.$$

➢Theorem (some properties of correlation coefficient).

(1) $-1 \le \rho_{XY} \le 1$. $(\Leftrightarrow |Cov(X,Y)|\le\sigma_X\sigma_Y)$

(2) $\rho_{XY}=\pm1$ if and only if $P(Y=aX+b)=1$.

(3) Furthermore, $\rho_{XY}=1$, if $a>0$ and $\rho_{XY}=-1$, if $a<0$.



Proof of (1). $0 \le Var\left(\frac{X}{\sigma_X}+\frac{Y}{\sigma_Y}\right)$

$$
\begin{aligned}
&= Var\left(\frac{X}{\sigma_X}\right)+Var\left(\frac{Y}{\sigma_Y}\right)+2Cov\left(\frac{X}{\sigma_X},\frac{Y}{\sigma_Y}\right) \\
&= \frac{Var(X)}{\sigma_X^2}+\frac{Var(Y)}{\sigma_Y^2}+2\frac{Cov(X,Y)}{\sigma_X\sigma_Y} \\
&= 1+1+2\rho_{XY} \qquad \Rightarrow \rho_{XY}\ge -1.
\end{aligned}
$$

Similarly,

$$0\le Var\left(\frac{X}{\sigma_X}-\frac{Y}{\sigma_Y}\right)=1+1-2\rho_{XY}\Rightarrow\rho_{XY}\le 1.$$

Proof of (2) and (3). We see from the proof of (1),

$$\rho_{XY}=1 \iff Var\left(\frac{X}{\sigma_X}-\frac{Y}{\sigma_Y}\right)=0.$$
$$\iff P\left(\frac{X}{\sigma_X}-\frac{Y}{\sigma_Y}=c\right)=1,$$
where c is a constant.
$$\iff P\left(Y=\frac{\sigma_Y}{\sigma_X}X+c\sigma_Y\right)=1.$$

Similarly, $\rho_{XY}=-1 \iff P\left(Y=-\frac{\sigma_Y}{\sigma_X}X+c\sigma_Y\right)=1.$

- **Q**: How to use expectations to (roughly) characterize random variables $X_1, \ldots, X_n$?
  - $g(X_1, \ldots, X_n)=X_i \Rightarrow E[g(\mathbf{X})]=\mu_{X_i}$: *mean* of X_i.
  - $g(X_1, \ldots, X_n)=(X_i-\mu_{X_i})^2 \Rightarrow E[g(\mathbf{X})]=\sigma^2_{X_i}$: *variance* of X_i.
  - $g(X_1, \ldots, X_n)=(X_i-\mu_{X_i})(X_j-\mu_{X_j})$ for $i\neq j$ $\Rightarrow E[g(\mathbf{X})]=\sigma_{X_iX_j}$: *covariance* of X_i and X_j.
  - $g(X_1, \ldots, X_n)=[(X_i-\mu_{X_i})/\sigma_{X_i}][(X_j-\mu_{X_j})/\sigma_{X_j}]$ for $i\neq j$ $\Rightarrow E[g(\mathbf{X})]=\rho_{X_iX_j}$: *correlation coefficient* of X_i and X_j.
  - Notes. $\mu_{X_i}, \sigma^2_{X_i}, \sigma_{X_iX_j}, \rho_{X_iX_j}$ are constants, not r.v.'s.

❖ **Reading**: textbook, Sec 7.1, 7.2, 7.4

Conditional Expectation

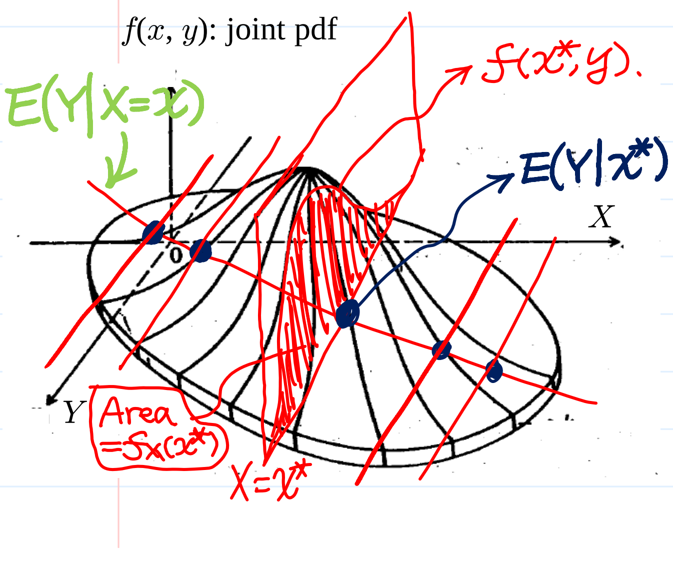
- Recall. $p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ or $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ is a pmf/pdf for $\mathbf{y}$.
- Definition. The conditional expectation of $h(\mathbf{Y})$ given $\mathbf{X}=\mathbf{x}$, where $h: \mathbb{R}^m \to \mathbb{R}^1$, is

$$E(h(\mathbf{Y})|\mathbf{X}=\mathbf{x})=\textstyle\sum_{\mathbf{y}\in\mathcal{Y}} h(\mathbf{y})p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}),$$

in the discrete case, or,

$$E(h(\mathbf{Y})|\mathbf{X}=\mathbf{x})=\textstyle\int_{\mathbb{R}^m} h(\mathbf{y})f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\,d\mathbf{y},$$

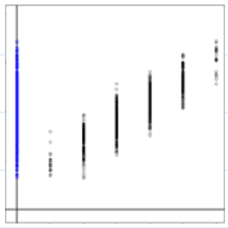
in the continuous case, provided that the sum or integral converges absolutely.



- $f(x, y)$: a joint pdf.
- Fix x^*, is $f(x^*, y)$ a pdf of y? i.e.,
$$\int_{-\infty}^{\infty} f(x^*,y)\,dy=f_X(x^*)\overset{?}{=}1$$
- $f_{Y|X}(y|x^*)=f(x^*, y)/f_X(x^*)$ is a pdf of y since $\dfrac{\int_{-\infty}^{\infty} f(x^*,y)\,dy}{f_X(x^*)}=1.$
- $E(Y|x^*)$: mean of $f_{Y|X}(y|x^*)$.
- Do it for any $x=x^*$, and get a function of $x \Rightarrow E(Y|x)$

➢ Some Notes.

- $E(h(\mathbf{Y})|\mathbf{X}=\mathbf{x})$ is a function of $\mathbf{x}$ and is free of $\mathbf{Y}$.
- If $\mathbf{X}$ and $\mathbf{Y}$ are independent, then $E(h(\mathbf{Y})|\mathbf{X}=\mathbf{x})=E[h(\mathbf{Y})]$.
- $E[h(\mathbf{X})|\mathbf{X}=\mathbf{x}]=h(\mathbf{x})$.
- Let $g(\mathbf{x})=E[h(\mathbf{Y})|\mathbf{X}=\mathbf{x}]$, where $g:\mathbb{R}^n\to \mathbb{R}$, then we write $E(h(\mathbf{Y})|\mathbf{X})$ when $\mathbf{x}$ (a fixed value) replaced by $\mathbf{X}$ (a r.v.) in g.
  - Notice that $g(\mathbf{X})$ is a random variable.

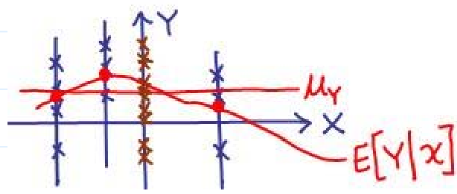


➢ Example. X=age (unit=year), Y=height (unit=cm)

- $Y|X=x$:a random variable (unit=cm) that represents the height distribution of people with age=x.
- $E(Y|X=x)$: a function maps from age (year) to average height (cm) of people with age=x. It is not a random variable.
- $E(Y|X)$: a random variable because it is a function of age, where age is treated as random. Notice that the unit of $E(Y|X)$ is "cm".
- $Var(Y|X=x)$ and $Var(Y|X)$ can be similarly defined.
- $E(Y)$: average height of *all* people;
  $Var(Y)$: variation of height of *all* people

- Theorem (Law of Total Expectation). For two random vectors $\mathbf{X}$ and $\mathbf{Y}$,

$$E_{\mathbf{X}}\{E_{\mathbf{Y}|\mathbf{X}}[h(\mathbf{Y})|\mathbf{X}]\}=E_{\mathbf{Y}}[h(\mathbf{Y})].$$



In particular, let $h(\mathbf{Y})=Y_i$, we have

$$E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]=E_{\mathbf{Y}}(Y_i).$$

Proof. (only prove it for the continuous case)

$$\begin{aligned}
E_{\mathbf{X}}\{E_{\mathbf{Y}|\mathbf{X}}[h(\mathbf{Y})|\mathbf{X}]\} &= \int_{\mathbb{R}^n} E_{\mathbf{Y}|\mathbf{X}}(h(\mathbf{Y})|\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \\
&= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^m} h(\mathbf{y}) f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\, d\mathbf{y}\right] f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \\
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} h(\mathbf{y}) \frac{f_{\mathbf{XY}}(\mathbf{x},\mathbf{y})}{f_{\mathbf{X}}(\mathbf{x})} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}d\mathbf{y} \\
&= \int_{\mathbb{R}^m} h(\mathbf{y}) \left[\int_{\mathbb{R}^n} f_{\mathbf{XY}}(\mathbf{x},\mathbf{y})\, d\mathbf{x}\right] d\mathbf{y} \\
&= \int_{\mathbb{R}^m} h(\mathbf{y}) f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y} = E_{\mathbf{Y}}[h(\mathbf{Y})].
\end{aligned}$$

➢ Example. If a sample of n balls is drawn without replacement from a box containing R red balls, W white balls, and N–R–W blue balls. Let

X = # of red balls in the sample,

Y = # of white balls in the sample,

then, the joint pmf of (X, Y) is

$$p_{X,Y}(x,y) = \frac{\binom{R}{x}\binom{W}{y}\binom{N-R-W}{n-x-y}}{\binom{N}{n}},$$

Find $E(Y)$.

Sol. Because $Y|X=x \sim$ hypergeometric$(n-x, N-R, W)$,

$$g(x) \equiv E(Y|X=x) = (n-x)[W/(N-R)].$$

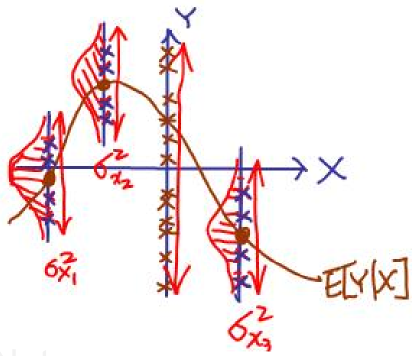
Because $X \sim$ hypergeometric$(n, N, R) \Rightarrow E(X)=n(R/N)$, and then

$$\begin{aligned} E(Y) &= E_X[E_{Y|X}(Y|X)] = E_X[g(X)] \\ &= E_X\left[(n-X)\frac{W}{N-R}\right] = \frac{W}{N-R}[n - E_X(X)] \\ &= \frac{W}{N-R}\left(n - n\frac{R}{N}\right) = n\frac{W}{N}. \end{aligned}$$

Note that $Y \sim$ hypergeometric$(n, N, W) \Rightarrow E(Y)=n(W/N)$.

- Theorem (Variance Decomposition). For two random vectors **X** and **Y**,

$$Var_{\mathbf{Y}}(Y_i) = Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})] + E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})].$$



Proof.

$$\begin{aligned} Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{x}) &= E_{\mathbf{Y}|\mathbf{X}}\{[Y_i - E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{x})]^2|\mathbf{x}\} \\ &= E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{x}) - [E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{x})]^2, \end{aligned}$$

and, $E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$

$$= E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{X})] - E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\}.$$

Also, $Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$

$$= E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\} - \{E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]\}^2.$$

Now, $Var_{\mathbf{Y}}(Y_i) = E_{\mathbf{Y}}(Y_i^2) - [E_{\mathbf{Y}}(Y_i)]^2$

$$\begin{aligned} &= E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{X})] - \{E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]\}^2 \\ &= E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{X})] - E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\} \\ &\quad + E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\} - \{E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]\}^2 \\ &= E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})] + Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]. \end{aligned}$$

➢ Corollary.
- $Var_{\mathbf{Y}}(Y_i) \geq E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$ and the equality holds if and only if $E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X}) = E_{\mathbf{Y}}(Y_i)$ with probability one.
- $Var_{\mathbf{Y}}(Y_i) \geq Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$ and the equality hold if and only if $Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})=0$ ($\Rightarrow Y_i = E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})$) with probability one.

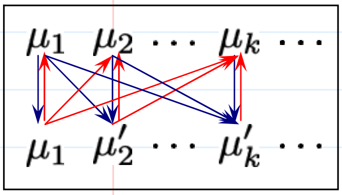
❖ **Reading**: textbook, Sec 7.5

Moment Generating Function

- Definition (Moment and Central Moment). If a random variable X has a cdf F_X, then
$$\mu_k \equiv E(X^k) = \int_{-\infty}^{\infty} x^k \, dF_X(x), \quad k = 1, 2, 3, \ldots,$$
are called the k^{th} *moments* of X provided that the integral converges absolutely, and
$$\mu'_k \equiv E[(X - \mu_X)^k] = \int_{-\infty}^{\infty} (x - \mu_X)^k \, dF_X(x), \quad k = 1, 2, 3, \ldots,$$
are called k^{th} *moment about the mean* μ_X or *central moment* of X provided that the integral converges absolutely.

➢ Some Notes.

- $\mu'_k = E[(X - \mu_X)^k] = E\left[\sum_{i=0}^{k} \binom{k}{i} (-\mu_X)^{n-i} X^i\right]$
$$= \sum_{i=0}^{k} \binom{k}{i} (-\mu_X)^{n-i} E(X^i) = \sum_{i=0}^{k} \binom{k}{i} (-\mu_X)^{n-i} \mu_i.$$

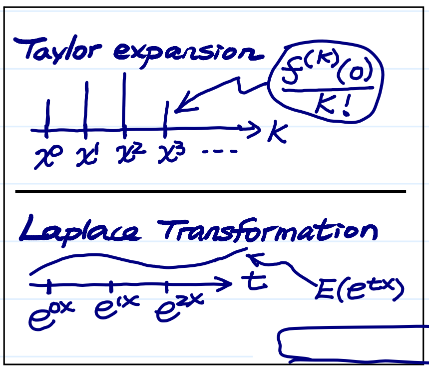


and,
$$\begin{aligned} \mu_k &= E(X^k) = E\{[(X - \mu_X) + \mu_X]^k\} \\ &= \sum_{i=0}^{k} \binom{k}{i} (\mu_X)^{n-i} E[(X - \mu_X)^i] \\ &= \sum_{i=0}^{k} \binom{k}{i} (\mu_X)^{n-i} \mu'_i. \end{aligned}$$

In particular,
$$\begin{aligned} E(X) &= \mu_X = \mu_1, \quad \text{and}, \\ Var(X) &= \sigma_X^2 = \mu_2 - \mu_1^2 = \mu'_2. \end{aligned}$$

- The (central) moments give a lot of useful information about the distribution, e.g., in addition to mean and variance,
  - Skewness (a measure of the asymmetry): μ'_3/σ^3.
  - Kurtosis (a measure of the "peakedness"): μ'_4/σ^4.

➢ Example (Uniform). If $X \sim$ Uniform(0, 1), then
$$\mu_k = \int_0^1 x^k \, dx = \frac{1}{k+1},$$
therefore,
$$\begin{aligned} \mu_X &= \mu_1 = 1/2, \quad \text{and}, \\ \sigma_X^2 &= \mu_2 - \mu_1^2 = 1/3 - (1/2)^2 = 1/12. \end{aligned}$$
And,
$$\begin{aligned} \mu'_k &= \int_0^1 (x - 1/2)^k \, dx = \frac{1}{k+1}\left[(1/2)^{k+1} - (-1/2)^{k+1}\right] \\ &= \begin{cases} 0, & k \text{ is odd}, \\ \frac{1}{(k+1)2^k}, & k \text{ is even}. \end{cases} \end{aligned}$$



- Definition (Moment Generating Function). If X is a random variable with the cdf F_X, then
$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \, dF_X(x),$$
is called the *moment generating function* (mgf) of X provided that the integral converges absolutely in some non-degenerate interval of t.

➢ Some Notes.

- The mgf is a function of the variable t.
- The mgf may only exist for some particular values of t.

➢ Example.

- If X is a discrete r.v. taking on values x_i with probability p_i, i=1, 2, 3, …, then $M_X(t) = \sum_{i=1}^{\infty} e^{tx_i} p_i$.
- If X ~ Poisson(λ), then for $-\infty<t<\infty$,

$$M_X(t) = \sum_{x=0}^{\infty}\left(e^{tx} \times \frac{e^{-\lambda}\lambda^x}{x!}\right)$$
$$= e^{-\lambda}\left(e^{\lambda e^t}\right)\sum_{x=0}^{\infty}\frac{e^{-(\lambda e^t)}(\lambda e^t)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}.$$

- If X ~ Exponential(λ), then for $t<\lambda$,

$$M_X(t) = \int_0^{\infty} e^{tx} \times \lambda e^{-\lambda x}\, dx$$
$$= \lambda\left(\frac{1}{\lambda-t}\right)\int_0^{\infty}(\lambda-t)e^{-(\lambda-t)x}\, dx = \frac{\lambda}{\lambda-t},$$

and $M_X(t)$ does not exist for $t \geq \lambda$.

- A list of some mgfs (exercise)
  - If X ~ Binomial(n, p),
  $$M_X(t) = (1-p+pe^t)^n, \text{ for } t < -\log(1-p).$$

  - If X ~ Negative Binomial(r, p),
  $$M_X(t) = \left[\frac{pe^t}{1-(1-p)e^t}\right]^r, \text{ for } t < -\log(1-p).$$
  - If X ~ Uniform(α, β), $M_X(t) = \frac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$.
  - If X ~ Gamma(α, λ),
  $$M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha}, \text{ for } t < \lambda.$$
  - If X ~ Beta(α, β), $M_X(t) = 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$
  - If X ~ Normal(μ, σ^2), $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$.

• Theorem (Uniqueness Theorem). Suppose that the mgfs $M_X(t)$ and $M_Y(t)$ of random variables X and Y exist for all $|t|<h$ for some $h>0$. If

$$M_X(t) = M_Y(t),$$

for $|t|<h$, then

$$F_X(z) = F_Y(z)$$

for all $z\in\mathbb{R}$, where F_X and F_Y are the cdfs of X and Y, respectively.

Proof. Skipped (by the uniqueness theorem of Laplace transform.)

➢Application of the uniqueness theorem

- When a moment generating function exists, there is a unique distribution corresponding to that mgf.
- This allows us to use mgfs to find distributions of transformed random variables in some cases.
- This technique is most commonly used for linear combinations of independent random variables

➢Example. If $M_X(t) = p_1 e^{a_1 t} + \cdots + p_k e^{a_k t}$, where $p_1+\cdots+p_k=1$, then X is a discrete r.v. and its pmf is

$$p_X(x) = \begin{cases} p_i, & \text{for } x = a_i,\ i = 1, \ldots, k, \\ 0, & \text{otherwise.} \end{cases}$$

- Theorem (Moments and MGF). If $M_X(t)$ exist for $|t|<h$ for some $h>0$, then

$$M_X(0)=1,$$

and,

$$M_X^{(k)}(0) = \mu_k, \quad k = 1, 2, 3, \ldots$$

Proof. First,

$$M_X(0) = \int_{-\infty}^{\infty} e^{0\cdot x}\, dF_X(x) = \int_{-\infty}^{\infty} 1\, dF_X(x) = 1.$$

$$\begin{aligned} M'_X(0) &= \frac{d}{dt} M_X(t)\Big|_{t=0} = \left[\frac{d}{dt}\int_{-\infty}^{\infty} e^{tx}\, dF_X(x)\right]\Bigg|_{t=0} \\ &= \int_{-\infty}^{\infty} \left(\frac{d}{dt} e^{tx}\Big|_{t=0}\right) dF_X(x) = \int_{-\infty}^{\infty} \left(x e^{tx}\big|_{t=0}\right) dF_X(x) \\ &= \int_{-\infty}^{\infty} x \cdot 1\, dF_X(x) = E(X) = \mu_1. \end{aligned}$$

$$\cdots = \cdots$$

$$\begin{aligned} M_X^{(k)}(0) &= \frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = \left[\frac{d^k}{dt^k}\int_{-\infty}^{\infty} e^{tx}\, dF_X(x)\right]\Bigg|_{t=0} \\ &= \int_{-\infty}^{\infty} \left(\frac{d^k}{dt^k} e^{tx}\Big|_{t=0}\right) dF_X(x) = \int_{-\infty}^{\infty} \left(x^k e^{tx}\big|_{t=0}\right) dF_X(x) \\ &= \int_{-\infty}^{\infty} x^k \cdot 1\, dF_X(x) = E(X^k) = \mu_k. \end{aligned}$$

➢Example. If $X \sim$ Exponential(λ), then $M_X(t) = \frac{\lambda}{\lambda - t}$.

Because

$$M_X^{(k)}(t) = \frac{k!\lambda}{(\lambda-t)^{k+1}},$$

we get

$$\mu_k = M_X^{(k)}(0) = \frac{k!}{\lambda^k}.$$

- Theorem (MGF for linear transformation). For constants a and b,

$$M_{a+bX}(t) = e^{at} M_X(bt).$$

Proof. $M_{a+bX}(t) = E[e^{t(a+bX)}] = e^{at} E[e^{(bt)X}] = e^{at} M_X(bt).$

- Theorem (MGF for sum of independent r.v.'s). If $X_1, \ldots, X_n$ are independent each with mgfs $M_1(t), \ldots, M_n(t)$, respectively, then the mgf of $S = X_1 + \cdots + X_n$ is

$$M_S(t) = M_1(t) \times \cdots \times M_n(t).$$

<u>Proof</u>. $M_S(t) = E(e^{tS}) = E[e^{t(X_1+\cdots+X_n)}]$

$$= E(e^{tX_1} \times \cdots \times e^{tX_n}) = E(e^{tX_1}) \times \cdots \times E(e^{tX_n})$$
$$= M_1(t) \times \cdots \times M_n(t).$$

➢Example. If $X_1, \ldots, X_n$ are i.i.d. ~ Geometric(p), then

$S=X_1+\cdots+X_n$ ~ Negative Binomial(n, p).

<u>Proof</u>. $M_S(t) = M_{X_1}(t) \times \cdots \times M_{X_n}(t)$

$$= \frac{pe^t}{1-(1-p)e^t} \times \cdots \times \frac{pe^t}{1-(1-p)e^t} = \left[\frac{pe^t}{1-(1-p)e^t}\right]^n.$$

➢Example. If $X_1, \ldots, X_n$ are independent and

X_i ~ Normal(μ_i, σ_i^2), for i=1, …, n.

Let $S = a_0+a_1X_1+\cdots +a_nX_n$, then

$$S \sim \text{Normal}(a_0 + a_1\mu_1 + \cdots + a_n\mu_n, a_1^2\sigma_1^2 + \cdots + a_n^2\sigma_n^2).$$

<u>Proof</u>. $M_S(t) = e^{a_0 t} \times \prod_{i=1}^n e^{\mu_i(a_i t) + \frac{\sigma_i^2 (a_i t)^2}{2}}$

$$= e^{(a_0+a_1\mu_1+\cdots+a_n\mu_n)t + \frac{(a_1^2\sigma_1^2+\cdots+a_n^2\sigma_n^2)t^2}{2}}.$$

- Definition (Joint Moment Generating Function). For random variables $X_1, \ldots, X_n$, their joint mgf is defined as

$$M_{X_1,\ldots,X_n}(t_1, \ldots, t_n) = E(e^{t_1X_1+\cdots+t_nX_n})$$

provided that the expectation exists.

➢Example. If $X_1, \ldots, X_m$ ~ Multinomial(n, m, p_1, …, p_m),

$$M_{X_1,\ldots,X_m}(t_1, \ldots, t_m)$$
$$= \sum_{x_1+\cdots+x_m=n} e^{t_1x_1+\cdots+t_mx_m} \binom{n}{x_1,\ldots,x_m} p_1^{x_1}\cdots p_m^{x_m}$$
$$= \sum_{x_1+\cdots+x_m=n} \binom{n}{x_1,\ldots,x_m} (p_1e^{t_1})^{x_1}\cdots(p_me^{t_m})^{x_m}$$
$$= \left(p_1e^{t_1} + \cdots + p_me^{t_m}\right)^n.$$

- Some Properties of Joint mgf

➢ $M_{X_1}(t) = M_{X_1,X_2,\ldots,X_n}(t, 0, \ldots, 0)$.

➢ uniqueness theorem

➢ $X_1, \ldots, X_n$ are independent if and only if

$$M_{X_1,\ldots,X_n}(t_1, \ldots, t_n) = M_{X_1}(t_1) \times \cdots \times M_{X_n}(t_n).$$

➢ $\frac{\partial^{k_1+\cdots+k_n}}{\partial t_1^{k_1}\cdots\partial t_n^{k_n}} M_{X_1,\ldots,X_n}(0, \ldots, 0) = E(X_1^{k_1} \times \cdots \times X_n^{k_n})$.

❖ **Reading**: textbook, Sec 7.7