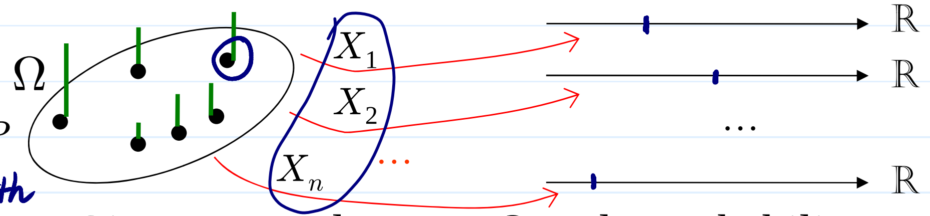
Jointly Distributed Random Variables

- Recall. In Chapters 4 and 5, focus on *univariate* random variable.
  - ➢However, often a single experiment will have more than one random variable which is of interest.

X_1: 身高
X_2: 体重
X_3: 血压.
how X_i's related to / associated with X_j?



➢Definition. Given a sample space Ω and a probability measure P defined on the subsets of Ω, random variables

$$X_1, X_2, \ldots, X_n: \Omega \to \mathbb{R}$$

must be functions defined on "same" sample space.

are said to be *jointly distributed*.

univariate $\mathbb{R}^1$ ↓ joint $\mathbb{R}^n$

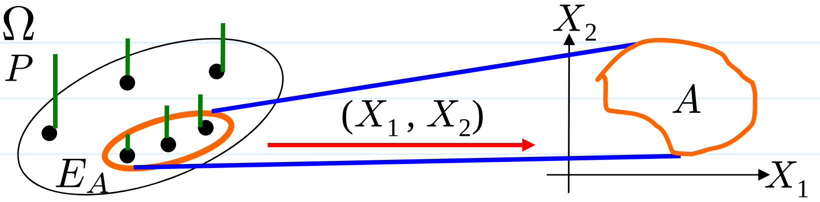
- We can regard n jointly distributed r.v.'s as a *random vector*

$$\mathbf{X}=(X_1, \ldots, X_n): \Omega \to \mathbb{R}^n.$$

- **Q**: For $A \subset \mathbb{R}^n$, how to define the probability of $\{\mathbf{X} \in A\}$ from P?

A occurs ⇔ E_A occurs

$P_{X_1,X_2}(A) = P(E_A)$



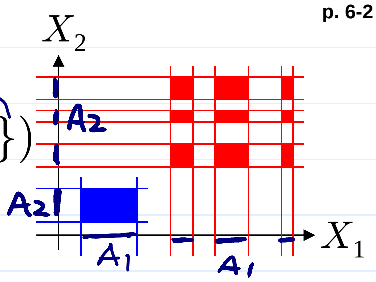
$P_{X_1,X_2}(A)$ =??

➢For $A \subset \mathbb{R}^n$,

$$P_{X_1,\ldots,X_n}(A) = P(\underbrace{\{\omega \in \Omega | (X_1(\omega), \ldots, X_n(\omega)) \in A\}}_{= E_A})$$

➢For $A_i \subset \mathbb{R}$, i=1, …, n,

$$P_{X_1,\ldots,X_n}(X_1 \in A_1, \cdots, X_n \in A_n) = P(\{\omega \in \Omega | X_1(\omega) \in A_1\} \cap \cdots \cap \{\omega \in \Omega | X_n(\omega) \in A_n\})$$

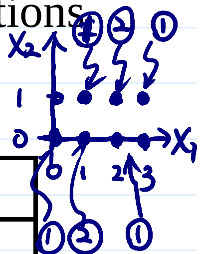


➢Definition. The probability measure of $\mathbf{X}$ ($P_{\mathbf{X}}$, defined on $\mathbb{R}^n$) is called the *joint distribution* of X_1, …, X_n. The probability measure of X_i (P_{X_i}, defined on ℝ) is called the *marginal distribution* of X_i.

聯合分配

邊際分配

- **Q**: Why need joint distribution? Why are marginal distributions not enough?
  - ➢Example (Coin Tossing, LNp.4-2).



$P(\{h,h,h\}) = \frac{1}{8}$

$P(\{ttt\} \cup \{tht\} \cup \{tth\} \cup \{thh\}) = 4/8 = 1/2$

X_2: # of head on 1st toss	X_1: total # of heads			
	0 (1/8)	**1** (3/8)	**2** (3/8)	**3** (1/8)
0 (1/2)	1/8 [1/16]	2/8 [3/16]	1/8 [3/16]	0 [1/16]
1 (1/2)	0 [1/16]	1/8 [3/16]	2/8 [3/16]	1/8 [1/16]

$P(\{h,h,h\}) = 1/8$

- blue numbers: joint distribution of X_1 and X_2
- (black numbers): marginal distributions
- [read numbers]: joint distribution of another (X_1', X_2')
- Some findings:
  - When joint distribution is given, its corresponding marginal distributions are known, e.g.,
    - $P(X_1=i)=P(X_1=i, X_2=0)+P(X_1=i, X_2=1)$, i=0, 1, 2, 3.
  - (X_1, X_2) and (X_1', X_2') have identical marginal distributions but different joint distributions.
    - When the marginal distributions are given, the corresponding joint distribution is still unknown. There could be many possible different joint distributions. (A special case: $X_1, \ldots, X_n$ are independent.)

when $X_1, \cdots, X_n$ independent, joint distribution can be obtained from marginal distribution

  - Joint distribution offers more information, e.g.,
    - When not observing X_1, the distribution of X_2 is: $P(X_2=0)=1/2$, $P(X_2=1)=1/2$ ⇒ marginal distribution
    - When X_1 was observed, say $X_1=1$, the distribution of X_2 is: $P(X_2=0|X_1=1)=(2/8)/(3/8)=2/3$ and $P(X_2=1|X_1=1)=(1/8)/(3/8)=1/3$ ⇒ the calculation requires the knowing of joint distribution

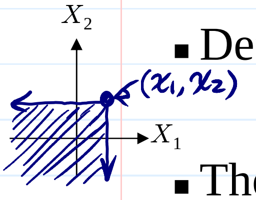
$\frac{P(X_1=1, X_2=0)}{P(X_1=1)}$

We can characterize the joint distribution of $\mathbf{X}$ in terms of its

1. Joint Cumulative Distribution Function (joint cdf)
2. Joint Probability Mass (Density) Function (joint pmf or pdf)
3. Joint Moment Generating Function (joint mgf, Chapter 7)

Compare with univariate case

➢ Joint Cumulative Distribution Function



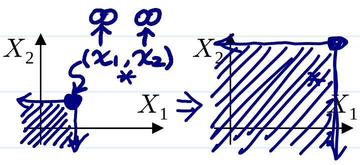
- Definition. The joint cdf of $\mathbf{X}=(X_1, \ldots, X_n)$ is defined as

$$F_{\mathbf{X}}(x_1, \ldots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n).$$

- Theorem. Suppose that $F_{\mathbf{X}}$ is a joint cdf. Then,

(i) $0 \le F_{\mathbf{X}}(x_1, \ldots, x_n) \le 1$, for $-\infty < x_i < \infty$, i=1, …, n.

(ii) $\lim_{x_1, x_2, \cdots, x_n \to \infty} F_{\mathbf{X}}(x_1, \ldots, x_n) = 1$



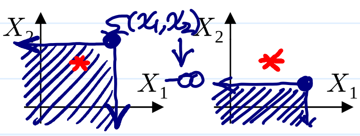
Proof. Let $z_{im} \uparrow \infty$, $1 \le i \le n$.

Let $A_m = (-\infty, z_{1m}) \times \cdots \times (-\infty, z_{nm})$.

Then, $A_m \uparrow \mathbb{R}^n \Rightarrow \lim P(A_m) = P(\mathbb{R}^n) = 1$.

(iii) For any $i \in \{1, \ldots, n\}$,

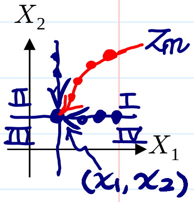
$$\lim_{x_i \to -\infty} F_{\mathbf{X}}(x_1, \ldots, x_n) = 0.$$



Proof. Let $z_{im} \downarrow -\infty$, for some i.

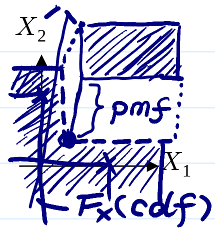
Let $A_m = (-\infty, x_1) \times \cdots \times (-\infty, z_{im}) \times \cdots \times (-\infty, x_n)$.

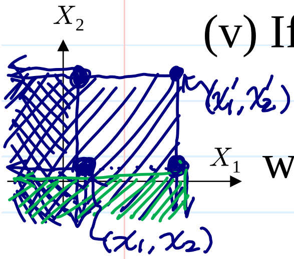
Then, $A_m \downarrow \emptyset \Rightarrow \lim P(A_m) = P(\emptyset) = 0$.



(iv) $F_{\mathbf{X}}$ is continuous from the right with respect to each of the coordinates, or any subset of them jointly, i.e., if $\mathbf{x}=(x_1, \ldots, x_n)$ and $\mathbf{z}_m=(z_{1m}, \ldots, z_{nm})$ such that $\mathbf{z}_m \downarrow \mathbf{x}$, then

$$F_{\mathbf{X}}(\mathbf{z}_m) \downarrow F_{\mathbf{X}}(\mathbf{x}).$$



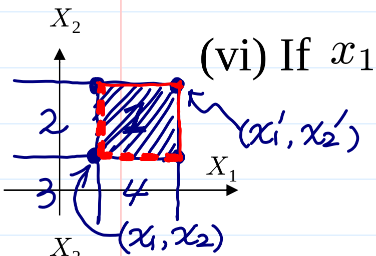


(v) If $x_i \le x_i', i = 1, \ldots, n,$ then

$$F_{\mathbf{X}}(x_1, \ldots, x_n) \le F_{\mathbf{X}}(t_1, \ldots, t_n) \le F_{\mathbf{X}}(x_1', \ldots, x_n').$$

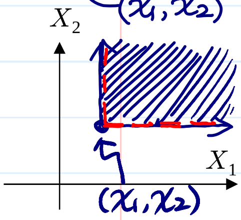
where $t_i \in \{x_i, x_i'\}, i = 1, 2, \ldots, n$. When n=2, we have

$$F_{X_1,X_2}(x_1, x_2) \le \left\{ \begin{array}{c} F_{X_1,X_2}(x_1, x_2') \\ F_{X_1,X_2}(x_1', x_2) \end{array} \right\} \le F_{X_1,X_2}(x_1', x_2').$$



(vi) If $x_1 \le x_1'$ and $x_2 \le x_2'$, then

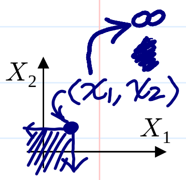
$$\begin{aligned} &P(x_1 < X_1 \le x_1', x_2 < X_2 \le x_2') \\ &= F_{X_1,X_2}(x_1', x_2') - F_{X_1,X_2}(x_1, x_2') \\ &\quad -F_{X_1,X_2}(x_1', x_2) + F_{X_1,X_2}(x_1, x_2). \end{aligned}$$



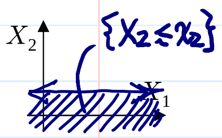
In particular, let $x_1' \uparrow \infty$ and $x_2' \uparrow \infty$, we get

$$\begin{aligned} &P(x_1 < X_1 < \infty, x_2 < X_2 < \infty) \\ &= 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{X_1,X_2}(x_1, x_2). \end{aligned}$$

(vii) The joint cdf of $\underbrace{X_1, \ldots, X_k}$, $k < n$, is can be any k r.v.'s in $\mathbf{X}$.



$$\begin{aligned} F_{X_1,\ldots,X_k}(x_1, \ldots, x_k) &= P(X_1 \le x_1, \ldots, X_k \le x_k) \\ &= P(X_1 \le x_1, \ldots, X_k \le x_k, \\ &\qquad -\infty < X_{k+1} < \infty, \ldots, -\infty < X_n < \infty) \\ &= \lim_{x_{k+1}, x_{k+2}, \cdots, x_n \to \infty} F_{\mathbf{X}}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n). \end{aligned}$$



In particular, the marginal cdf of $\underbrace{X_1}$ is can be any r.v. in $\mathbf{X}$.

$$\begin{aligned} F_{X_1}(x) &= P(X_1 \le x) \\ &= \lim_{x_2, x_3, \cdots, x_n \to \infty} F_{\mathbf{X}}(x, x_2, x_3, \ldots, x_n). \end{aligned}$$

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- Theorem. A function $F_{\mathbf{X}}(x_1, \ldots, x_n)$ can be a joint cdf if $F_{\mathbf{X}}$ satisfies (i)-(v) in the previous theorem.

➢Joint Probability Mass Function

- Definition. Suppose that $X_1, \ldots, X_n$ are discrete random variables. The joint pmf of $\mathbf{X}=(X_1, \ldots, X_n)$ is defined as

$$p_{\mathbf{X}}(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n).$$

- Theorem. Suppose that $p_{\mathbf{X}}$ is a joint pmf. Then,

(a) $p_{\mathbf{X}}(x_1, \ldots, x_n) \ge 0$, for $-\infty < x_i < \infty$, $i = 1, \ldots, n$.

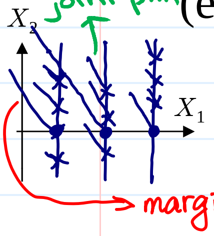
(b) There exists a finite or countably infinite set $\mathcal{X} \subset \mathbb{R}^n$ such that $p_{\mathbf{X}}(x_1, \ldots, x_n) = 0$, for $(x_1, \ldots, x_n) \notin \mathcal{X}$.

(c) $\sum_{\mathbf{x} \in \mathcal{X}} p_{\mathbf{X}}(\mathbf{x}) = 1$, where $\mathbf{x} = (x_1, \ldots, x_n)$.

(d) For $A \subset \mathbb{R}^n$,
$$P(\mathbf{X} \in A) = \sum_{\mathbf{x} \in A \cap \mathcal{X}} p_{\mathbf{X}}(\mathbf{x}).$$

can be any k r.v.'s in X

joint pmf

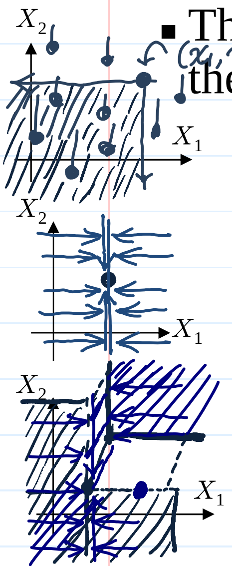


marginal pmf

(e) The joint pmf of $X_1, \ldots, X_k$, $k < n$, is
$$p_{X_1,\ldots,X_k}(x_1, \ldots, x_k) = P(X_1 = x_1, \ldots, X_k = x_k)$$
$$= P(X_1 = x_1, \ldots, X_k = x_k, -\infty < X_{k+1} < \infty, \ldots, -\infty < X_n < \infty)$$
$$= \sum_{\substack{(x_1,\ldots,x_n) \in \mathcal{X} \\ -\infty < x_{k+1} < \infty, \ldots, -\infty < x_n < \infty}} p_{\mathbf{X}}(x_1, \ldots, x_k, \underset{-\infty}{\overset{\infty}{x_{k+1}}}, \ldots, \underset{-\infty}{\overset{\infty}{x_n}}).$$

In particular, the marginal ~~cdf~~ pmf of X_1 is — can be any r.v. in X.
$$p_{X_1}(x) = P(X_1 = x)$$
$$= \sum_{\substack{(x, x_2, \ldots, x_n) \in \mathcal{X} \\ \boxed{-\infty < x_2 < \infty}, \ldots, -\infty < x_n < \infty}} p_{\mathbf{X}}(x, \underset{-\infty}{\overset{\infty}{x_2}}, \underset{-\infty}{\overset{\infty}{x_3}}, \ldots, \underset{-\infty}{\overset{\infty}{x_n}}).$$

- Theorem. A function $p_{\mathbf{X}}(x_1, \ldots, x_n)$ can be a joint pmf if $p_{\mathbf{X}}$ satisfies (a)-(c) in the previous theorem.



- Theorem. If $F_{\mathbf{X}}$ and $p_{\mathbf{X}}$ are the joint cdf and joint pmf of $\mathbf{X}$, then
$$F_{\mathbf{X}}(x_1, \ldots, x_n) = \sum_{\substack{(t_1,\ldots,t_n) \in \mathcal{X} \\ t_1 \le x_1, \ldots, t_n \le x_n}} p_{\mathbf{X}}(t_1, \ldots, t_n), \quad \text{and}$$
$\overset{\shortparallel}{P(X \in A)}$

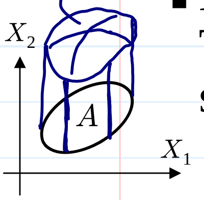
$A = (-\infty, x_1] \times (-\infty, x_2] \times \cdots \times (-\infty, x_n]$
$$p_{\mathbf{X}}(\mathbf{x}) = F_{\mathbf{X}}(\mathbf{x}) - F_{\mathbf{X}}(\mathbf{x}-), \quad \text{where } \mathbf{x} = (x_{10}, \ldots, x_{n0}), \text{ and}$$
$$\begin{aligned}
F^{(1)}_{\mathbf{X}}(x_2, \ldots, x_n) &\equiv F_{\mathbf{X}}(x_{10}, x_2, \ldots, x_n) - F_{\mathbf{X}}(x_{10}-, x_2, \ldots, x_n) \\
F^{(2)}_{\mathbf{X}}(x_3, \ldots, x_n) &\equiv F^{(1)}_{\mathbf{X}}(x_{20}, x_3, \ldots, x_n) - F^{(1)}_{\mathbf{X}}(x_{20}-, x_3, \ldots, x_n) \\
\cdots &\equiv \cdots \\
F_{\mathbf{X}}(\mathbf{x}) - F_{\mathbf{X}}(\mathbf{x}-) &\equiv F^{(n-1)}_{\mathbf{X}}(x_{n0}) - F^{(n-1)}_{\mathbf{X}}(x_{n0}-)
\end{aligned}$$

Note: pmf pdf, $\Sigma \leftrightarrow \int$

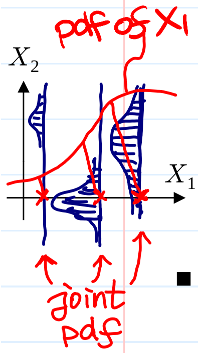
Joint Probability Density Function

- Definition. A function $f_{\mathbf{X}}(x_1, \ldots, x_n)$ can be a joint pdf if
  (1) $f_{\mathbf{X}}(x_1, \ldots, x_n) \ge 0$, for $-\infty < x_i < \infty$, $i=1, \ldots, n$.
  (2) $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = 1.$

pdf surface



- Definition. Suppose that $X_1, \ldots, X_n$ are continuous r.v.'s. The joint pdf of $\mathbf{X} = (X_1, \ldots, X_n)$ is a function $f_{\mathbf{X}}(x_1, \ldots, x_n)$ satisfying (1) and (2) above, and for any event $A \subset \mathbb{R}^n$,
$$P(\mathbf{X} \in A) = \int \cdots \int_A f_{\mathbf{X}}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$



- Theorem. Suppose that $f_{\mathbf{X}}$ is the joint pdf of $\mathbf{X}=(X_1, \ldots, X_n)$. Then, the joint pdf of $X_1, \ldots, X_k$, $k < n$, is — can be any k r.v.'s in $\mathbf{X}$

$$f_{X_1,\ldots,X_k}(x_1,\ldots,x_k) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1,\ldots,x_k,\underbrace{x_{k+1},\ldots,x_n}_{\text{can be any r.v. in } \mathbf{X}})\,dx_{k+1}\cdots dx_n.$$

In particular, the marginal pdf of X_1 is

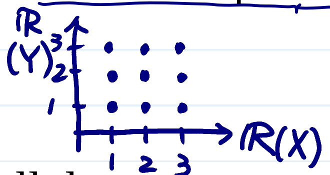
$$f_{X_1}(x) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{\mathbf{X}}(x, x_2, \ldots, x_n)\,dx_2\cdots dx_n.$$

- Theorem. If $F_{\mathbf{X}}$ and $f_{\mathbf{X}}$ are the joint cdf and joint pdf of $\mathbf{X}$, then

$$F_{\mathbf{X}}(x_1,\ldots,x_n) = P(\mathbf{X}\in A),\ A=(-\infty,x_1]\times(-\infty,x_2]\times\cdots\times(-\infty,x_n]$$
$$= \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n} f_{\mathbf{X}}(t_1,\ldots,t_n)\,dt_1\cdots dt_n, \quad \text{and}$$
$$f_{\mathbf{X}}(x_1,\ldots,x_n) = \frac{\partial^n}{\partial x_1\cdots\partial x_n}F_{\mathbf{X}}(x_1,\ldots,x_n).$$

at the $\boxed{\text{continuity points}}$ of $f_{\mathbf{X}}$.

- Examples. $\overset{\mathrm{I}}{b_1}, \overset{\mathrm{II}}{b_2}, \overset{\mathrm{II}}{b_3}, \overset{\mathrm{III}}{b_4}, \overset{\mathrm{III}}{b_5}, \overset{\mathrm{III}}{b_6}$, $\Omega=\{(b_1,b_2),(b_1,b_3),\ldots,(b_6,b_5)\}$ $\#\Omega=6\times5=30$.
  - ➢Experiment. Two balls are drawn without replacement from a box (6 balls) with 1 ball labeled one, 2 balls labeled two, 3 balls labeled three. (jointly distributed)



Let (r.v.) X = label on the 1st ball drawn,
Y = label on the 2nd ball drawn.

- The joint pmf and marginal pmfs of (X, Y) are

Q: What are the joint pmf & marginal pmf of (X', Y') if drawn with replacement? (remind: sampling exp't in LN p. 3-6~7)

$p(x,y)$ (joint pmf)		X = 1	2	3	$p_Y(y)$ (marginal pmf)
Y	1	0	2/30	3/30	1/6
	2	2/30	2/30	6/30	2/6
	3	3/30	6/30	6/30	3/6
$p_X(x)$ (marginal pmf)		1/6	2/6	3/6	

$\frac{3}{6}\times\frac{2}{5}=\frac{6}{30}$

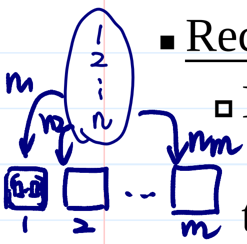
$P(X=3, Y=3) = P(X=3)P(Y=3|X=3)$

Q: When balls drawn without replacement, why do X and Y have same marginal distributions?

- **Q**: $P(|X-Y|=1)=??$

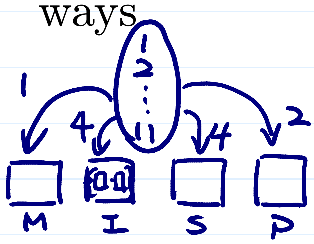
$P(|X-Y|=1) = P(X=1, Y=2) + P(X=2, Y=1) + P(X=2, Y=3) + P(X=3, Y=2) = 8/15.$ (12/8)

- ➢Multinomial Distribution $\xleftrightarrow{\text{c.f.}}$ binomial distribution
  - Recall. Partitions



  - ▫ If $n\geq1$ and $n_1, \ldots, n_m \geq 0$ are integers for which
$$n_1 + \cdots + n_m = n,$$
then a set of n elements may be partitioned into m subsets of sizes $n_1, \ldots, n_m$ in

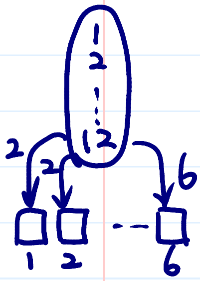
$$\binom{n}{n_1, \cdots, n_m} = \frac{n!}{n_1! \times \cdots \times n_m!} \quad \text{ways}$$



- Example: MISSISSIPPI

$$\binom{11}{4,1,2,4} = \frac{11!}{4!1!2!4!}.$$

- Example (Die Rolling) $\xleftrightarrow{\text{c.f.}}$ Coin Tossing
  - **Q**: If a balanced (6-sided) die is rolled 12 times, P(each face appears twice)=??
  - Sample space of rolling the die once (basic experiment):
  $$\Omega_0 = \{1, 2, 3, 4, 5, 6\}.$$
  - The sample space for the 12 trials is
  $$\Omega = \Omega_0 \times \cdots \times \Omega_0 = \Omega_0^{12}$$
  An outcome $\omega \in \Omega$ is $\omega = (i_1, i_2, \ldots, i_{12})$, where $1 \le i_1, \ldots, i_{12} \le 6$.



  - There are 6^{12} possible outcomes in Ω, i.e., $\#\Omega = 6^{12}$.
  - Among all possible outcomes, there are $\binom{12}{2,2,2,2,2,2} = \frac{12!}{(2!)^6}$ of which each face appears twice.
  - P(each face appears twice) $= \frac{12!}{(2!)^6}\left(\frac{1}{6}\right)^{12}$. $\left(\frac{1}{6}\right)^{12} = \frac{1}{\#\Omega}$, $\left(\frac{1}{6}\right)^{12} = \frac{1}{6} \times \frac{1}{6} \times \cdots \times \frac{1}{6}$

- Generalization.
  - Consider a basic experiment which can result in one of m types of outcomes. Denote its sample space as
  $$\Omega_0 = \{1, 2, \ldots, m\}.$$
  Let $p_i = P$(outcome i appears),

  then, (i) $p_1, \ldots, p_m \ge 0$, and

  (ii) $p_1 + \cdots + p_m = 1$.
  - Repeat the basic experiment n times. Then, the sample space for the n trials is
  $$\Omega = \Omega_0 \times \cdots \times \Omega_0 = \Omega_0^n$$
  Let X_i = # of trials with outcome i, $i=1, \ldots, m$,

  Then, (i) $(X_1, \ldots, X_m): \Omega \to \mathbb{R}^m$, $X_i: \Omega \to \mathbb{R}$, and ← jointly distributed.

  (ii) $X_1 + \cdots + X_m = n$.
  - The joint pmf of $X_1, \ldots, X_m$ is

$\sum_{\mathbf{x}} p_{\mathbf{X}}(\mathbf{x}) \overset{?}{=} 1$

$$\begin{aligned} p_{\mathbf{X}}(x_1, \ldots, x_m) &= P(X_1 = x_1, \ldots, X_m = x_m) \\ &= \binom{n}{x_1, \cdots, x_m} p_1^{x_1} \times \cdots \times p_m^{x_m}. \end{aligned}$$

for $x_1, \ldots, x_m \ge 0$ and $x_1 + \cdots + x_m = n$.

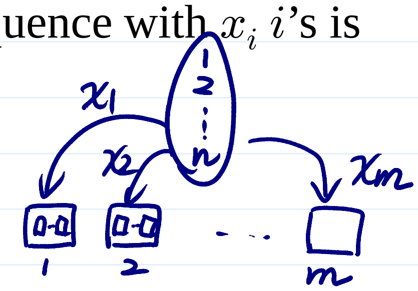
Proof. The probability of any sequence with x_i i's is

$$p_1^{x_1} \times \cdots \times p_m^{x_m},$$

and there are

$$\binom{n}{x_1, \cdots, x_m}$$

such sequences.



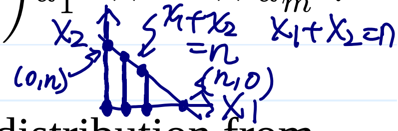
- The distribution of a random vector **X**=(X_1, … , X_m) with the above joint pmf is called the *multinomial* distribution with parameters n, m, and p_1, …, p_m, denoted by Multinomial(n, m, p_1, …, p_m).

$\sum_{\underset{\sim}{x}} P_{\underset{\sim}{X}}(\underset{\sim}{x})$
$=(p_1+\cdots+p_m)^n$
$=1^n=1$

binomial
head tail
multinomial
1 2 ⋯ m

- The multinomial distribution is called after the Multinomial Theorem:

$$(a_1+\cdots+a_m)^n = \sum_{\substack{x_i\in\{0,\ldots,n\};\ i=1,\ldots,m \\ x_1+\cdots+x_m=n}} \binom{n}{x_1, \cdots, x_m} \overset{p_1}{a_1^{x_1}} \times \cdots \times \overset{p_m}{a_m^{x_m}}. \quad (*)$$



- It is a generalization of the binomial distribution from 2 types of outcomes to m types of outcomes.

$m=2$, (X_1, X_2) binomial & $X_2=n-X_1$

- Some Properties.
  - Because $X_i = n - (X_1+\cdots+X_{i-1}+X_{i+1}+\cdots+X_m)$, and

$$p_i = 1 - (p_1+\cdots+p_{i-1}+p_{i+1}+\cdots+p_m),$$

wlog, we can write

$\in \mathbb{R}^m$, but its dimension $=m-1$

$$(X_1,\ldots, X_{m-1}, X_m) \to (X_1,\ldots, X_{m-1}, n-(X_1+\cdots+X_{m-1}))$$

  - Marginal Distribution. Suppose that

$$(X_1, \ldots, X_m) \sim \text{Multinomial}(n, m, p_1, \ldots, p_m).$$

m types
1 2 ⋯ k k+1 ⋯ m
type (k+1)'
$p_{k+1}+p_{k+2}+\cdots+p_m$

For $1\le k<m$, the distribution of

$$(X_1, \ldots, X_k, X_{k+1}+\cdots+X_m)$$

by (e) in LNp. 6-7. & (*) in LNp.6-14

is Multinomial(n, $k+1$, p_1, …, p_k, $p_{k+1}+\cdots+p_m$).

In particular, $X_i \sim$ Binomial(n, p_i) $(X_1, \cdots, X_m)$

$(X_1, X_2+\cdots+X_m)$

  - Mean and Variance.

$$E(X_i)=np_i \text{ and } Var(X_i)=np_i(1-p_i)$$

multinomial $(n, 2, p_1, p_2+\cdots+p_m)$

for $i = 1, \ldots, m$.

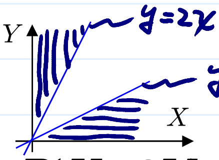
➢Example.

check whether $f(x,y)$ is a joint pdf?

- Suppose that the joint pdf of 2 continuous r.v.'s (X, Y) is

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)}, & x \geq 0,\ y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Q: $P(Y \geq 2X$ or $X \geq 2Y)$=??



⇒ mutually exclusive.

- The event $\{Y \geq 2X\} \cup \{X \geq 2Y\}$ is
- So, $P(Y \geq 2X$ or $X \geq 2Y)=P(Y \geq 2X)+P(X \geq 2Y)$=2/3 because

$\iint_A f(x,y)\,dxdy = P((X,Y) \in A)$

$$P(Y \geq 2X) = \int_0^\infty \left[\int_{2x}^\infty \lambda^2 e^{-\lambda(x+y)}\, dy \right] dx$$

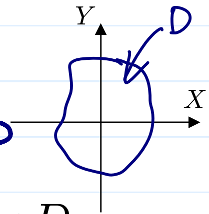
$$= \int_0^\infty -\lambda e^{-\lambda(x+y)} \Big|_{y=2x}^\infty dx = \int_0^\infty \lambda e^{-3\lambda x}\, dx$$

$$= (-1/3) e^{-3\lambda x} \Big|_{x=0}^\infty = 1/3.$$

and similarly, we can get $P(X \geq 2Y)$=1/3 (exercise).

$\iint_{\mathbb{R}^2} f(x,y)\,dxdy = c \iint_{\mathbb{R}^2} 1_D\, dxdy = c \cdot \iint_D 1\, dxdy = c \cdot \text{Area}(D) = 1$

➢Example. Consider two continuous r.v.'s X and Y.

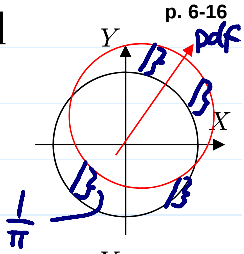


- Uniform Distribution over a region D. If $D \subset \mathbb{R}^2$ and $0 < \alpha = \text{Area}(D) < \infty$, then

indicator function

$$f(x,y) = c \cdot \mathbf{1}_D(x,y) \qquad \mathbf{1}_D(x,y) = \begin{cases} 1, & (x,y) \in D \\ 0, & \text{o.w.} \end{cases}$$

is a joint pdf when $c=1/\alpha$, called the uniform pdf over D.



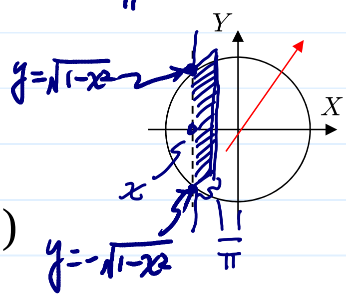
- Let $D = \{(x, y): x^2+y^2 \leq 1\}$, then $\alpha=\text{Area}(D)=\pi$ and

$$f(x,y) = \tfrac{1}{\pi} \mathbf{1}_D(x,y)$$

is a joint pdf.

- Marginal distribution. The marginal pdf of X is

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \frac{2}{\pi}\sqrt{1-x^2}$$



for $-1 \leq x \leq 1$, and $f_X(x)$=0, otherwise.
(exercise: Find the marginal distribution of Y.)

❖ **Reading**: textbook, Sec 6.1

Independent Random Variables

cf (independent events (LNp.3-13~22)

- Recall.

➢When the joint distribution is given, the marginal distributions are known.

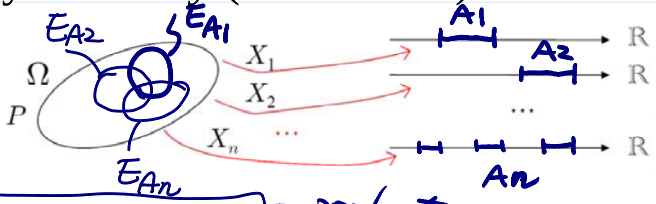
➢The converse statement does not hold in general.

➢However, when random variables are independent,

marginal distributions + independence ⇒ joint distribution.

- Definition. The random variables $X_1, \ldots, X_n$ are called (*mutually*) *independent* if and only if for *any* (measurable) sets $A_i \subset \mathbb{R}$, $i=1, \ldots, n$, the events $\{X_1 \in A_1\}, \ldots, \{X_n \in A_n\}$ are independent. That is,

for calculation purpose.



$$P(\boxed{X_{i_1} \in A_{i_1}, X_{i_2} \in A_{i_2}, \cdots, X_{i_k} \in A_{i_k}})$$
product set.
$$= \quad P(X_{i_1} \in A_{i_1}) \times P(X_{i_2} \in A_{i_2}) \times \cdots \times P(X_{i_k} \in A_{i_k}),$$

for *any* $1 \le i_1 < i_2 < \cdots < i_k \le n$; $\boxed{k=2, \ldots, n}$.

- If $X_1, \ldots, X_n$ are independent, for $1 \le k < n$,

for interpretation purpose

$$P(X_{k+1} \in A_{k+1}, \ldots, X_n \in A_n | X_1 \in A_1, \ldots, X_k \in A_k)$$
$$= \quad P(X_{k+1} \in A_{k+1}, \ldots, X_n \in A_n)$$

(exercise)

provided that $P(X_1 \in A_1, \ldots, X_k \in A_k) > 0$. In other words, $X_1, \ldots, X_k$ do not carry information about $X_{k+1}, \ldots, X_n$.

- Theorem (Factorization Theorem). The random variables $\mathbf{X}=(X_1, \ldots, X_n)$ are independent if and only if one of the following conditions holds.

(1) $F_{\mathbf{X}}(x_1, \ldots, x_n) = F_{X_1}(x_1) \times \cdots \times F_{X_n}(x_n)$, where $F_{\mathbf{X}}$ is the joint cdf of $\mathbf{X}$ and F_{X_i} is the marginal cdf of X_i for $i=1,\ldots,n$.

(2) Suppose that $X_1, \ldots, X_n$ are discrete random variables. $p_{\mathbf{X}}(x_1, \ldots, x_n) = p_{X_1}(x_1) \times \cdots \times p_{X_n}(x_n)$, where $p_{\mathbf{X}}$ is the joint pmf of $\mathbf{X}$ and p_{X_i} is the marginal pmf of X_i for $i=1,\ldots,n$.

(3) Suppose that $X_1, \ldots, X_n$ are continuous random variables. $f_{\mathbf{X}}(x_1, \ldots, x_n) = f_{X_1}(x_1) \times \cdots \times f_{X_n}(x_n)$, where $f_{\mathbf{X}}$ is the joint pdf of $\mathbf{X}$ and f_{X_i} is the marginal pdf of X_i for $i=1,\ldots,n$.

Proof.

independent $\Rightarrow$ (1). $F_{\mathbf{X}}(x_1, \ldots, x_n) = P(X_1 \le x_1, \ldots, X_n \le x_n)$
$$= P(X_1 \in (-\infty, x_1], \ldots, X_n \in (-\infty, x_n])$$
$$= P(X_1 \in (-\infty, x_1]) \times \cdots \times P(X_n \in (-\infty, x_n])$$
$$= F_{X_1}(x_1) \times \cdots \times F_{X_n}(x_n)$$

by the definition of indep.

independent $\Leftarrow$ (1). Out of the scope of this couse so skip.

use the property of σ-field

---

independent $\Rightarrow$ (2). $p_{\mathbf{X}}(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n)$
$$= P(X_1 \in \{x_1\}, \ldots, X_n \in \{x_n\})$$
$$= P(X_1 \in \{x_1\}) \times \cdots \times P(X_n \in \{x_n\})$$
$$= p_{X_1}(x_1) \times \cdots \times p_{X_n}(x_n)$$

by the definition of indep

(2) ⇒ (1). Thm in LNp.6-8.

$$F_{\mathbf{X}}(x_1,\ldots,x_n)=\sum_{\substack{(t_1,\ldots,t_n)\in\mathcal{X}\\ t_1\le x_1,\ldots,t_n\le x_n}} p_{\mathbf{X}}(t_1,\ldots,t_n)$$

$$=\sum_{\substack{(t_1,\ldots,t_n)\in\mathcal{X}\\ t_1\le x_1}}\cdots\boxed{\sum_{\substack{(t_1,\ldots,t_n)\in\mathcal{X}\\ t_n\le x_n}}} p_{X_1}(t_1)\times\cdots\times p_{X_n}(t_n)$$

$$=\sum_{\substack{(t_1,\ldots,t_n)\in\mathcal{X}\\ t_1\le x_1}} p_{X_1}(t_1)\times\cdots\times\sum_{\substack{(t_1,\ldots,t_n)\in\mathcal{X}\\ t_n\le x_n}} p_{X_n}(t_n)=F_{X_1}(x_1)\times\cdots\times F_{X_n}(x_n)$$

(3) ⇒ (1).

$$F_{\mathbf{X}}(x_1,\ldots,x_n)=\int_{-\infty}^{x_n}\cdots\int_{-\infty}^{x_1} f_{\mathbf{X}}(t_1,\ldots,t_n)\,dt_1\cdots dt_n$$

by(3)

$$=\int_{-\infty}^{x_n}\cdots\int_{-\infty}^{x_1} f_{X_1}(t_1)\times\cdots\times f_{X_n}(t_n)\,dt_1\cdots dt_n$$

$$=\int_{-\infty}^{x_1} f_{X_1}(t_1)\,dt_1\times\cdots\times\int_{-\infty}^{x_n} f_{X_n}(t_n)\,dt_n=F_{X_1}(x_1)\times\cdots\times F_{X_n}(x_n)$$

(3) ⇐ (1).

$$f_{\mathbf{X}}(x_1,\ldots,x_n)=\frac{\partial^n}{\partial x_1\cdots\partial x_n}F_{\mathbf{X}}(x_1,\ldots,x_n).$$

by(1)

$$=\frac{\partial^n}{\partial x_1\cdots\partial x_n}F_{X_1}(x_1)\times\cdots\times F_{X_n}(x_n)$$

$$=\frac{\partial}{\partial x_1}F_{X_1}(x_1)\times\cdots\times\frac{\partial}{\partial x_n}F_{X_n}(x_n)=f_{X_1}(x_1)\times\cdots\times f_{X_n}(x_n)$$

➢Remark. It follows from the Multiplication Law (LNp.3-7)

$$F_{\mathbf{X}}(x_1,\ldots,x_n)=P(X_1\le x_1,\ldots,X_n\le x_n)$$

$$=P(X_1\le x_1) \qquad (=F_{X_1}(x_1))$$

$$\times P(X_2\le x_2|X_1\le x_1) \qquad \left(\overset{?}{=}P(X_2\le x_2)=F_{X_2}(x_2)\right)$$

X_2 indep of X_1

$$\times P(X_3\le x_3|X_1\le x_1,X_2\le x_2) \qquad \left(\overset{?}{=}P(X_3\le x_3)=F_{X_3}(x_3)\right)$$

X_3 indep of X_1 X_2

$$\times\cdots$$

12/13

$$\times P(X_n\le x_n|X_1\le x_1,\ldots,X_{n-1}\le x_{n-1}) \qquad \left(\overset{?}{=}P(X_n\le x_n)=F_{X_n}(x_n)\right)$$

X_n indep of $X_1,\ldots,X_{n-1}$

that the independence can be established *sequentially*.

➢Example. If A_1, …, A_n are independent events, then $\mathbf{1}_{A_1},\ldots,\mathbf{1}_{A_n}$, are independent random variables. For example,

indicator function

joint pmf

$$P(\mathbf{1}_{A_1}=1,\mathbf{1}_{A_2}=0,\mathbf{1}_{A_3}=1)$$

by Thm in LNp.3-18.

$$=P(A_1\cap A_2^c\cap A_3)=P(A_1)P(A_2^c)P(A_3)$$

$$=P(\mathbf{1}_{A_1}=1)P(\mathbf{1}_{A_2}=0)P(\mathbf{1}_{A_3}=1)$$

← product of marginal pmf.

➢Example. If $\mathbf{X}$=(X_1, …, X_n) are independent and

$$Y_i=g_i(X_i),\ i=1,\ldots,n,$$

then Y_1, …, Y_n are independent.

indep.

生日 身高

g_1 ↓ g_2 ↓

星座 座位前後

indep.

generalization

$$1=i_0<i_1<\cdots<i_k=n$$

$$Y_1 = g_1(X_1,\ldots,X_{i_1}),$$

$$Y_2 = g_2(X_{i_1+1},\ldots,X_{i_2}),$$

$$\cdots$$

$$Y_k = g_k(X_{i_{k-1}+1},\ldots,X_{i_k}).$$

Proof. Let $A_i(y) = \{x : g_i(x) \le y\}$, i=1, …, n, then

joint cdf

$\{\omega\in\Omega: X_1(\omega)\in A_1(y_1)\}$

$\{\omega\in\Omega: Y_1(\omega)\le y\}$, $Y_1(\omega) = g_1(X_1(\omega))$

$$\begin{aligned}
F_{\mathbf{Y}}(y_1,\ldots,y_n) &= P(Y_1\le y_1,\ldots,Y_n\le y_n)\\
&= P(X_1\in A_1(y_1),\ldots,X_n\in A_n(y_n))\\
&= P(X_1\in A_1(y_1))\times\cdots\times P(X_n\in A_n(y_n))\\
&= P(Y_1\le y_1)\times\cdots\times P(Y_n\le y_n)\\
&= F_{Y_1}(y_1)\times\cdots\times F_{Y_n}(y_n).
\end{aligned}$$

— product of marginal cdf.

- Theorem. $\mathbf{X}$=(X_1, …, X_n) are independent if and only if there exist univariate functions $g_i(x)$, i=1, …, n, such that
  - (a) when X_1, …, X_n are discrete r.v.'s with joint pmf $p_{\mathbf{X}}$,
    $p_{\mathbf{X}}(x_1, \ldots, x_n) \propto g_1(x_1)\times\cdots\times g_n(x_n)$, $-\infty<x_i<\infty$, i=1,…,n.
  - (b) when X_1, …, X_n are continuous r.v.'s with joint pdf $f_{\mathbf{X}}$,
    $f_{\mathbf{X}}(x_1, \ldots, x_n) \propto g_1(x_1)\times\cdots\times g_n(x_n)$, $-\infty<x_i<\infty$, i=1,…,n.

Sketch of proof for (b).

$$\begin{aligned}
f_{X_1}(x_1) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1,x_2,\ldots,x_n)\,dx_2\cdots dx_n\\
&\propto \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g_1(x_1)g_2(x_2)\cdots g_n(x_n)\,dx_2\cdots dx_n \propto g_1(x_1).
\end{aligned}$$

Similarly, $f_{X_2}(x_2)\propto g_2(x_2),\ldots,f_{X_n}(x_n)\propto g_n(x_n)$

$$\begin{aligned}
&\Rightarrow\ f_{X_1}(x_1)\cdots f_{X_n}(x_n)\propto g_1(x_1)\cdots g_n(x_n)\\
&\Rightarrow\ f_{\mathbf{X}}(x_1,\ldots,x_n)\propto f_{X_1}(x_1)\cdots f_{X_n}(x_n)\\
&\Rightarrow\ f_{\mathbf{X}}(x_1,\ldots,x_n) = c\cdot f_{X_1}(x_1)\cdots f_{X_n}(x_n)
\end{aligned}$$

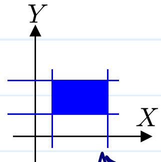
for some constant c.

Because $\displaystyle\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1,x_2,\ldots,x_n)\,dx_1\cdots dx_n = 1$, and

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{X_1}(x_1)\cdots f_{X_n}(x_n)\,dx_1\cdots dx_n = 1,\quad \Rightarrow \boldsymbol{c=1}.$$

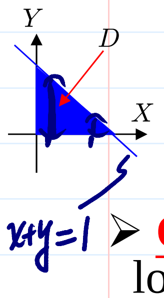
➢Example.

- If the joint pdf of (X, Y) is
  $$f(x,y)\propto e^{-2x}e^{-3y},\quad 0<x<\infty,\ 0<y<\infty,$$
  and $f(x, y)$=0, otherwise, i.e.,
  $$f(x,y)\propto e^{-2x}e^{-3y}\mathbf{1}_{(0,\infty)}(x)\mathbf{1}_{(0,\infty)}(y),$$
  $1_A(x)\cdot 1_B(y)$
  then X and Y are independent. Note that the region in which the joint pdf is nonzero can be expressed in the form $\{(x, y)$: $x\in A$, $y\in B\}$.



product set.

- Suppose that the joint pdf of (X, Y) is



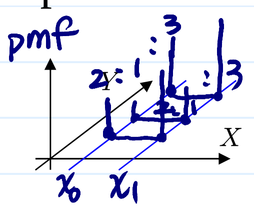
$$f(x,y) \propto xy,\ \ 0<x<1,\ 0<y<1,\ 0<x+y<1,$$

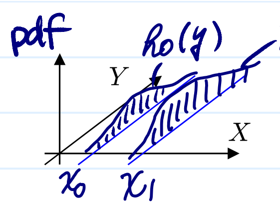
and $f(x, y)$=0, otherwise, i.e., $f(x,y) \propto xy \cdot \mathbf{1}_D(x,y)$,

not a function of the form $g_1(x)\cdot g_2(y)$

X and Y are not independent.

➢ **Q**: For independent X and Y, how should their joint pdf/pmf look like?





$\frac{h_1(y)}{h_0(y)}$ = a constant

❖ **Reading**: textbook, Sec 6.2

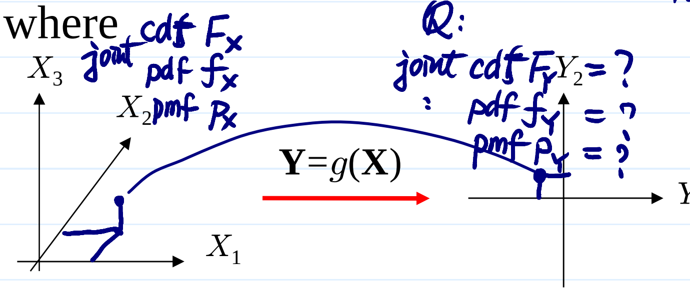
Transformation

Recall: transformation for univariate r.v. (LNp. 4-10, 5-6 ~10)

- **Q**: Given the joint distribution of $\mathbf{X}=(X_1, \ldots, X_n)$, how to find the distribution $\mathbf{Y}=(Y_1, \ldots, Y_k)$, where

$$Y_1=g_1(X_1, \ldots, X_n),$$
$$\ldots,$$
$$Y_k=g_k(X_1, \ldots, X_n),$$

denoted by $\mathbf{Y}=g(\mathbf{X})$, $g:\mathbb{R}^n \to \mathbb{R}^k$.



➢The following methods are useful:

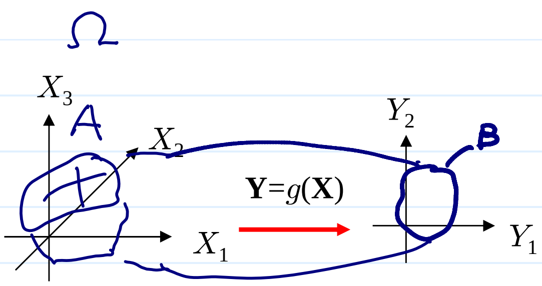
1. Method of Events
2. Method of Cumulative Distribution Function
3. Method of Probability Density Function
4. Method of Moment Generating Function (chapter 7)

➢Method of Events

- Theorem. The distribution of $\mathbf{Y}$ is determined by the distribution of $\mathbf{X}$ as follows: for any event $B \subset \mathbb{R}^k$,

$$P_{\mathbf{Y}}(\mathbf{Y} \in B) = P_{\mathbf{X}}(\mathbf{X} \in A),$$

where $A = g^{-1}(B) \subset \mathbb{R}^n$.



- Example. Let $\mathbf{X}$ be a discrete random vector taking values $\mathbf{x}_i=(x_{1i}, x_{2i}, \ldots, x_{ni})$, i=1, 2, …, with joint pmf $p_{\mathbf{X}}$. Then, $\mathbf{Y}=g(\mathbf{X})$ is also a discrete random vector. Suppose that $\mathbf{Y}$ takes values on $\mathbf{y}_j$, j= 1, 2, …. To determine the joint pmf of $\mathbf{Y}$, by taking $B=\{\mathbf{y}_j\}$, we have

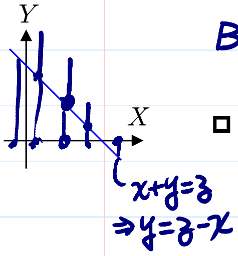
$$A = \{\mathbf{x}_i : g(\mathbf{x}_i) = \mathbf{y}_j\}$$

and hence, the joint pmf of $\mathbf{Y}$ is

$$p_{\mathbf{Y}}(\mathbf{y}_j) = P_{\mathbf{Y}}(\{\mathbf{y}_j\}) = P_{\mathbf{X}}(A) = \textstyle\sum_{\mathbf{x}_i \in A} p_{\mathbf{X}}(\mathbf{x}_i).$$

X, Y discrete r.v.'s

- Example. Let X and Y be random variables with the joint pmf $p(x, y)$. Find the distribution of $Z=X+Y$.



  - $\underbrace{\{Z=z\}}_{B} = \{(X, Y) \in \underbrace{\{(x, y): x+y=z\}}_{A}\}$

$$p_Z(z) = P_Z(\{z\}) = P(X+Y=z) = \sum_{x \in \mathcal{X}_X} p(x, z-x).$$

$\mathcal{X}_X$ = range of X

  - When X and Y are independent,

$$p(x, y) = p_X(x)p_Y(y),$$

So,

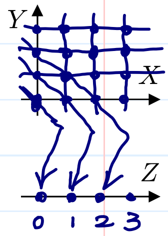
$$p_Z(z) = \sum_{x \in \mathcal{X}_X} p_X(x)p_Y(z-x).$$

which is referred to as the *convolution* of p_X and p_Y.

  - (Exercise) $Z=X-Y$

- Theorem. If X and Y are independent, and $X \sim$ Poisson(λ_1), $Y \sim$ Poisson(λ_2), then

$$Z = X + Y \sim \text{Poisson}(\lambda_1+\lambda_2).$$



Proof. For z=0, 1, 2, …, the pmf $p_Z(z)$ of Z is

$$p_Z(z) = \sum_{x=0}^{z} p_X(x)p_Y(z-x) = \sum_{x=0}^{z} \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^{z-x}}{(z-x)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{z!}\left(\sum_{x=0}^{z}\frac{z!}{x!(z-x)!}\lambda_1^x\lambda_2^{z-x}\right) = \frac{e^{-(\lambda_1+\lambda_2)}}{z!}(\lambda_1+\lambda_2)^z.$$

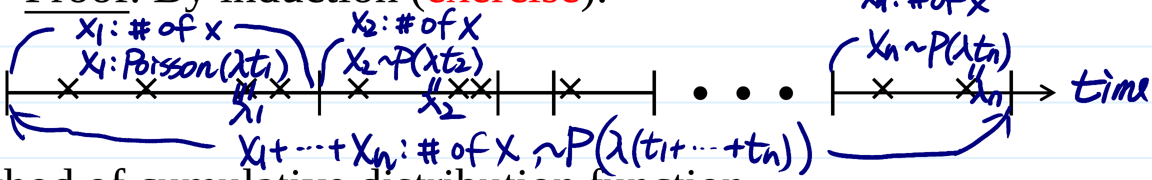
by Binomial Thm (LNp. 4-20)

indep? (∵ generalization in LNp. 6-20)
$X_1+\cdots+X_{n-1} \sim$ Poisson($\lambda_1+\cdots+\lambda_{n-1}$)
$X_n \sim$ Poisson(λ_n)

  - Corollary. If $X_1, \ldots, X_n$ are independent, and $X_i \sim$Poisson(λ_i), i=1, …, n, then

$$X_1+\cdots+X_n \sim \text{Poisson}(\lambda_1+\cdots+\lambda_n).$$

Proof. By induction (exercise).



- Method of cumulative distribution function

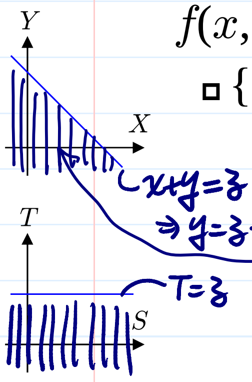
a special case of method of events.

1. In the $(X_1, \ldots, X_n)$ space, find the region (A) that corresponds to $\{Y_1 \le y_1, \ldots, Y_k \le y_k\}$ (B)
2. Find $F_{\mathbf{Y}}(y_1, \ldots, y_k)=P(Y_1 \le y_1, \ldots, Y_k \le y_k)$ by summing the joint pmf or integrating the joint pdf of $X_1, \ldots, X_n$ over the region identified in 1.

replace B by $\{Y_1 \le y_1, \ldots, Y_k \le y_k\}$

3. (for continuous case) Find the joint pdf of $\mathbf{Y}$ by differentiating $F_{\mathbf{Y}}(y_1, \ldots, y_k)$, i.e.,

$$f_{\mathbf{Y}}(y_1, \ldots, y_k) = \frac{d^k}{dy_1 \cdots dy_k} F_{\mathbf{Y}}(y_1, \ldots, y_k).$$



- Example. X and Y are random variables with joint pdf $f(x, y)$. Find the distribution of $Z=X+Y$.   X,Y continuous r.v.'s.
  - $\{Z \le z\} = \{(X, Y) \in \{(x, y): x+y \le z\}\}$. So,

$$F_Z(z) = P(Z \le z) = P(X+Y \le z)$$
$$= \boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{z-x}} f(x,y)\, dydx$$
$$= \int_{-\infty}^{z}\int_{-\infty}^{\infty} f(s, t-s)\, dsdt \qquad \left(\text{set } \begin{cases} x = s \\ y = t-s \end{cases}\right)$$

$\begin{cases} s = x \\ t = y+x \end{cases}$

$$\begin{vmatrix} dx/ds & dx/dt \\ dy/ds & dy/dt \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$dxdy = dsdt.$

and $f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)dx$

  - When X and Y are independent,

$$f(x, y) = f_X(x) f_Y(y).$$

So, $F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_X(x) f_Y(y)\, dydx$

integration by part

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{z-x} f_Y(y)\, dy\right] f_X(x)\, dx$$

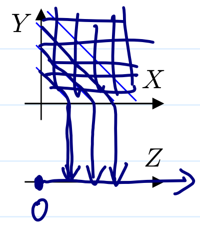
$$\int_{-\infty}^{\infty} F_X(z-y) f_Y(y) dy = \int_{-\infty}^{\infty} F_Y(z-x) \boxed{f_X(x)}\, dx \qquad \frac{d}{dx}F_X(x)$$

which is referred to as the *convolution* of F_X and F_Y, and

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx \quad \text{c.f. convolution of pmf (LN p 6-25)}$$

which is referred to as the *convolution* of f_X and f_Y.

  - (exercise) $Z=X-Y$.



- Theorem. If X and Y are independent, and $X \sim \text{Gamma}(\alpha_1, \lambda)$, $Y \sim \text{Gamma}(\alpha_2, \lambda)$, then

$$Z = X + Y \sim \text{Gamma}(\alpha_1+\alpha_2, \lambda).$$

Proof. For $z \ge 0$,

$$f_Z(z) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^z x^{\alpha_1-1}(z-x)^{\alpha_2-1} e^{-\lambda z}\, dx \qquad \lambda z = \lambda(x+(z-x))$$

let $y = \frac{x}{z}$
$\Rightarrow x = zy$
$\frac{dx}{dy} = z$
$\Rightarrow dx = z\,dy$

$$= \frac{\lambda^{\alpha_1+\alpha_2} e^{-\lambda z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1 z^{(\alpha_1-1)+(\alpha_2-1)+1}\underbrace{y^{\alpha_1-1}(1-y)^{\alpha_2-1}\, dy}_{\text{Beta function (LN p.5-21)}}$$

$$= \frac{\lambda^{\alpha_1+\alpha_2} z^{(\alpha_1+\alpha_2)-1} e^{-\lambda z}}{\cancel{\Gamma(\alpha_1)\Gamma(\alpha_2)}} \times \frac{\cancel{\Gamma(\alpha_1)\Gamma(\alpha_2)}}{\Gamma(\alpha_1+\alpha_2)}.$$

and $f_Z(z) = 0$, for $z < 0$.

  - Corollary. If $X_1, \ldots, X_n$ are independent, and $X_i \sim \text{Gamma}(\alpha_i, \lambda)$, $i=1, \ldots, n$, then

$$X_1 + \cdots + X_n \sim \text{Gamma}(\alpha_1 + \cdots + \alpha_n, \lambda).$$

Proof. By induction (exercise).

  - (exercise) Corollary. If $X_1, \ldots, X_n$ are independent, and $X_i \sim$ Exponential(λ), $i=1, \ldots, n$, then

Gamma(1, λ)

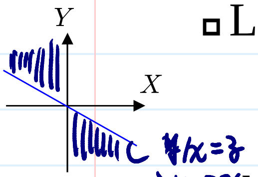
$$X_1 + \cdots + X_n \sim \text{Gamma}(n, \lambda).$$

check textbook p.256

■ (exercise) Theorem. If $X_1, \ldots, X_n$ are independent, and $X_i \sim \text{Normal}(\mu_i, \sigma_i^2)$, $i=1, \ldots, n$, then

$$X_1 + \cdots + X_n \sim \text{Normal}(\mu_1+\cdots+\mu_n, \sigma_1^2+\cdots+\sigma_n^2).$$

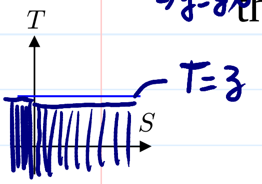
■ Example. X and Y are random variables with joint pdf $f(x, y)$. Find the distribution of $Z=Y/X$.



□ Let

$$\begin{aligned} Q_z \overset{=\{Z\le z\}}{=} &\{(x,y) : y/x \le z\} \\ = &\{(x,y) : x<0, y \ge zx\} \\ &\cup \{(x,y) : x>0, y \le zx\} \end{aligned}$$

then, $F_Z(z) = \int\int_{Q_z} f(x,y)\, dxdy$



$$\begin{aligned} &= \int_{-\infty}^{0}\int_{xz}^{\infty} + \int_0^{\infty}\int_{-\infty}^{xz} f(x,y)\, dydx \\ &= \int_{-\infty}^{0}\int_{z}^{-\infty} + \int_0^{\infty}\int_{-\infty}^{z} s f(s, st)\, dtds \quad st \\ &= \boxed{\int_{-\infty}^{0}\int_{-\infty}^{z}} (-s) f(s,st)\, dtds \quad \int_{-\infty}^{z}\int_{-\infty}^{0} |s| f(s,st)\, dsdt \\ &\quad + \boxed{\int_0^{\infty}\int_{-\infty}^{z}} s f(s,st)\, dtds \quad \int_{-\infty}^{z}\int_{0}^{\infty} |s| f(s,st)\, dsdt \\ &= \int_{-\infty}^{z}\int_{-\infty}^{\infty} |s| f(s,st)\, dsdt \end{aligned}$$

$$\left(= \int_{-\infty}^{z}\int_{-\infty}^{\infty} |s| f_X(s) f_Y(st)\, dsdt \right.$$
when X and Y are independent)

$$\left(\text{set } \begin{cases} x = s \\ y = st \end{cases}\right)$$

$s=x$, $t=y/x$

$$\begin{vmatrix} dx/ds & dx/dt \\ dy/ds & dy/dt \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ t & s \end{vmatrix} = s$$

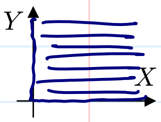
$dxdy = s\, dsdt.$

and, $f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} |x| f(x, xz)\, dx$

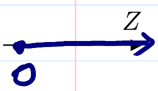
$\left(= \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xz)\, dx\right.$
when X and Y are independent)

□ (exercise) $Z=XY$

□ If X and Y are independent, $X \sim \text{exponential}(\lambda_1)$, $Y \sim \text{exponential}(\lambda_2)$, and $Z=Y/X$. The pdf of Z is



$$\begin{aligned} f_Z(z) &= \int_0^{\infty} x \left(\lambda_1 e^{-\lambda_1 x}\right)\left[\lambda_2 e^{-\lambda_2 (xz)}\right] dx \\ &= \frac{\lambda_1\lambda_2\Gamma(2)}{(\lambda_1+\lambda_2 z)^2}\int_0^{\infty} \underbrace{\frac{(\lambda_1+\lambda_2 z)^2}{\Gamma(2)} x^{2-1} e^{-(\lambda_1+\lambda_2 z)x}}_{\text{pdf of Gamma}(2,\lambda_1+\lambda_2 z)}\, dx \\ &= \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2 z)^2} \end{aligned}$$



for $z \ge 0$, and 0 for $z < 0$.

And, the cdf of Z is

$$\begin{aligned} F_Z(z) &= \int_0^z f_Z(t)\, dt = \int_0^z \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2 t)^2}\, dt \\ &= -\frac{\lambda_1\lambda_2}{\lambda_2}(\lambda_1+\lambda_2 t)^{-1}\Big|_0^z = 1 - \frac{\lambda_1}{\lambda_1+\lambda_2 z} \end{aligned}$$

for $z \ge 0$, and 0 for $z < 0$.

➢ Method of probability density function

a special case of method of cdf (see proof of the Thm)

- Theorem. Let $\mathbf{X}=(X_1, \ldots, X_n)$ be continuous random variables with the joint pdf $f_{\mathbf{X}}$. Let

$$\mathbf{Y}=(Y_1, \ldots, Y_n) = g(\mathbf{X}),$$

where g is 1-to-1, so that its inverse exists and is denoted by

$$\mathbf{x}=g^{-1}(\mathbf{y}) = w(\mathbf{y}) = (w_1(\mathbf{y}), w_2(\mathbf{y}), \ldots, w_n(\mathbf{y})).$$

Assume w have continuous partial derivatives, and let

$$\underbrace{J}_{\text{Jacobian}} = \begin{vmatrix} \frac{\partial w_1(\mathbf{y})}{\partial y_1} & \frac{\partial w_1(\mathbf{y})}{\partial y_2} & \cdots & \frac{\partial w_1(\mathbf{y})}{\partial y_n} \\ \frac{\partial w_2(\mathbf{y})}{\partial y_1} & \frac{\partial w_2(\mathbf{y})}{\partial y_2} & \cdots & \frac{\partial w_2(\mathbf{y})}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial w_n(\mathbf{y})}{\partial y_1} & \frac{\partial w_n(\mathbf{y})}{\partial y_2} & \cdots & \frac{\partial w_n(\mathbf{y})}{\partial y_n} \end{vmatrix}_{n\times n}$$

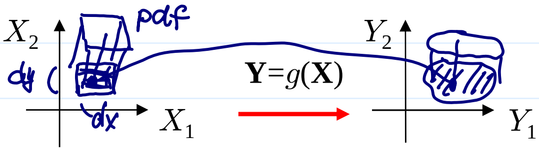
Note: become J^{-1} if use $\partial g/\partial x_i$

Then

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(g^{-1}(\mathbf{y})) \times |J| \xleftrightarrow{\text{c.f.}} \text{Thm in LN p.5-8}$$

for $\mathbf{y}$ s.t. $\mathbf{y}=g(\mathbf{x})$ for some $\mathbf{x}$, and $f_{\mathbf{Y}}(\mathbf{y})=0$, otherwise.

(**Q**: What is the role of $|J|$?)



Proof.

$$F_{\mathbf{Y}}(y_1, \ldots, y_n) = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n} f_{\mathbf{Y}}(t_1, \ldots, t_n)\, dt_n \cdots dt_1$$
$$= \int \cdots \int_{\substack{(x_1,\ldots,x_n): \\ Y_1 = g_1(x_1,\ldots,x_n) \le y_1 \\ \cdots \\ Y_n = g_n(x_1,\ldots,x_n) \le y_n}} f_{\mathbf{X}}(x_1, \ldots, x_n)\, dx_n \cdots dx_1.$$

It then follows from an exercise in advanced calculus that

$$f_{\mathbf{Y}}(y_1, \ldots, y_n) = \frac{\partial^n}{\partial y_1 \cdots \partial y_n} F_{\mathbf{Y}}(y_1, \ldots, y_n)$$
$$= f_{\mathbf{X}}(w_1(\mathbf{y}), \ldots, w_n(\mathbf{y})) \times |J|.$$

- Remark. When the dimensionality of $\mathbf{Y}$ (denoted by k) is less than n, we can choose another $n-k$ transformations $\mathbf{Z}$ such that $(\mathbf{Y}, \mathbf{Z})=g(\mathbf{X})$ satisfy the assumptions in above theorem. By integrating out the last $n-k$ arguments in the pdf of $(\mathbf{Y}, \mathbf{Z})$, the pdf of $\mathbf{Y}$ can be obtained.

- Example. X_1 and X_2 are random variables with joint pdf $f_{\mathbf{X}}(x_1, x_2)$. Find the distribution of $Y_1=X_1/(X_1+X_2) \equiv g_1(x_1, x_2)$
  - Let $Y_2=\underline{X_1+X_2}$, then

$g_2(x_1, x_2)$ add one more transformation

$$x_1 = y_1 y_2 \equiv w_1(y_1, y_2)$$
$$x_2 = y_2 - y_1 y_2 \equiv w_2(y_1, y_2).$$

Since $\frac{\partial w_1}{\partial y_1} = y_2$, $\frac{\partial w_1}{\partial y_2} = y_1$, $\frac{\partial w_2}{\partial y_1} = -y_2$, $\frac{\partial w_2}{\partial y_2} = 1 - y_1$,

$$J = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{vmatrix} = y_2 - y_1y_2 + y_1y_2 = y_2, \text{ and } |J| = |y_2|.$$

Therefore, $f_{\mathbf{Y}}(y_1, y_2) = f_{\mathbf{X}}(y_1y_2, y_2 - y_1y_2)|y_2|$,

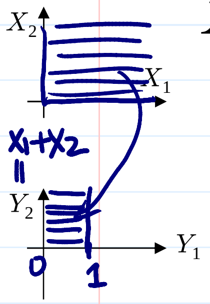
and, $f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{\mathbf{Y}}(y_1, y_2)\, dy_2$

$$= \int_{-\infty}^{\infty} f_{\mathbf{X}}(y_1y_2, y_2 - y_1y_2)|y_2|\, dy_2.$$

$$(= \int_{-\infty}^{\infty} f_{X_1}(y_1y_2) f_{X_2}(y_2 - y_1y_2)|y_2|\, dy_2$$

when X_1 and X_2 are independent)

- Theorem. If X_1 and X_2 are independent, and $X_1 \sim \text{Gamma}(\alpha_1, \lambda)$, $X_2 \sim \text{Gamma}(\alpha_2, \lambda)$, then



$$Y_1 = X_1/(X_1+X_2) \sim \text{Beta}(\alpha_1, \alpha_2).$$

Proof. For $x_1, x_2 \geq 0$, the joint pdf of $\mathbf{X}$ is

$$f_{\mathbf{X}}(x_1, x_2) = \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} x_1^{\alpha_1 - 1} e^{-\lambda x_1} \times \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} x_2^{\alpha_2 - 1} e^{-\lambda x_2}$$

$$= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x_1^{\alpha_1-1} x_2^{\alpha_2-1} e^{-\lambda(x_1+x_2)}.$$

So, for $0 \leq y_1 \leq 1$,

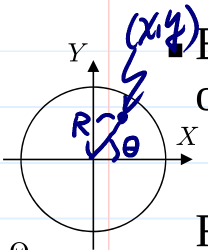
$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1}(y_1y_2) f_{X_2}(y_2 - y_1y_2)|y_2|\, dy_2$$

$$= \int_0^{\infty} \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (y_1y_2)^{\alpha_1-1} (y_2 - y_1y_2)^{\alpha_2-1} e^{-\lambda y_2} \cdot y_2\, dy_2$$

$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1}$$

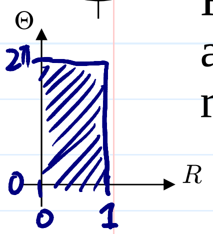
$$\times \int_0^{\infty} \underbrace{\frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} y_2^{(\alpha_1+\alpha_2)-1} e^{-\lambda y_2}}_{\text{pdf of Gamma}(\alpha_1+\alpha_2,\lambda)}\, dy_2.$$

and $f_{Y_1}(y_1) = 0$, otherwise.



- Example. Suppose that X and Y have a uniform distribution over the region $D = \{(x, y): x^2 + y^2 \leq 1\}$, i.e., their joint pdf is

$$f_{X,Y}(x, y) = \tfrac{1}{\pi} \mathbf{1}_D(x, y).$$



Find the joint distribution of (R, Θ) and examine whether R and Θ are independent, where (R, Θ) is the polar coordinate representation of (X, Y), i.e.,

$$X = R\cos(\Theta) \equiv w_1(R, \Theta),$$
$$Y = R\sin(\Theta) \equiv w_2(R, \Theta).$$

- Since $\frac{\partial w_1}{\partial r} = \cos(\theta)$, $\quad \frac{\partial w_1}{\partial \theta} = -r\sin(\theta)$,

$\frac{\partial w_2}{\partial r} = \sin(\theta)$, $\quad \frac{\partial w_2}{\partial \theta} = r\cos(\theta)$,

$$J = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r,$$

and $|J| = |r| = r$.

- For $0 \le r \le 1$ and $0 \le \theta \le 2\pi$, the joint pdf of (R, Θ) is

product set

$$f_{R,\Theta}(r,\theta) = f_{X,Y}(r\cos(\theta), r\sin(\theta)) \times |J| = \frac{1}{\pi} r = \frac{1}{2\pi} \cdot 2r$$

and $f_{R,\Theta}(r,\theta) = 0$, otherwise.

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marginal

$\Theta \sim \text{Uniform}(0, 2\pi)$

$R \sim \text{pdf. } 2r\, I_{(0,1)}(r)$

- By the theorem in LNp.6-21, (R, Θ) are independent.

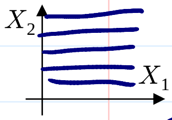
- Example. Let $X_1, \ldots, X_n$ be independent and identically distributed exponential(λ). Let

$$Y_i = X_1 + \cdots + X_i,\ i = 1, \ldots, n.$$

Find the distribution of $\mathbf{Y}=(Y_1, \ldots, Y_n)$.

LNp.6-28.

(Note. It has been shown that $Y_i \sim \text{Gamma}(i, \lambda)$, $i=1, \ldots, n$.)



- The joint pdf of $X_1, \ldots, X_n$ is

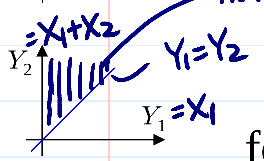
not a product set.

Note: not "x"

$$f_{\mathbf{X}}(x_1, \ldots, x_n) = \prod_{i=1}^n f_{X_i}(x_i)$$

Note: not "nx"

$$= \prod_{i=1}^n \left(\lambda e^{-\lambda x_i}\right) = \lambda^n e^{-\lambda(x_1 + \cdots + x_n)}.$$



for $0 \le x_i < \infty$, $i=1, \ldots, n$.

$Y_1 \le Y_2 \le \cdots \le Y_n$

- Since
$$\begin{aligned} x_1 &= y_1 &&\equiv w_1(y_1, \ldots, y_n), \\ x_2 &= y_2 - y_1 &&\equiv w_2(y_1, \ldots, y_n), \\ &\cdots \\ x_n &= y_n - y_{n-1} &&\equiv w_n(y_1, \ldots, y_n), \end{aligned}$$

$\Rightarrow g$ is 1-to-1 $\Rightarrow g^{-1}$ exists

we have
$$\frac{\partial w_i}{\partial y_j} = \begin{cases} 1, & \text{if } j = i, \\ -1, & \text{if } j = i-1, \\ 0, & \text{otherwise,} \end{cases}$$

$$J = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = 1, \text{ and } |J| = 1.$$

- For $0 \le y_1 \le \cdots \le y_{i-1} \le \underset{\substack{\| \\ y}}{y_i} \le y_{i+1} \le \cdots \le y_n < \infty$,

$$\begin{aligned} f_{\mathbf{Y}}(y_1, \ldots, y_n) &= f_{\mathbf{X}}(y_1, y_2 - y_1, \ldots, y_n - y_{n-1}) \times |J| \\ &= \lambda^n e^{-\lambda y_n}. \end{aligned}$$

$\leftarrow$ Q: $Y_1, \ldots, Y_n$ indep.?

and $f_{\mathbf{Y}}(y_1, \ldots, y_n) = 0$, otherwise.

- The marginal pdf of Y_i is

$$\begin{aligned} f_{Y_i}(y) &= \int_0^y \int_{y_1}^y \cdots \int_{y_{i-2}}^y \boxed{\int_y^\infty \int_{y_{i+1}}^\infty \cdots \int_{y_{n-1}}^\infty} \lambda^n e^{-\lambda y_n}\, dy_n \cdots dy_{i+2} dy_{i+1} dy_{i-1} \cdots dy_2 dy_1 \\ &= \int_0^y \cdots \int_{y_{i-2}}^y \underbrace{\boxed{\lambda^i e^{-\lambda y}}}_{\text{a constant}} dy_{i-1} \cdots dy_1 \\ &= \lambda^i e^{-\lambda y} \frac{y^{i-1}}{\underbrace{(i-1)!}_{\Gamma(i)}} \end{aligned}$$

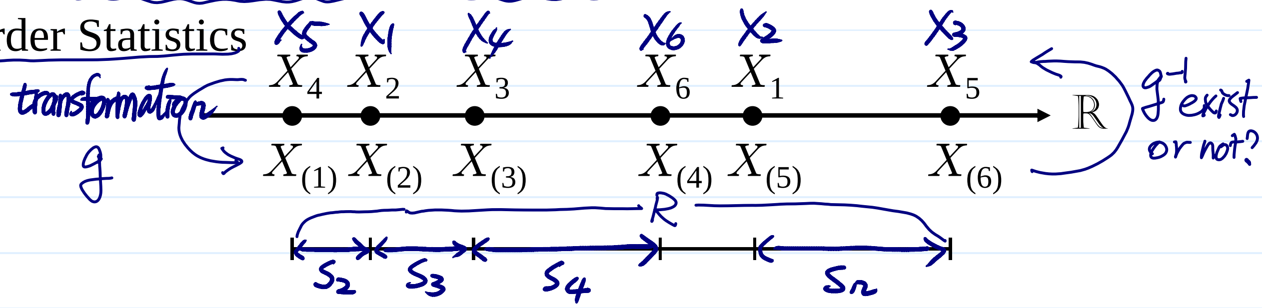
for $y \ge 0$, and $f_{Y_i}(y) = 0$, otherwise.

➢Method of moment generating function.

- Based on the *uniqueness theorem* of moment generating function to be explained later in Chapter 7
- Especially useful to identify the distribution of sum of independent random variables. $X_1, \cdots, X_n$ indep. $\Rightarrow Y = X_1 + \cdots + X_n$

順序統計量

➢Order Statistics



➢Definition. Let $X_1, \ldots, X_n$ be random variables. We sort the X_i's and denote by $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ the *order statistics*. Using the notation,

e.g. $X_1, \cdots, X_n \sim$ Uniform(a, b) a, b unknown.

$X_{(1)} = \min(X_1, \ldots, X_n)$ is the *minimum*,

$X_{(n)} = \max(X_1, \ldots, X_n)$ is the *maximum*,

function of order statistics

$R \equiv X_{(n)} - X_{(1)}$ is called *range*,

$S_j \equiv X_{(j)} - X_{(j-1)}$, j=2, …, n, are called *spacings*.

Q: What are the joint distributions of various order statistics and their marginal distributions?

➢Definition. $X_1, \ldots, X_n$ are called *i.i.d.* (**i**ndependent, **i**dentically **d**istributed) with cdf F/pdf f/pmf p if random variables $X_1, \ldots, X_n$ are independent and have a common marginal distribution with cdf F/pdf f/pmf p.

- Remark. For order statistics, we only consider the case that $X_1, \ldots, X_n$ are i.i.d. intuition, when $X_{(1)} = x$, $X_{(2)} \ge x$
  - Note. Although $X_1, \ldots, X_n$ are independent, their order statistics $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ are not independent in general.

Q: Which of the methods 1, 2, 3 in LN p.6-24 will you choose to find the distribution of $X_{(1)}$ & $X_{(n)}$?

➢Theorem. Suppose that $X_1, \ldots, X_n$ are i.i.d. with cdf F.

1. The cdf of $X_{(1)}$ is $1-[1-F(x)]^n$ and the cdf of $X_{(n)}$ is $[F(x)]^n$.
2. If **X** are continuous and F has a pdf f, then the pdf of $X_{(1)}$ is $nf(x)[1-F(x)]^{n-1}$ and the pdf of $X_{(n)}$ is $nf(x)[F(x)]^{n-1}$.

Proof. By the method of cumulative distribution function,

$$1 - F_{X_{(1)}}(x) = P(X_{(1)} > x) = P(X_1 > x, \ldots, X_n > x)$$
$$\underset{\because \text{ independence}}{=} P(X_1 > x) \cdots P(X_n > x) \underset{\because \text{ identical}}{=} [1 - F(x)]^n.$$

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x) = P(X_1 \le x, \ldots, X_n \le x)$$
$$= P(X_1 \le x)\cdots P(X_n \le x) = [F(x)]^n.$$

$$f_{X_{(1)}}(x) = \frac{d}{dx}F_{X_{(1)}}(x) = n[1-F(x)]^{n-1}\left(\frac{d}{dx}F(x)\right).$$

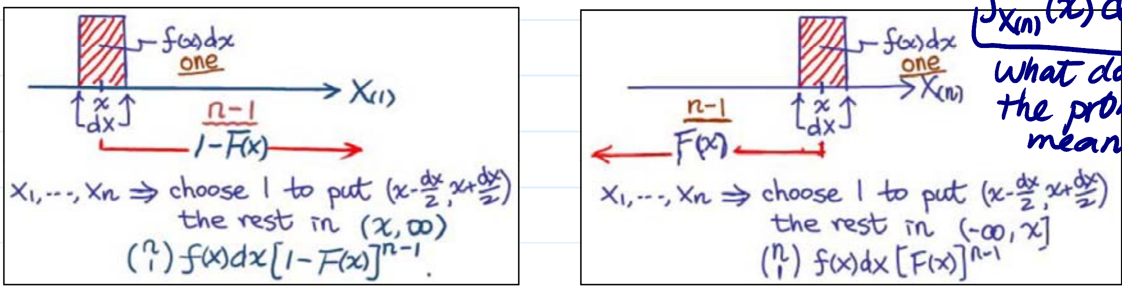
$$f_{X_{(n)}}(x) = \frac{d}{dx}F_{X_{(n)}}(x) = n[F(x)]^{n-1}\left(\frac{d}{dx}F(x)\right).$$

$\to f(x)$

- Graphical interpretation for the pdfs of $X_{(1)}$ and $X_{(n)}$.

$f_{X_{(1)}}(x)dx$ what does the prob. mean?

$f_{X_{(n)}}(x)dx$ what does the prob. mean?



- Example. n light bulbs are placed in service at time t=0, and allowed to burn continuously. Denote their lifetimes by $X_1, \ldots, X_n$, and suppose that they are i.i.d. with cdf F. If burned out bulbs are not replaced, then the room goes dark at time
  $$Y = \max(X_1, \ldots, X_n).$$
  the cdf of X_i
  - If n=5 and F is exponential with λ = 1 per month, then

Note: We can derive any marginal distributions and the dist. of functions of $X_{(1)}, \ldots, X_{(n)}$ from the joint pdf/pmf

$$F(x) = 1-e^{-x}, \text{ for } x \ge 0, \text{ and } 0, \text{ for } x < 0.$$

- The cdf of Y is
  $$F_Y(y) = (1-e^{-y})^5, \text{ for } y \ge 0, \text{ and } 0, \text{ for } y < 0,$$
  and its pdf is $5(1-e^{-y})^4e^{-y}$, for $y \ge 0$, and 0, for $y < 0$.
- The probability that the room is still lighted after two months is $P(Y > 2) = 1-F_Y(2) = 1-(1-e^{-2})^5$.

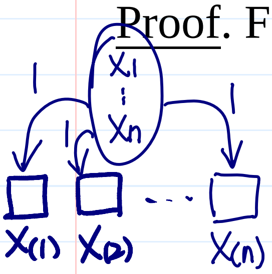
12/22

➢ Theorem. Suppose that $X_1, \ldots, X_n$ are i.i.d. with pdf f/pmf p. Then, the joint pmf/pdf of $X_{(1)}, \ldots, X_{(n)}$ is

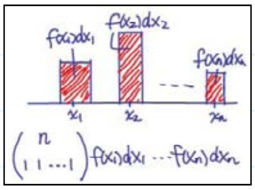
$$p_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! \times p(x_1) \times \cdots \times p(x_n),$$
$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! \times f(x_1) \times \cdots \times f(x_n),$$

for $x_1 \le x_2 \le \cdots \le x_n$, and 0 otherwise. not a product set.

Proof. For $x_1 \le x_2 \le \cdots \le x_n$,



$$p_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = P(X_{(1)} = x_1, \ldots, X_{(n)} = x_n)$$
$$= \sum_{\substack{(i_1,\ldots,i_n):\\ \text{permutations of}\\ (1,\ldots,n)}} P(X_1 = x_{i_1}, \ldots, X_n = x_{i_n})$$
$$\overset{\text{i.i.d.}}{=} \sum_{\substack{(i_1,\ldots,i_n):\\ \text{permutations of}\\ (1,\ldots,n)}} p(x_1) \times \cdots \times p(x_n)$$
$$= n! \times p(x_1) \times \cdots \times p(x_n).$$

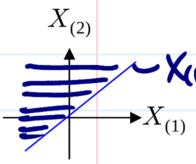


What does the prob. mean?

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1,\ldots,x_n)\,dx_1\cdots dx_n$$
$$\approx P\left(x_1-\frac{dx_1}{2}<X_{(1)}<x_1+\frac{dx_1}{2},\ldots,\ x_n-\frac{dx_n}{2}<X_{(n)}<x_n+\frac{dx_n}{2}\right)$$
$$=\sum_{\substack{(i_1,\ldots,i_n):\\ \text{permutations of}\\ (1,\ldots,n)}} P\left(x_{i_1}-\frac{dx_{i_1}}{2}<X_1<x_{i_1}+\frac{dx_{i_1}}{2},\ldots,\ x_{i_n}-\frac{dx_{i_n}}{2}<X_n<x_{i_n}+\frac{dx_{i_n}}{2}\right)$$

$\because$ iid

$$\approx \sum_{\substack{(i_1,\ldots,i_n):\\ \text{permutations of}\\ (1,\ldots,n)}} f(x_1)\times\cdots\times f(x_n)\,dx_1\cdots dx_n$$
$$= n!\times f(x_1)\times\cdots\times f(x_n)\,dx_1\cdots dx_n.$$



- **Q**: Examine whether $X_{(1)}, \ldots, X_{(n)}$ are independent using the Theorem in LNp.6-21.

➢Theorem. If $X_1, \ldots, X_n$ are i.i.d. with cdf F and pdf f, then

1. The pdf of the k^{th} order statistic $X_{(k)}$ is ← Note: can be derived from the joint pdf of $X_{(1)},\ldots,X_{(n)}$ (exercise)

$$f_{X_{(k)}}(x)=\binom{n}{1,k-1,n-k}f(x)F(x)^{k-1}[1-F(x)]^{n-k}.$$

2. The cdf of $X_{(k)}$ is

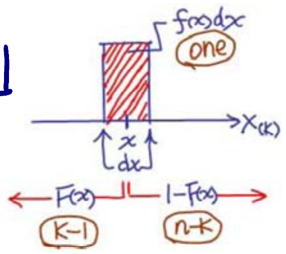
$$F_{X_{(k)}}(x)=\sum_{m=k}^{n}\binom{n}{m}[F(x)]^m[1-F(x)]^{n-m}.$$

use $F(x)=\int_{-\infty}^{x}f(t)\,dt$ to prove (exercise)

use $f(x)=\frac{d}{dx}F(x)$ to prove (exercise)

Proof.

$f_{X_{(k)}}(x)\,dx$ What does the prob. mean?

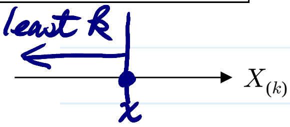


$X_1,\ldots,X_n \Rightarrow$ choose 1 to place in $(x-\frac{dx}{2}, x+\frac{dx}{2})$
" $k-1$ " " " $(-\infty, x)$
" $n-k$ " " " (x,∞)

$$\binom{n}{1\ \ k-1\ \ n-k}f(x)dx\,[F(x)]^{k-1}[1-F(x)]^{n-k}$$

$$F_{X_{(k)}}(x)=P(X_{(k)}\le x)$$
$$=P(\text{at least } k \text{ of the } X_i\text{'s are} \le x)$$
$$=\sum_{m=k}^{n}P(\text{exact } m \text{ of the } X_i\text{'s are} \le x)$$
$$=\sum_{m=k}^{n}\binom{n}{m}[F(x)]^m[1-F(x)]^{n-m}$$

mutually exclusive



Q: What if $X_1,\ldots,X_n$ are i.i.d discrete r.v.'s?

➢Theorem. If $X_1, \ldots, X_n$ are i.i.d. with cdf F and pdf f, then

1. The joint pdf of $X_{(1)}$ and $X_{(n)}$ is ← can be derived from the joint pdf of $X_{(1)},\ldots,X_{(n)}$ (exercise)

$$f_{X_{(1)},X_{(n)}}(s,t)=n(n-1)f(s)f(t)[F(t)-F(s)]^{n-2},$$

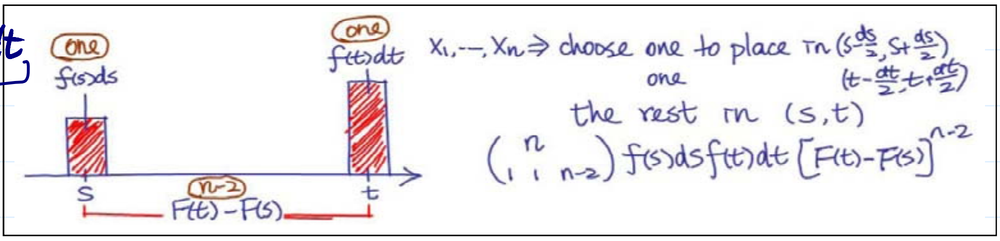
for $s\le t$, and 0 otherwise.

2. The pdf of the range $R=X_{(n)}-X_{(1)}$ is ← exercise given in LNp.6-27

$$f_R(r)=\int_{-\infty}^{\infty}f_{X_{(1)},X_{(n)}}(u,u+r)\,du,$$

for $r\ge 0$, and 0 otherwise.

$f_{X_{(1)},X_{(n)}}(s,t)\,ds\,dt$ What does the prob. mean?



➢Theorem. If $X_1, \ldots, X_n$ are i.i.d. with cdf F and pdf f, then

1. The joint pdf of $X_{(i)}$ and $X_{(j)}$, where $1 \le i < j \le n$, is

can be derived from the joint pdf of $X_{(1)}, \ldots, X_{(n)}$

$$f_{X_{(i)},X_{(j)}}(s,t) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} f(s)f(t) \times [F(s)]^{i-1}[F(t)-F(s)]^{j-i-1}[1-F(t)]^{n-j},$$

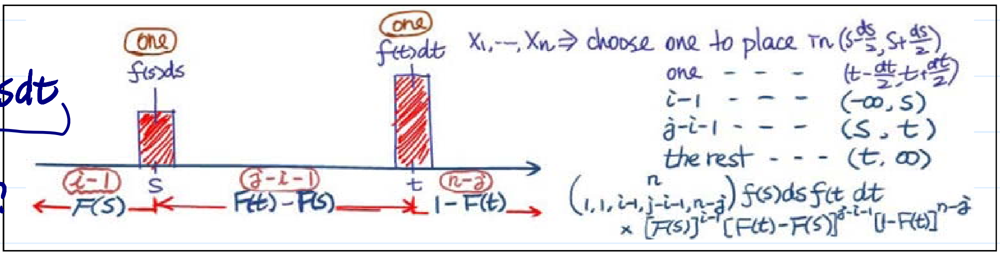
for $s \le t$, and 0 otherwise.

2. The pdf of the j^{th} spacing $S_j = X_{(j)} - X_{(j-1)}$ is

$$f_{S_j}(s) = \int_{-\infty}^{\infty} f_{X_{(j-1)},X_{(j)}}(u, u+s)\,du,$$

for $s \ge 0$, and zero otherwise.

$f_{X_{(i)},X_{(j)}}(s,t)\,ds\,dt$ What does the prob. mean?



❖ **Reading**: textbook, Sec 6.3, 6.6, 6.7

Conditional Distribution

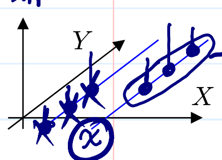
Recall: conditional prob. (LNp.3-1~13)

- Definition. Let **X** ($\in \mathbb{R}^n$) and **Y** ($\in \mathbb{R}^m$) be discrete random vectors and (**X**, **Y**) have a joint pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$, then the *conditional joint pmf* of **Y** given **X**=**x** is defined as

$P(B|A) = P(A \cap B)/P(A)$

$$p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) \equiv P(\underbrace{\{\mathbf{Y}=\mathbf{y}\}}_{B}|\underbrace{\{\mathbf{X}=\mathbf{x}\}}_{A}) = \frac{P(\{\mathbf{X}=\mathbf{x},\mathbf{Y}=\mathbf{y}\})}{P(\{\mathbf{X}=\mathbf{x}\})} = \frac{p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})}{p_{\mathbf{X}}(\mathbf{x})} = \frac{\text{joint}}{\text{marginal}}$$

$p_{X,Y}$



sum of the prob. $f_X(x)$ ($\ne$ one)

if $p_{\mathbf{X}}(\mathbf{x})>0$. The probability is defined to be zero if $p_{\mathbf{X}}(\mathbf{x})=0$.

➢Some Notes. Note: x & y play different roles in $P_{Y|X}$

- For each fixed **x**, $p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ is a joint pmf for **y**, since

$$\sum_{\mathbf{y}} p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = \frac{1}{p_{\mathbf{X}}(\mathbf{x})}\sum_{\mathbf{y}} p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = \frac{1}{p_{\mathbf{X}}(\mathbf{x})} \times p_{\mathbf{X}}(\mathbf{x}) = 1.$$

- For an event B of **Y**, the probability that $\mathbf{Y} \in B$ given **X**=**x** is

$$P(\mathbf{Y} \in B|\mathbf{X}=\mathbf{x}) = \sum_{\mathbf{u} \in B} p_{\mathbf{Y}|\mathbf{X}}(\mathbf{u}|\mathbf{x}).$$

$B =$

- The *conditional joint cdf* of **Y** given **X**=**x** can be similarly defined from the conditional joint pmf $p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$, i.e.,

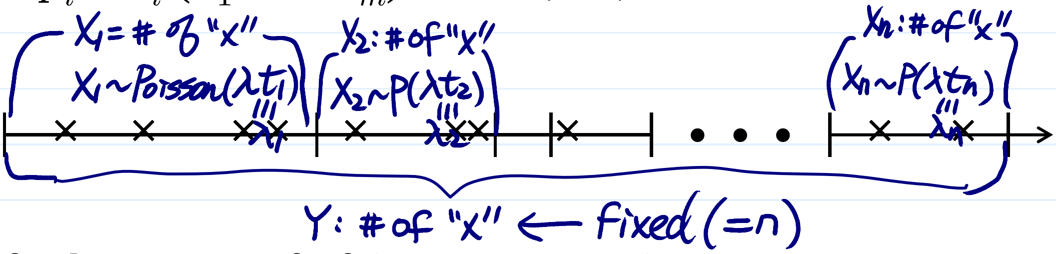
$\{Y_1 \le y_1, \ldots, Y_m \le y_m\}$

$$F_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = P(\mathbf{Y} \le \mathbf{y}|\mathbf{X}=\mathbf{x}) = \sum_{\mathbf{u} \le \mathbf{y}} p_{\mathbf{Y}|\mathbf{X}}(\mathbf{u}|\mathbf{x}).$$

➢Theorem. Let $X_1, \ldots, X_m$ be independent and X_i~Poisson(λ_i), i=1, …, m. Let Y=X_1+⋯+X_m, then

$$(X_1, \ldots, X_m | Y=n) \sim \text{Multinomial}(n, m, p_1, \ldots, p_m),$$

where $p_i = \lambda_i/(\lambda_1+\cdots+\lambda_m)$ for i=1, …, m.



Proof. The joint pmf of ($X_1, \ldots, X_m, Y$) is

$$p_{\mathbf{X},Y}(x_1,\ldots,x_m,n) = P(\{X_1=x_1,\ldots,X_m=x_m\}\cap\{Y=n\})$$
$$= \begin{cases} P(X_1=x_1,\ldots,X_m=x_m), & \text{if } x_1+\cdots+x_m=n,\\ 0, & \text{if } x_1+\cdots+x_m\neq n.\end{cases}$$

Furthermore, the distribution of Y is Poisson(λ_1+⋯+λ_m), i.e., LNp.6-26.

$$p_Y(n) = P(Y=n) = \frac{e^{-(\lambda_1+\cdots+\lambda_m)}(\lambda_1+\cdots+\lambda_m)^n}{n!}.$$

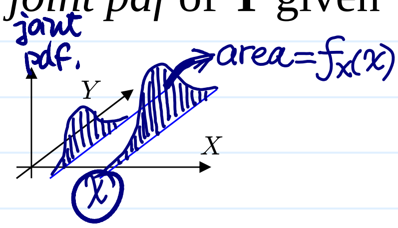
Therefore, for $\mathbf{x}$=($x_1, \ldots, x_m$) wheres $x_i\in\{0,1,2,\ldots\}$, i=1, …, m, and $x_1+\cdots+x_m=n$, the conditional joint pmf of $\mathbf{X}$ given Y=n is

$$p_{\mathbf{X}|Y}(\mathbf{x}|n) = \frac{p_{\mathbf{X},Y}(x_1,\ldots,x_m,n)}{p_Y(n)} = \frac{\prod_{i=1}^m \frac{\cancel{e^{-\lambda_i}}\lambda_i^{x_i}}{x_i!}}{\frac{\cancel{e^{-(\lambda_1+\cdots+\lambda_m)}}(\lambda_1+\cdots+\lambda_m)^{n = x_1+\cdots+x_m}}{n!}}$$

$$= \frac{n!}{x_1!\times\cdots\times x_m!}\times\left(\frac{\lambda_1}{\lambda_1+\cdots+\lambda_m}\right)^{x_1}\times\cdots\times\left(\frac{\lambda_m}{\lambda_1+\cdots+\lambda_m}\right)^{x_m}.$$

• Definition. Let $\mathbf{X}$ $\in\mathbb{R}^n$ and $\mathbf{Y}$ $\in\mathbb{R}^m$ be continuous random vectors and ($\mathbf{X}$, $\mathbf{Y}$) have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$, then the *conditional joint pdf* of $\mathbf{Y}$ given $\mathbf{X}$=$\mathbf{x}$ is defined as

$$f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) \equiv \frac{f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})}{f_{\mathbf{X}}(\mathbf{x})}, \quad \begin{matrix}\leftarrow \text{joint}\\ \text{marginal.}\end{matrix}$$



if $f_{\mathbf{X}}(\mathbf{x})$>0, and 0 otherwise.

➢Some Notes.

- P($\mathbf{X}$=$\mathbf{x}$)=0 for a continuous random vector $\mathbf{X}$. $\{x_1-\frac{\Delta x_1}{2}<X_1\le x_1+\frac{\Delta x_1}{2}, \ldots, x_n-\frac{\Delta x_n}{2}<X_n\le x_n+\frac{\Delta x_n}{2}\}$
- The definition of $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ comes from

joint cdf ← cf $P(\underbrace{\mathbf{Y}\le\mathbf{y}}_{B}\,|\,\underbrace{\mathbf{x}-(\Delta\mathbf{x}/2)<\mathbf{X}\le\mathbf{x}+(\Delta\mathbf{x}/2)}_{A})$

$\{Y_1\le y_1,\ldots,Y_m\le y_m\}$ =

$$= \frac{\int_{-\infty}^{\mathbf{y}}\int_{\mathbf{x}-\frac{\Delta\mathbf{x}}{2}}^{\mathbf{x}+\frac{\Delta\mathbf{x}}{2}} f_{\mathbf{X},\mathbf{Y}}(\mathbf{u},\mathbf{v})\,d\mathbf{u}d\mathbf{v}}{\int_{\mathbf{x}-\frac{\Delta\mathbf{x}}{2}}^{\mathbf{x}+\frac{\Delta\mathbf{x}}{2}} f_{\mathbf{X}}(\mathbf{t})\,d\mathbf{t}} \begin{matrix} = P(A\cap B) \\ = P(A)\end{matrix}$$

become "=" when $\Delta x\to0$.

$$\approx \frac{\int_{-\infty}^{\mathbf{y}} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{v})\cancel{\Delta\mathbf{x}}\,d\mathbf{v}}{f_{\mathbf{X}}(\mathbf{x})\cancel{\Delta\mathbf{x}}} = \int_{-\infty}^{\mathbf{y}}\frac{f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})}{f_{\mathbf{X}}(\mathbf{x})}d\mathbf{y} \leftarrow \text{joint pdf.}$$

Note: x & y play different roles

- For each fixed $\mathbf{x}$, $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ is a joint pdf for $\mathbf{y}$, since
$$\int_{-\infty}^{\infty} f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\, d\mathbf{y} = \frac{1}{f_{\mathbf{X}}(\mathbf{x})}\int_{-\infty}^{\infty} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})\, d\mathbf{y} = \frac{1}{f_{\mathbf{X}}(\mathbf{x})} \times f_{\mathbf{X}}(\mathbf{x}) = 1.$$
$\mathbb{R}^m$

- For an event B of $\mathbf{Y}$, we can write
$$P(\mathbf{Y} \in B|\mathbf{X}=\mathbf{x}) = \int_B f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\, d\mathbf{y}.$$

- The *conditional joint cdf* of $\mathbf{Y}$ given $\mathbf{X}=\mathbf{x}$ can be similarly defined from the conditional joint pdf $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$, i.e.,
$$F_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = P(\mathbf{Y} \le \mathbf{y}|\mathbf{X}=\mathbf{x}) = \int_{-\infty}^{\mathbf{y}} f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\, d\mathbf{y}.$$

➢ Example. If X and Y have a joint pdf
$$f(x,y) = \frac{2}{(1+x+y)^3},$$
for $0 \le x, y < \infty$, then
$$f_X(x) = \int_0^{\infty} f(x,y)\, dy = -\frac{1}{(1+x+y)^2}\Big|_0^{\infty} = \frac{1}{(1+x)^2},$$
for $0 \le x < \infty$. So,
$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2(1+x)^2}{(1+x+y)^3},$$
and,
$$\begin{aligned} P(Y > c|X=x) &= \int_c^{\infty} \frac{2(1+x)^2}{(1+x+y)^3}\, dy \\ &= -\frac{(1+x)^2}{(1+x+y)^2}\Big|_{y=c}^{\infty} = \frac{(1+x)^2}{(1+x+c)^2}. \end{aligned}$$

- Mixed Distribution: The definition of conditional distribution can be similarly generalized to the case in which some random variables are discrete and the others continuous (see a later example). joint marginal

Recall: The 3 Laws in LNp.3-7~9.

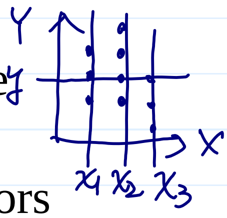
- Theorem (Multiplication Law). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X}, \mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$, then
$$p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) \times p_{\mathbf{X}}(\mathbf{x}), \quad \text{or}$$
$$f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) \times f_{\mathbf{X}}(\mathbf{x}).$$
$P_{X_1,\dots,X_n}(x_1,\dots,x_n)$
$= P_{X_1}(x_1)\cdot P_{X_2|X_1}(x_2|x_1)\cdot P_{X_3|X_1,X_2}(x_3|x_1,x_2)$ ---

Proof. By the definition of conditional distribution.

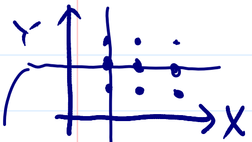
- Theorem (Law of Total Probability). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X}, \mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$, then
$$\sum_x P_{X,Y}(x,y) \quad p_{\mathbf{Y}}(\mathbf{y}) = \sum_{\mathbf{x}=-\infty}^{\infty} p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) p_{\mathbf{X}}(\mathbf{x}), \quad \text{or}$$
$\{X=x\}$ form a partition
$$f_{\mathbf{Y}}(\mathbf{y}) = \int_{-\infty}^{\infty} f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}.$$

Proof. By the definition of marginal distribution and the multiplication law.

$\{X=x_1\}$



- Theorem (Bayes Theorem). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X}, \mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$/or a joint pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$, then



$$p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) = \frac{p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})p_{\mathbf{X}}(\mathbf{x})}{\sum_{\mathbf{x}=-\infty}^{\infty} p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})p_{\mathbf{X}}(\mathbf{x})}, \quad \text{or}$$

$p_{X|Y}(x|y) \leftarrow p_{Y|X}(y|x)$

$$f_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) = \frac{f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})f_{\mathbf{X}}(\mathbf{x})}{\int_{-\infty}^{\infty} f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x}}.$$

Proof. By the definition of conditional distribution, multiplication law, and the law of total probability.

Example.

Note: X: continuous, $Y_1, \cdots, Y_n$: discrete

$\to$ 1st $\to$ 2nd — o (LN p.3-10)

$(Y_1, \cdots, Y_n)$

- Suppose that $X \sim$ Uniform(0, 1), and $(Y_1, \ldots, Y_n|X=x)$ are i.i.d. with Bernoulli(x), i.e.,
$$p_{\mathbf{Y}|X}(y_1, \ldots, y_n|x) = x^{y_1+\cdots+y_n}(1-x)^{n-(y_1+\cdots+y_n)},$$
for $y_1, \ldots, y_n \in \{0, 1\}$.
- By the multiplication law, for $y_1, \ldots, y_n \in \{0, 1\}$ and $0<x<1$,
joint
$$p_{\mathbf{Y},X}(y_1, \ldots, y_n, x) = x^{y_1+\cdots+y_n}(1-x)^{n-(y_1+\cdots+y_n)}.$$
- Suppose that we observed $Y_1=1, \ldots, Y_n=1$. By the law of total probability, $P(Y_1 = 1, \ldots, Y_n = 1) = p_{\mathbf{Y}}(1, \ldots, 1)$
marginal of $Y_1, \cdots, Y_n$
$$= \int_0^1 p_{\mathbf{Y}|X}(1, \ldots, 1|x)f_X(x)\,dx$$
$$= \int_0^1 x^n\,dx = \left.\frac{1}{n+1}x^{n+1}\right|_0^1 = \frac{1}{n+1}.$$

- And, by Bayes' Theorem,
$$f_{X|\mathbf{Y}}(x|Y_1 = 1, \ldots, Y_n = 1)$$
$$= \frac{p_{\mathbf{Y}|X}(1, \ldots, 1|x)f_X(x)}{p_{\mathbf{Y}}(1, \ldots, 1)} = (n+1)x^n.$$
$\frac{\text{joint}}{\text{marginal}}$

for $0<x<1$, i.e., $(X|Y_1=1, \ldots, Y_n=1) \sim$ ~~Gamma~~ Beta$(n+1, 1)$.

c.f. marginal distribution of $X \sim U(0,1)$

- If there were an $(n+1)$st Bernoulli trial Y_{n+1}, $E[X|Y_1=1, \cdots, Y_n=1] = \frac{n+1}{n+2}$
$$P(Y_{n+1} = 1|Y_1 = 1, \ldots, Y_n = 1)$$
$$= \frac{P(Y_1 = 1, \ldots, Y_{n+1} = 1)}{P(Y_1 = 1, \ldots, Y_n = 1)} = \frac{1/(n+2)}{1/(n+1)} = \frac{n+1}{n+2}.$$
$E[X|Y_1=y_1, \cdots, Y_n=y_n] = \frac{(y_1+\cdots+y_n)+1}{n+2}$
- (exercise) In general, it can be shown that
$(X|Y_1=y_1, \ldots, Y_n=y_n) \sim$ ~~Gamma~~ Beta$((y_1+\cdots+y_n)+1, n-(y_1+\cdots+y_n)+1)$.

• Theorem (Independent). Let **X** and **Y** be random vectors and (**X**, **Y**) have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$. Then, **X** and **Y** are independent, i.e.,
$$p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y}) = p_{\mathbf{X}}(\mathbf{x}) \times p_{\mathbf{Y}}(\mathbf{y}), \quad \text{or}$$
$$f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y}) = f_{\mathbf{X}}(\mathbf{x}) \times f_{\mathbf{Y}}(\mathbf{y}),$$
if and only if

joint/marginal =
$$\begin{cases} p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = p_{\mathbf{Y}}(\mathbf{y}), & \text{or} \\ f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = f_{\mathbf{Y}}(\mathbf{y}). \end{cases}$$

Proof. By the definition of conditional distribution.

➢ intuition.

- the 2 graphs in LNp.6-23
- $p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ (or $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$) offers information of $\mathbf{Y}$ when $\mathbf{X} = \mathbf{x}$; $p_{\mathbf{Y}}(\mathbf{y})$ (or $f_{\mathbf{Y}}(\mathbf{y})$) offers information of $\mathbf{Y}$ when $\mathbf{X}$ not observed.

❖ **Reading**: textbook, Sec 6.4, 6.5