2.

(a)

$$P(0,0)=\frac{8\cdot 7}{13\cdot 12}=\frac{14}{39}$$

$$P(0,1)=\frac{8\cdot 5}{13\cdot 12}=\frac{10}{39}$$

$$P(1,1)=\frac{5\cdot 4}{13\cdot 12}=\frac{5}{39}$$

(b)

$$P(0,0,0)=\frac{8\cdot 7\cdot 6}{13\cdot 12\cdot 11}=\frac{28}{143}$$

$$P(0,0,1)=P(0,1,0)=P(1,0,0)=\frac{8\cdot 7\cdot 5}{13\cdot 12\cdot 11}=\frac{70}{429}$$

$$P(0,1,1)=P(1,0,1)=P(1,1,0)=\frac{8\cdot 5\cdot 4}{13\cdot 12\cdot 11}=\frac{40}{429}$$

$$P(1,1,1)=\frac{5\cdot 4\cdot 3}{13\cdot 12\cdot 11}=\frac{5}{143}$$

6.

(a,b):爲瑕疵在五個傳輸器中出現的排列順序

N_1:直到第一個瑕疵被檢測出來所需要的檢測數。

N_2:直到第二個瑕疵被檢測出來所額外需要的檢測數。

$P(N_1,N_2)$爲相對應的機率

$$(1,2)\Rightarrow P(1,1)=\frac{2\cdot 1}{5\cdot 4}=\frac{1}{10}$$

$$(1,3)\Rightarrow P(1,2)=\frac{2\cdot 3\cdot 1}{5\cdot 4\cdot 3}=\frac{1}{10}$$

$$(1,4),(1,5)\Rightarrow P(1,3)=2\cdot\frac{2\cdot 3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2}=\frac{2}{10}$$

$$(2,3)\Rightarrow P(2,1)=\frac{3\cdot 2\cdot 1}{5\cdot 4\cdot 3}=\frac{1}{10}$$

$$(2,4),(2,5)\Rightarrow P(2,2)=2\cdot\frac{3\cdot 2\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2}=\frac{2}{10}$$

$$(3,4),(3,5)\Rightarrow P(3,1)=2\cdot\frac{3\cdot 2\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2}=\frac{2}{10}$$

$$(4,5)\Rightarrow P(4,0)=\frac{3\cdot 2\cdot 1}{5\cdot 4\cdot 3}=\frac{1}{10}$$

(a,b)	(1,2)	(1,3)	(1,4)	(1,5)	(2,3)	(2,4)	(2,5)	(3,4)	(3,5)	(4,5)
(N_1,N_2)	(1,1)	(1,2)	(1,3)	(1,3)	(2,1)	(2,2)	(2,2)	(3,1)	(3,1)	(4,0)
P(N_1,N_2)	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

Hence the joint pmf of (N_1,N_2):

(N_1,N_2)	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(3,1)	(4,0)
P(N_1,N_2)	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{1}{10}$

8.

(a)

$$\int_0^\infty \int_{-y}^{y} c(y^2 - x^2)e^{-y}\, dxdy = 1$$

$$\Rightarrow c \cdot \int_0^\infty \frac{4}{3} y^3 e^{-y} dy = 1$$

$$\Rightarrow c = \frac{1}{8}$$

(b)

$$f_Y(y) = \int_{-y}^{y} \frac{1}{8}(y^2 - x^2)e^{-y} dx = \frac{1}{6} y^3 e^{-y} \qquad 0 < y < \infty$$

$$f_X(x) = \int_{|x|}^{\infty} \frac{1}{8}(y^2 - x^2)e^{-y}\, dy = \lim_{b\to\infty} \frac{1}{8}\left[-y^2e^{-y} - 2ye^{-y} - 2e^{-y} + x^2e^{-y}\right]_{y=|x|}^{y=b}$$

$$= \frac{1}{4} e^{-|x|} \cdot (1 + |x|) \qquad -\infty < x < \infty$$

9.

(a)

$$f(x,y) = \frac{6}{7}(x^2 + \frac{xy}{2}) \qquad 0 < y < 2 \;\;,\; 0 < x < 1$$

$$\int_0^1 \int_0^2 f(x,y) dydx = \frac{6}{7} \int_0^1 \int_0^2 x^2 + \frac{xy}{2}\, dydx = \frac{6}{7} \int_0^1 2x^2 + x\, dx = \frac{6}{7} \cdot \frac{7}{6} = 1$$

(b)

$$f_X(x) = \int_0^2 \frac{6}{7}(x^2 + \frac{xy}{2})\, dy = \frac{6}{7}(2x^2 + x) \qquad 0 < x < 1$$

(c)

$$P(X > Y) = \int_0^1 \int_0^x \frac{6}{7}(x^2 + \frac{xy}{2})\, dydx = \frac{6}{7} \int_0^1 x^3 + \frac{1}{4}x^3\, dx = \frac{6}{7}(\frac{1}{4} + \frac{1}{16}) = \frac{15}{56}$$

(d)

$$P\left(Y>\frac{1}{2}\mid X<\frac{1}{2}\right)=\frac{P\left(Y>\frac{1}{2}\ ,\ X<\frac{1}{2}\right)}{P\left(X<\frac{1}{2}\right)}$$

$$=\frac{\int_{\frac{1}{2}}^{2}\int_{0}^{\frac{1}{2}}x^2+\frac{xy}{2}\,dxdy}{\int_{0}^{\frac{1}{2}}2x^2+x\,dx}=\frac{\frac{23}{128}}{\frac{5}{24}}=\frac{23\cdot 24}{5\cdot 128}=\frac{69}{80}=0.8625$$

18.

$$X\sim Unif(0,\frac{L}{2})$$

$$Y\sim Unif(\frac{L}{2},L)$$

$$f_{X,Y}(x,y)=\frac{4}{L^2}\qquad 0<x<\frac{L}{2}\ ,\ \frac{L}{2}<y<L$$

$$P(Y-X>\frac{L}{3})=\frac{4}{L^2}\cdot\left[\int_{0}^{\frac{L}{6}}\int_{\frac{L}{2}}^{L}dydx+\int_{\frac{L}{6}}^{\frac{L}{2}}\int_{x+\frac{L}{3}}^{L}dydx\right]$$

$$=\frac{4}{L^2}\cdot\left(\frac{L^2}{12}+\frac{L^2}{9}\right)=\frac{7}{9}$$

22.

(a)

$$f_X(x)=\int_0^1 x+y\,dy=x+\frac{1}{2}\quad 0<x<1$$

$$f_Y(y)=\int_0^1 x+y\,dx=y+\frac{1}{2}\quad 0<y<1$$

$Hence\ f_{X,Y}(x,y)\neq f_X(x)\cdot f_Y(y)$

$\Rightarrow X\ and\ Y\ are\ not\ independent.$

(*b*)

$$f_X(x)=\int_0^1 x+y\,dy = x+\frac{1}{2} \quad 0<x<1$$

(*c*)

$$P(X+Y<1)=\int_0^1\int_0^{1-x} x+y\,dydx$$

$$=\int_0^1 x(1-x)+\frac{1}{2}(1-x)^2\,dx=\frac{1}{2}-\frac{1}{3}+\frac{1}{6}=\frac{1}{3}$$

Theoretical Exercises

11.

$$I=P(X_1<X_2<X_3<X_4<X_5)=\iiint\limits_{x_1<x_2<x_3<x_4<x_5}\iint f(x_1,x_2,x_3,x_4,x_5)dx_1dx_2dx_3dx_4dx_5$$

$$=\iiint\limits_{x_1<x_2<x_3<x_4<x_5}\iint f(x_1)f(x_2)f(x_3)f(x_4)f(x_5)dx_1dx_2dx_3dx_4dx_5$$

$$=\int_0^1\int_0^{u_5}\int_0^{u_4}\int_0^{u_3}\int_0^{u_2}du_1du_2du_3du_4du_5 \qquad \text{(i)} \qquad \begin{Bmatrix} let\ \ u_i=F(x_i)\ \ ,i=1,\cdots,5 \\ du_i=dF(x_i)=f(x_i)dx_i \end{Bmatrix}$$

$$=\frac{1}{5!}=\frac{1}{120} \qquad \text{(ii)}$$

(*a*)

By (*i*), *I does not depend on F*

(*b*)

By (*ii*), $I=\dfrac{1}{120}$

(*c*)

Ω:5個字母任意排列。A:5個字母按字母先後順序排列。

P(Ω)=1，P(A)=P($X_1<X_2<X_3<X_4<X_5$) $=1/120$

24.

$X \sim Exp(\lambda)$

$P([X]=n, X-[X]\le x) = P(n \le X \le n+x)$

$= P(X < n+x) - P(X<n)$

$= e^{-n\lambda}(1-e^{-\lambda x}) = e^{-n\lambda}(1-e^{-\lambda})\dfrac{1-e^{-\lambda x}}{1-e^{-\lambda}}$

1^o $(1-e^{-\lambda x})$的部份, *for* $0<x<1$,

$P(X \le x \mid 0<x<1) = \dfrac{1-e^{-\lambda x}}{1-e^{-\lambda}}$

2^o $e^{-n\lambda}$的部份,可依序代值觀察出 $e^{-\lambda}, e^{-2\lambda}, e^{-3\lambda}$,

隨著n增加,每次多乘$p=e^{-\lambda}$,

$[X]+1 \sim Geo(p)$ *with* $p = e^{-\lambda}$

$P([X]+1=n+1) = (1-p)p^n = (1-e^{-\lambda})e^{-n\lambda}$,

by 1^o and 2^o, X-[X] and [X] are independent