Note: observed data, always discrete

Continuous Random Variables

- Recall: For *discrete* random variables, only a *finite* or *countably infinite* number of possible values with positive probability. (>0)
  - ➢Often, there is interest in random variables that can take (at least theoretically) on an *uncountable* number of possible values, e.g.,

(0, 400) ← the weight of a randomly selected person in a population,

(0, ∞) ← the length of time that a randomly selected light bulb works,

(−a, a) for some a ← the error in experimentally measuring the speed of light.

- ➢Example (Uniform Spinner, LNp.2-14):
  - $\Omega = (-\pi, \pi]$
  - For $(a, b] \subset \Omega$, $P((a, b]) = (b-a)/(2\pi)$
  - Consider the random variables:

range of X is (−π, π]
range of Y is (−∞, ∞)

$$X: \Omega \to \mathbb{R}, \quad \text{and} \quad X(\omega) = \omega \text{ for } \omega \in \Omega,$$
$$Y: \Omega \to \mathbb{R}, \quad \text{and} \quad Y(\omega) = tan(\omega) \text{ for } \omega \in \Omega.$$

Then, X and Y are random variables that takes on an uncountable number of possible values.

- Notice that: $P(\{x\}) = 0$

distribution of X (Y)
① any fixed value has prob. zero
② Instead, positive prob. is assigned to arbitrary interval

LNp.2-15 → $P_X(\{X = x\}) = 0$, for any $x \in \mathbb{R}$, $P_Y(Y=y)=0$

But, for $-\pi \le a < b \le \pi$,

$$P_X(\{X \in (a, b]\}) = P((a, b]) = (b-a)/(2\pi) > 0.$$

Why it happen? uncountable sum

discrete	continuous
pmf	pdf
Σ	∫

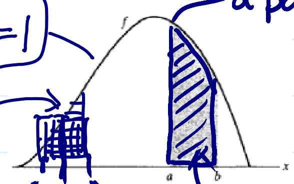
Q: Can we still define a probability mass function for X? No. If not, what can play a *similar* role like pmf for X?

- Probability Density Function and Continuous Random Variable
  - ➢Definition. A function f: $\mathbb{R} \to \mathbb{R}$ is called a probability density function (pdf) if

LNp.4-5 (ii)(iii) c.f.

1. $f(x) \ge 0$, for all $x \in (-\infty, \infty)$, and
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

area $= P(x - \frac{dx}{2} < X < x + \frac{dx}{2}) \approx f(x) \cdot dx$



total area = 1 ; a pdf ; f(x) ; $x - \frac{dx}{2}$; dx ; $x + \frac{dx}{2}$; x ; $P(a \le X \le b)$

- ➢Definition: A random variable X is called *continuous* if there exists a pdf f such that for any set B of real numbers

LNp.4-5 (IV) ← c.f. →

$$P_X(\{X \in B\}) = \int_B f(x)\, dx. \leftrightarrow \text{area}$$

  - For example, $P_X(a \le X \le b) = \int_a^b f(x)dx$.

c.f. Last Thm in LNp.45

➢ Theorem. If f is a pdf, then there must exist a continuous random variable with pdf f. proof: Let $F(x) = \int_{-\infty}^{x} f(t)dt$, then show that $F(x)$ is a cdf (exercise). Then, by the thm in LNp.4-9.

➢ Some properties

The corresponding cdf is a continuous function, i.e., no jump

- $P_X(\{X = x\}) = \int_x^x f(y)dy = 0$ for any $x \in \mathbb{R}$
- It does not matter if the intervals are open or close, i.e.,

$$P(X \in [a, b]) = P(X \in (a, b]) = P(X \in [a, b)) = P(X \in (a, b)).$$

- It is important to remember that the value a pdf $f(x)$ is NOT a probability itself
  * area is probability
  * but $f(x)$ is not. $\overset{\text{c.f.}}{\longleftrightarrow}$ pmf
- It is quite possible for a pdf to have value greater than 1 $\overset{\text{c.f.}}{\longleftrightarrow}$ pmf $\leq$ 1
- **Q**: How to interpret the value of a pdf $f(x)$? For small dx,

$$P\left(x - \tfrac{dx}{2} \leq X \leq x + \tfrac{dx}{2}\right) = \int_{x-\frac{dx}{2}}^{x+\frac{dx}{2}} f(y)dy \approx f(x) \cdot dx.$$

$\Rightarrow f(x)$ is a measure of how likely it is that X will be near x

i.e. if $f(a) > f(b)$ Then X is more likely to appear near a than near b.

c.f. Ans in LNp.44

➢ We can characterize the distribution of a continuous random variable in terms of its

1. Probability Density Function (pdf)
2. Cumulative Distribution Function (cdf) ← Note. cdf defined for any r.v. (LNp.4-6)
3. Moment Generating Function (mgf, Chapter 7)

- Relation between the pdf and the cdf

➢ Theorem. If F_X and f_X are the cdf and the pdf of a $\boxed{\text{continuous}}$ random variable X, respectively, then

- $F_X(x) \overset{\text{definition of cdf}}{=} P(X \leq x) = \int_{-\infty}^{x} f_X(y)dy$ for all $-\infty < x < \infty$
  "$P(X \in (-\infty, x])$"
- $f_X(x) = F'_X(x) = \frac{d}{dx}F_X(x)$ at continuity points of f_X

c.f. $\boxed{\text{relationship between cdf \& pmf LNp.4-7(6)}}$

when f_X is given $\Rightarrow F_X$ is known

" F_X " " $\Rightarrow f_X$ " "

mixed?

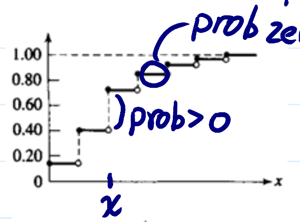
➢ Some Notes

- For $-\infty \leq a < b \leq \infty$

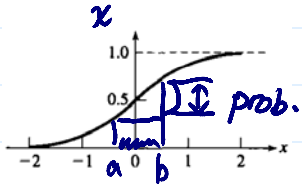
LNp.4-8,(5) → $\boxed{P(a < X \leq b) = F_X(b) - F_X(a)} = \int_a^b f_X(x)dx = F(x)\Big|_a^b$

$P(X \in (a, b])$ //

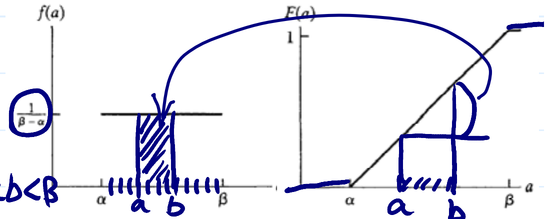
- The cdf for continuous random variables has the same interpretation and properties as in the discrete case ↳ LNp.4-7~9



- The only difference is in plotting F_X. In the discrete case, there are jumps. In the continuous case, F_X is a (absolutely) continuous non-decreasing function.



➢Example (Uniform Distributions)



- If $-\infty<\alpha<\beta<\infty$, then

$$f(x)=\begin{cases}\frac{1}{\beta-\alpha}, & \text{if } \alpha<x\le\beta,\\ 0, & \text{otherwise,}\end{cases}$$

$\alpha<a<b<\beta$

$\int_a^b f(x)dx=\frac{1}{\beta-\alpha}(b-a)$

① For interval $\subset(\alpha,\beta)$ of same length they have same prob.

② prob $\propto$ length of interval.

is a pdf since

1. $f(x)\ge 0$ for all $x\in\mathbb{R}$
2. $\int_{-\infty}^{\infty} f(x)dx=\int_\alpha^\beta \frac{1}{\beta-\alpha}dx=\frac{1}{\beta-\alpha}(\beta-\alpha)=1.$

- Its corresponding cdf is

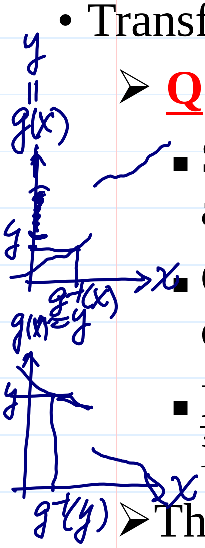
$$F(x)=\int_{-\infty}^{x} f(y)dy=\begin{cases}0, & \text{if } x\le\alpha,\\ \frac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha<x\le\beta,\\ 1, & \text{if } x>\beta.\end{cases}$$

- (exercise) Conversely, it can be easily checked that F is a cdf and $f(x)=F'(x)$ except at $x=\alpha$ and $x=\beta$ (Derivative does not exist when $x=\alpha$ and $x=\beta$, but it does not matter.)

check LNp.4-7&8 (2)(3)(4)

we can assign $f(\alpha)=0$, $f(\beta)=0$ or other positive value.

- An example of Uniform distribution is the r.v. X in the Uniform Spinner example where $\alpha=-\pi$ and $\beta=\pi$.



- Transformation $\overset{\text{cf.}}{\Longleftrightarrow}$ pmf case, LNp.4-10

➢ **Q**: $Y=g(X)$, how to find the distribution of Y?

- Suppose that X is a continuous random variable with cdf F_X and pdf f_X.

i.e. $g(a)>g(b)$, if $a>b$
$g(a)<g(b)$, if $a>b$

- Consider $Y=g(X)$, where g is a strictly monotone (increasing or decreasing) function. Let R_Y be the range of g.
- Note. Any strictly monotone function has an inverse function, i.e., g^{-1} exists on R_Y.

➢The cdf of Y, denoted by F_Y

1.Suppose that g is a strictly increasing function. For $y\in R_Y$,

true for any r.v.

Q: what if $y\notin R_Y$?

$$\begin{aligned}F_Y(y)&=P(Y\le y)\\&=P(g(X)\le y)=P(X\le g^{-1}(y))\\&=F_X(g^{-1}(y)).\end{aligned}$$

2.Suppose that g is a strictly decreasing function. For $y\in R_Y$,

$P(X\in(-\infty,g^{-1}(y)))=F_X(g^{-1}(y)-)$

$$\begin{aligned}F_Y(y)&=P(Y\le y)\\&=P(g(X)\le y)=P(X\ge g^{-1}(y))=1-P(X<g^{-1}(y))\\&=1-F_X(g^{-1}(y)).\end{aligned}$$

$\because X$ is a continuous r.v.

$=1-F_X(g^{-1}(y)-)$ in general

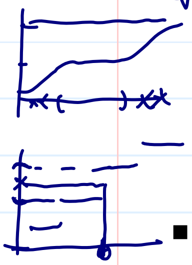
* nondecreasing
* no jumps
* one-to-one
* strictly increasing

- **Theorem.** Let X be a continuous random variable whose cdf F_X possesses a unique inverse F_X^{-1}. Let $Z=F_X(X)$, then Z has a uniform distribution on [0, 1].

$g \Rightarrow g^{-1}(\cdot) = F_X^{-1}$

Proof. For $0 \le z \le 1$, $F_Z(z) = F_X(F_X^{-1}(z)) = z$.

$\begin{cases} 0, & \text{if } z<0 \\ z \\ 1, & \text{if } z>1 \end{cases}$ cdf of uniform distribution with $\alpha=0$, $\beta=1$



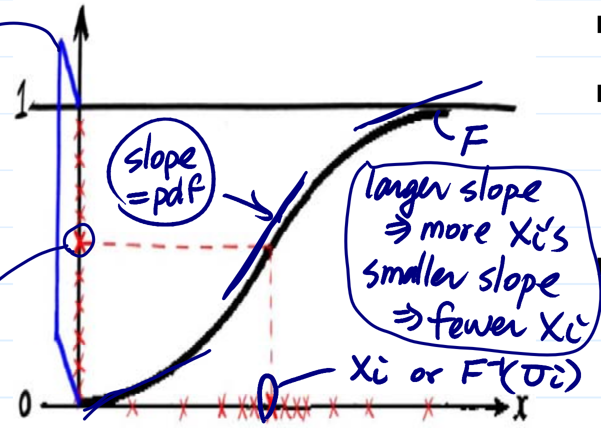
- **Theorem.** Let U be a uniform random variable on [0, 1] and F is a cdf which possesses a unique inverse F^{-1}. Let $X=F^{-1}(U)$, then the cdf of X is F.

$=g \Rightarrow g^{-1}=F$

Proof. $F_X(x) = F_U(F(x)) = P(U \le F(x)) = F(x)$.

- The 2 theorems are useful for pseudo-random number generation in computer simulation.

pdf of uniform $\alpha=0$ $\beta=1$

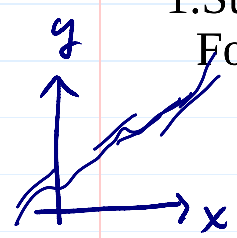


- X is r.v. $\Rightarrow$ $F(X)$ is r.v.
- $X_1, \ldots, X_n$: r.v.'s with cdf F $\Rightarrow$ $F(X_1), \ldots, F(X_n)$: r.v.'s with distribution Uniform(0, 1)
- $U_1, \ldots, U_n$: r.v.'s with distribution Uniform(0, 1) $\Rightarrow$ $F^{-1}(U_1), \ldots, F^{-1}(U_n)$: r.v.'s with cdf F

➢ The pdf of Y, denoted by f_Y

1. Suppose that g is a $\boxed{\textit{differentiable}}$ strictly increasing function. For $y \in R_Y$,

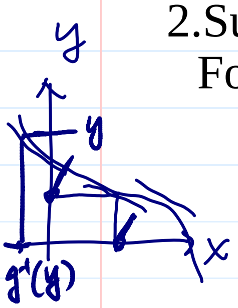
g^{-1} is also strictly increasing.



$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(g^{-1}(y))$$
$$= f_X(g^{-1}(y))\frac{dg^{-1}(y)}{dy} = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|.$$

2. Suppose that g is a $\boxed{\textit{differentiable}}$ strictly decreasing function. For $y \in R_Y$,

g^{-1} is also strictly decreasing.



$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}(1 - F_X(g^{-1}(y)))$$
$$= -f_X(g^{-1}(y))\frac{dg^{-1}(y)}{dy} = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|.$$

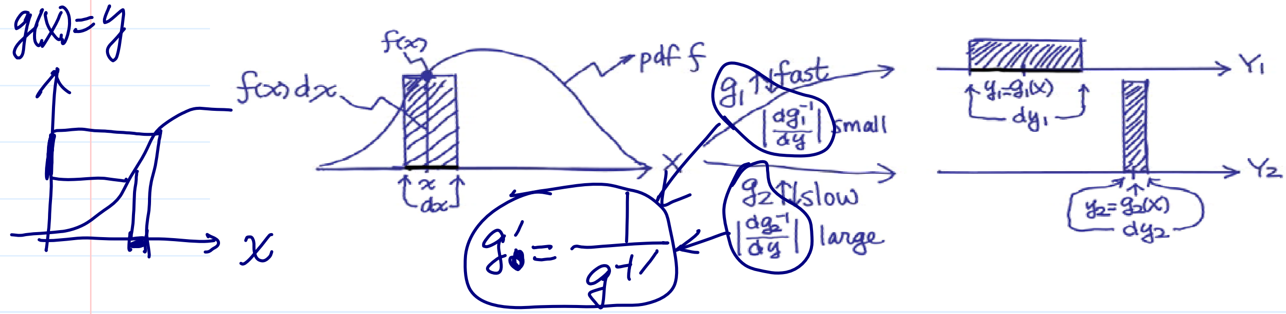
- **Theorem.** Let X be a continuous random variable with pdf f_X. Let $Y=g(X)$, where g is differentiable and strictly monotone. Then, the pdf of Y, denoted by f_Y, is

c.f. Thm in LNp.440

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|,$$

for y such that $y=g(x)$ for some x, and $f_Y(y)=0$ otherwise.

- **Q**: What is the role of $|dg^{-1}(y)/dy|$? How to interpret it?



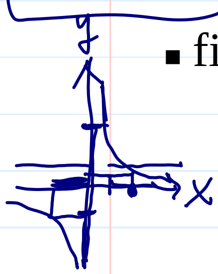
➢ Some Examples. Given the pdf f_X of random variable X,

- find the pdf f_Y of $Y=aX+b$, where $a\neq 0$. strictly monotone.

$$y = g(x) = ax + b \;\Rightarrow\; x = g^{-1}(y) = \frac{y-b}{a} \;\Rightarrow\; \left|\frac{d}{dy}g^{-1}(y)\right| = \frac{1}{|a|}$$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{|a|}$$

piecewise monotone no one-to-one



- find the pdf f_Y of $Y=1/X$.

$$y = g(x) = \frac{1}{x} \;\Rightarrow\; x = g^{-1}(y) = \frac{1}{y} \;\Rightarrow\; \left|\frac{d}{dy}g^{-1}(y)\right| = |-y^{-2}| = \frac{1}{y^2}$$

$$f_Y(y) = f_X\left(\frac{1}{y}\right)\cdot\frac{1}{y^2}$$

$y<0$,
$F_Y(y) = P(Y \le y)$
$= P(\frac{1}{y} \le X \le 0)$
$= F_X(0) - F_X(\frac{1}{y})$

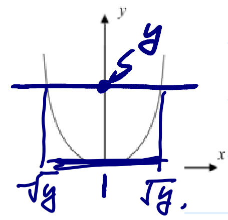
$y>0$,
$F_Y(y) = P(Y \le y)$
$= P(\{X \le 0\} \cup \{X \ge \frac{1}{y}\})$
$= F_X(0) + 1 - F_X(\frac{1}{y})$

c.f. Example in LN p.4-10

- find the cdf F_Y and pdf f_Y of $Y=X^2$.

piecewise monotone not one-to-one

□
$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= P(X \in (-\infty, \sqrt{y}]) - P(X \in (-\infty, -\sqrt{y})) \\ &= \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}), & \text{if } y > 0, \\ 0, & \text{if } y \le 0. \end{cases} \end{aligned}$$



□ For $y > 0$,

increasing part

decreasing part

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right] \\ &= f_X(\sqrt{y})\cdot\frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\cdot\frac{-1}{2\sqrt{y}} = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} \end{aligned}$$

For $y \le 0$, $f_Y(y) = 0$.

← piecewise monotone.

- Expectation, Mean, and Variance

c.f. Definition in LN p.4-11

➢ Definition. If X has a pdf f_X, then the *expectation* of X is defined by

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

Σ

prob. that X near x

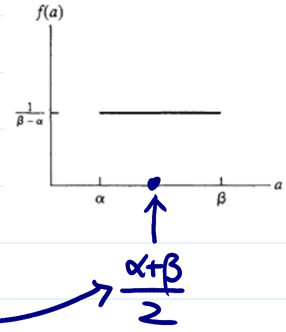
$\int_{-\infty}^{\infty} |x| f_X(x)\,dx < \infty$

provided that the integral converges absolutely.

c.f. equally likely Example LNp.4-11

- Example (Uniform Distributions). If

$$f_X(x)=\begin{cases}\frac{1}{\beta-\alpha}, & \text{if } \alpha<x\le\beta,\\ 0, & \text{otherwise,}\end{cases}$$



then

$$\begin{aligned}E(X) &= \int_\alpha^\beta x\cdot\frac{1}{\beta-\alpha}\,dx=\frac{1}{2}\cdot\frac{x^2}{\beta-\alpha}\Big|_\alpha^\beta\\ &= \frac{1}{2}\cdot\frac{\beta^2-\alpha^2}{\beta-\alpha}=\frac{\alpha+\beta}{2}.\end{aligned}$$

➢Some properties of expectation

c.f. discrete case LNp.4-13

- Expectation of Transformation. If $Y=g(X)$, then

$$E(Y)=\int_{-\infty}^{\infty}y\cdot f_Y(y)\,dy=\int_{-\infty}^{\infty}g(x)\cdot f_X(x)\,dx,$$

provided that the integral converges absolutely.

proof. (homework) $Y=aX+b$

c.f.

- Expectation of Linear Function. $E(aX+b)=a\cdot E(X)+b$, since

$$\begin{aligned}E(aX+b) &= \int_{-\infty}^{\infty}(ax+b)f_X(x)\,dx\\ &= a\int_{-\infty}^{\infty}x\cdot f_X(x)\,dx+b\int_{-\infty}^{\infty}f_X(x)\,dx\\ &= a\cdot E(X)+b.\end{aligned}$$

c.f. Definition in LNp.4-14

➢Definition. If X has a pdf f_X, then the expectation of X is also called the *mean* of X or f_X and denoted by μ_X, so that

$$\mu_X=E(X)=\int_{-\infty}^{\infty}x\cdot f_X(x)\,dx.$$

The *variance* of X is defined as

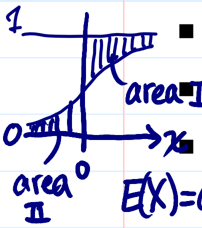
$$Var(X)=E[(X-\mu_x)^2]=\int_{-\infty}^{\infty}(x-\mu_X)^2\cdot f_X(x)\,dx,$$

and denoted by σ_X^2. The σ_X is called the *standard deviation*.

➢Some properties of mean and variance

check LNp.4-12, 4-16

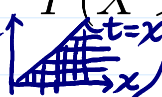
- The mean and variance for continuous random variables have the same intuitive interpretation as in the discrete case.
- $Var(X)=E(X^2)-[E(X)]^2$ c.f. LNp.4-16, discrete case. proof: (exercise)
- Variance of Linear Function. $Var(aX+b)=a^2\cdot Var(X)$ c.f. LNp.4-5
- Theorem. For a nonnegative continuous random variable X, discrete case proof: Recall HW (exercise) $EX=\sum_{n=1}^{\infty}P(X>n)$



E(X)=area I − area II

cdf of X

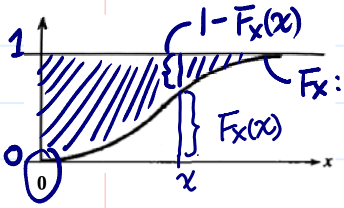
$$E(X)=\int_0^\infty 1-F_X(x)dx=\int_0^\infty P(X>x)dx.$$

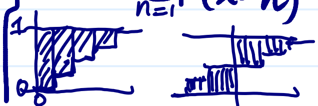


Proof.

$$\begin{aligned}E(X) &= \int_0^\infty x\cdot f_X(x)\,dx\\ &= \int_0^\infty\left(\int_0^x 1\,dt\right)f_X(x)\,dx\\ &= \int_0^\infty\int_0^x f_X(x)\,dt\,dx\\ &= \int_0^\infty\int_t^\infty f_X(x)dx\,dt=\int_0^\infty 1-F_X(t)dt.\end{aligned}$$

converge absolutely





- Example (Uniform Distributions)

$$E(X^2)=\int_{-\infty}^{\infty}\frac{x^2}{\beta-\alpha}dx=\frac{1}{3}\frac{x^3}{\beta-\alpha}\Big|_{\alpha}^{\beta}=\frac{\beta^3-\alpha^3}{3(\beta-\alpha)}=\frac{\beta^2+\alpha\beta+\alpha^2}{3}.$$

$$\begin{aligned}Var(X)&=E(X^2)-[E(X)]^2=\frac{\beta^2+\alpha\beta+\alpha^2}{3}-\left(\frac{\alpha+\beta}{2}\right)^2\\&=\frac{4(\beta^2+\alpha\beta+\alpha^2)-3(\beta^2+2\alpha\beta+\alpha^2)}{12}=\frac{(\beta-\alpha)^2}{12}.\end{aligned}$$

← If $\beta-\alpha$ large, variance large.

❖ **Reading**: textbook, Sec 5.1, 5.2, 5.3, 5.7

Some Common Continuous Distributions

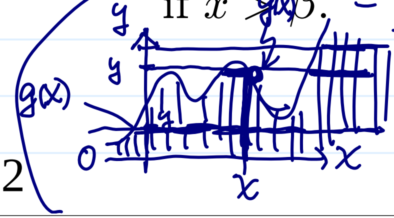
- Uniform Distribution

a uniform r.v. can only take values in a finite interval (α, β).

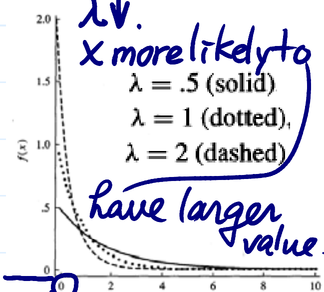
- Summary for $X \sim$ Uniform(α, β)
  - Pdf: $$f(x)=\begin{cases}1/(\beta-\alpha), & \text{if } \alpha<x\le\beta,\\ 0, & \text{otherwise},\end{cases}$$
  - Cdf: $$F(x)=\begin{cases}0, & \text{if } x\le\alpha,\\ (x-\alpha)/(\beta-\alpha), & \text{if } \alpha<x\le\beta,\\ 1, & \text{if } x>\beta.\end{cases}$$
  - Parameters: $-\infty<\alpha<\beta<\infty$
  - Mean: $E(X)=(\alpha+\beta)/2$
  - Variance: $Var(X)=(\beta-\alpha)^2/12$

$Y: g(X)$, Y nonnegative

$$\begin{aligned}E(Y)&=\int_0^\infty P(Y>y)\,dy\\&=\int_0^\infty P(g(X)>y)\,dy\\&=\int_0^\infty\int_{\{x:g(x)>y\}}f_X(x)\,dx\,dy\\&=\int_{-\infty}^{\infty}\int_0^{g(x)}f_X(x)\,dy\,dx\\&=\int_{-\infty}^{\infty}g(x)f_X(x)\,dx\end{aligned}$$



- Exponential Distribution
  - For $\lambda>0$, the function
$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & \text{if } x\ge 0,\\ 0, & \text{if } x<0,\end{cases}$$
is a pdf since (1) $f(x)\ge 0$ for all $x\in\mathbb{R}$, and (2)
$$\int_{-\infty}^{\infty}f(x)\,dx=\int_0^\infty\lambda e^{-\lambda x}\,dx=-e^{-\lambda x}\Big|_0^\infty=1.$$
    - The distribution of a random variable X with this pdf is called the *exponential* distribution with parameter λ.
  - The cdf of an exponential r.v. is $F(x)=0$ for $x<0$, and for $x\ge 0$,
$$F(x)=P(X\le x)=\int_0^x\lambda e^{-\lambda y}\,dy=-e^{-\lambda y}\Big|_0^x=1-e^{-\lambda x}.$$



$\lambda\downarrow$, X more likely to have larger value.

the pdf is discontinuous at $x=0$. $\frac{dF}{dx}$ does not exist at $x=0$.

$\frac{dF}{dx}$ (exercise)

  - Theorem. The mean and variance of an exponential distribution with parameter λ are
$$\mu=1/\lambda \quad\text{and}\quad \sigma^2=1/\lambda^2.$$

$E(X^2)-(EX)^2$

check LN p.5-18 for its definition

Proof.
$$\begin{aligned}E(X)&=\int_0^\infty x\lambda e^{-\lambda x}dx=\int_0^\infty\frac{y}{\cancel{\lambda}}(\cancel{\lambda}e^{-y})\frac{1}{\lambda}dy\\&=\frac{1}{\lambda}\int_0^\infty ye^{-y}dy=\frac{1}{\lambda}\Gamma(2)=\frac{1}{\lambda}.\\E(X^2)&=\int_0^\infty x^2\lambda e^{-\lambda x}dx=\int_0^\infty\left(\frac{y}{\lambda}\right)^2(\cancel{\lambda}e^{-y})\frac{1}{\cancel{\lambda}}dy\\&=\frac{1}{\lambda^2}\int_0^\infty y^2e^{-y}dy=\frac{1}{\lambda^2}\Gamma(3)=\frac{2}{\lambda^2}.\end{aligned}$$

$y=\lambda x \Rightarrow x=\frac{1}{\lambda}y$

$\frac{dx}{dy}=\frac{1}{\lambda}\Rightarrow dx=\frac{1}{\lambda}dy$

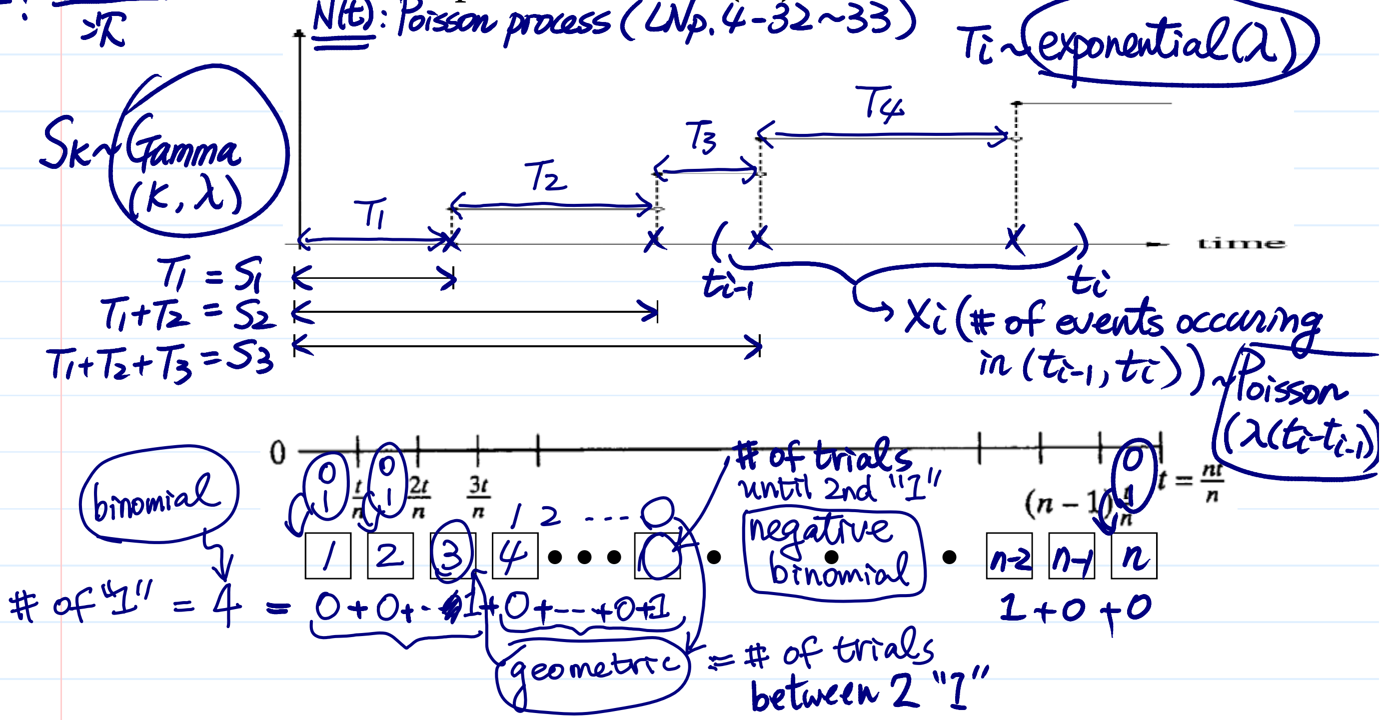
➢ Some properties

- The exponential distribution is often used to model the length of time until an event occurs or lifetime.

λ: $\frac{次}{單位時間}$

$\frac{1}{\lambda}$: $\frac{單位時間}{次}$

- ▫ The parameter λ is called the rate and is the average number of events that occur in unit time. This gives an intuitive interpretation of $E(X)=1/\lambda$.



- Theorem (relationship between exponential, gamma, and Poisson distributions, Sec. 9.1). Let
  1. $T_1, T_2, T_3, \ldots$, be independent and ~ exponential(λ),
  2. $S_k=T_1+\cdots+T_k$, $k=1, 2, 3, \ldots$,
  3. X_i be the number of S_k's that falls in the time interval $(t_{i-1}, t_i]$, $i=1, \ldots, m$.

X_1 X_2 / t_0 t_1 t_2 - -

Then, (i) $X_1, \ldots, X_m$ are independent,

(ii) X_i~Poisson($\lambda(t_i - t_{i-1})$),

(iii) S_k ~ Gamma(k, λ). (prove in chapter 7)

(iv) The reverse statement is also true.

→ $X_1, X_2, X_3, \ldots$ define $S_k \to T_i$
X_i indep Poisson $\Rightarrow T_i \sim$ exponential

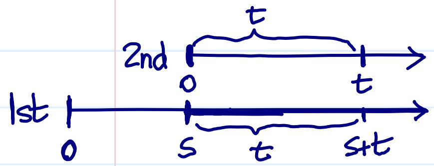
- ▫ The rate parameter λ is the same for both the Poisson and exponential random variable.

lifetime = X ~ exponential

- ▫ The exponential distribution can be thought of as the continuous analogue of the geometric distribution.

- Theorem. The exponential distribution (like the geometric distribution) is *memoryless*, i.e., for $s, t \geq 0$,

$$P(X > s+t | X > s) = P(X > t).$$



$P(X_1 > s+t | X_1 > s) = P(X_2 > t).$

Proof.

$$P(X > s+t | X > s) = \frac{P(\{X > s+t\} \cap \{X > s\})}{P(\{X > s\})} = \frac{P(\{X > s+t\})}{P(\{X > s\})}$$

$$= \frac{1 - F_X(s+t)}{1 - F_X(s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

cdf in LNp.5-14.

intuitive explanation: check the graph in LNp.5-15

- This means that the distribution of the waiting time to the next event remains the same regardless of how long we have already been waiting.
- This only happens when events occur (or not) totally at random, i.e., independent of past history. ← connection between exponential & geometrics
- Notice that it does not mean the two events $\{X > s+t\}$ and $\{X > t\}$ are independent.

Q: Why negative binomial does not possess the memoryless property?

→ if $\{X>s+t\}$ & $\{X>t\}$ independent
$P(X>s+t | X>t)$
$= P(X>s+t)$

➢ Summary for $X \sim$ Exponential(λ)

- Pdf: $$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$
- Cdf: $$F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$
- Parameters: $\lambda > 0$.
- Mean: $E(X) = 1/\lambda$.
- Variance: $Var(X) = 1/\lambda^2$.

Note:
① X = Y
② X dist = Y dist

- Gamma Distribution

➢ Gamma Function

- Definition. For $\alpha > 0$, the *gamma function* is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

- $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$ (exercise) check textbook, p228 exercise 5.21.
- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

Proof. By integration by parts, (f, g′)

$$\Gamma(\alpha+1) = \int_0^\infty \boxed{x^\alpha}\boxed{e^{-x}}\, dx$$
$$= -x^\alpha e^{-x}\Big|_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x}\, dx = \alpha\Gamma(\alpha).$$

- $\Gamma(\alpha) = (\alpha-1)!$ if α is an integer

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) = \cdots = (\alpha-1)(\alpha-2)\cdots\Gamma(1) = (\alpha-1)!$$

- $\Gamma(\alpha/2) = \frac{\sqrt{\pi}(\alpha-1)!}{2^{\alpha-1}(\frac{\alpha-1}{2})!}$ if α is an odd integer

$$\Gamma(\tfrac{\alpha}{2}) = (\tfrac{\alpha-2}{2})\Gamma(\tfrac{\alpha}{2}-1) = \cdots = (\tfrac{\alpha-2}{2 \cdot 2})(\tfrac{\alpha-4}{2 \cdot 2})\cdots\tfrac{1}{2}\Gamma(\tfrac{1}{2})$$

α−1, α−3, α−1/2, α−3/2

- Gamma function is a generalization of the factorial functions

➢ For $\alpha, \lambda > 0$, the function

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} \boxed{x^{\alpha-1} e^{-\lambda x}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

is a pdf since (1) $f(x) \geq 0$ for all $x \in \mathbb{R}$, and (2)

$y=\lambda x \Rightarrow x=\frac{y}{\lambda}$

$\frac{dx}{dy}=\frac{1}{\lambda} \Rightarrow dx=\frac{1}{\lambda}dy$

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y}\, dy = 1.$$

- The distribution of a random variable X with this pdf is called the *gamma* distribution with parameters α and λ.

➢ The cdf of gamma distribution can be expressed in terms of the incomplete gamma function, i.e., $F(x)$=0 for x<0, and for $x \geq 0$,

if α: integer

$$F(x) = \int_0^x \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y} dy = \frac{1}{\Gamma(\alpha)} \boxed{\int_0^{\lambda x} \underbrace{z^{\alpha-1}}_{f} \underbrace{e^{-z}}_{g'} dz} \equiv \frac{\gamma(\alpha, \lambda x)}{\Gamma(\alpha)}.$$

$F(x) = 1-\sum_{k=0}^{\alpha-1} e^{-\lambda x}\frac{(\lambda x)^k}{k!}$

$F_X(x) = 1-F_Y(\alpha-1)$, $X \sim$ Gamma(α,λ), $Y \sim$ Poisson(λx)

➢ Theorem. The mean and variance of a gamma distribution with parameter α and λ are

intuitive explanation

$$\mu = \alpha/\lambda \quad \text{and} \quad \sigma^2 = \alpha/\lambda^2.$$

$\alpha \cdot \left(\frac{1}{\lambda}\right)$ $\qquad$ $E(X^2)-(EX)^2$

Proof. $E(X) = \int_0^\infty x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx$

discrete time version

$$= \frac{\cancel{\lambda^\alpha}}{\cancel{\Gamma(\alpha)}} \frac{\Gamma(\alpha+1)}{\lambda^{\cancel{\alpha}+1}} \int_0^\infty \underbrace{\frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\lambda x}}_{\text{pdf of Gamma}(\alpha+1,\lambda)}\, dx = \frac{\alpha}{\lambda}.$$

$\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$

negative binomial (r, p) $\qquad$ binomial (n, p)

$$E(X^2) = \int_0^\infty x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx$$

$$= \frac{\cancel{\lambda^\alpha}}{\cancel{\Gamma(\alpha)}} \frac{\Gamma(\alpha+2)}{\lambda^{\cancel{\alpha}+2}} \int_0^\infty \underbrace{\frac{\lambda^{\alpha+2}}{\Gamma(\alpha+2)} x^{(\alpha+2)-1} e^{-\lambda x}}_{\text{pdf of Gamma}(\alpha+2,\lambda)}\, dx = \frac{\alpha(\alpha+1)}{\lambda^2}.$$

$\Gamma(\alpha+2) = (\alpha+1)\alpha\,\Gamma(\alpha)$

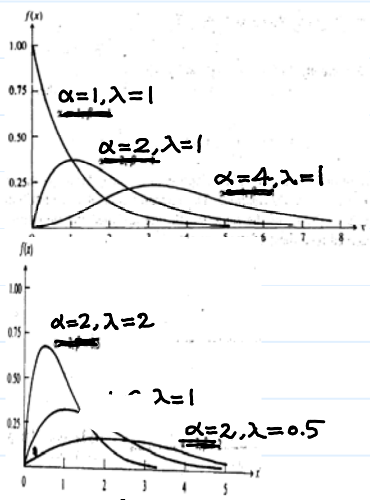
$P(X>n) = P(Y<r)$ LNp.4-23

11/24

- (exercise) $E(X^k) = \frac{\Gamma(\alpha+k)}{\lambda^k \Gamma(\alpha)}$, for $0 < k$, and

$$E\left(\frac{1}{X^k}\right) = \frac{\lambda^k \Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad \text{for } 0 < k < \alpha.$$

$=X^{-k}$



➢ Some properties

check graph in LNp.515

- The gamma distribution can be used to model the waiting time until a number of random events occurs

integer ⇒ α

will be proved in Chapter 7 using mgf

  - When α=1, it is exponential(λ)
  - $T_1, \ldots, T_n$: independent exponential(λ) r.v.'s ⇒ $T_1+\cdots+T_n \sim$ Gamma(n, λ)
  - Gamma distribution can be thought of as a continuous analogue of the negative binomial distribution
    - A summary

	Discrete Time Version	Continuous Time Version
number of events	binomial	Poisson
waiting time until 1 event occurs	geometric	exponential
waiting time until r events occur	negative binomial	gamma

waiting (memoryless)

~~waiting~~ time until αth events

- α is called shape parameter and λ scale parameter (**Q**: how to interpret α and λ from the viewpoint of waiting time?) 次/單位時間

- A special case of the gamma distribution occurs when $\alpha = n/2$ and $\lambda = 1/2$ for some positive integer n. This is known as the Chi-squared distribution with n degrees of freedom (Chapter 6) → sum of $(\text{Normal}(0,1))^2$

➢ Summary for $X \sim \text{Gamma}(\alpha, \lambda)$

- Pdf: $$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$
- Cdf: $F(x) = \gamma(\alpha, \lambda x)/\Gamma(\alpha)$.
- Parameters: $\alpha, \lambda > 0$.
- Mean: $E(X) = \alpha/\lambda$.
- Variance: $Var(X) = \alpha/\lambda^2$.

$\frac{1}{\binom{n}{x}} = \frac{x!(n-x)!}{n!}$

$\frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}$ || if α, β integers

c.f.

• Beta Distribution

➢ Beta Function: $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

(exercise) $\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty u^{\alpha-1}e^{-u}du \int_0^\infty v^{\beta-1}e^{-v}dv$

let $u = zt$, $v = z(1-t)$

➢ For $\alpha, \beta > 0$, the function

binomial pmf $\binom{n}{x}p^x(1-p)^{n-x} \overset{\text{c.f.}}{\longleftrightarrow}$ $$f(x) = \begin{cases} \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{=\frac{1}{B(\alpha,\beta)}} \underbrace{x^{\alpha-1}(1-x)^{\beta-1}}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

is a pdf (exercise).

- The distribution of a random variable X with this pdf is called the *beta* distribution with parameters α and β.

➢ The cdf of beta distribution can be expressed in terms of the *incomplete beta function*, i.e., $F(x)=0$ for $x<0$, $F(x)=1$ for $x>1$, and for $0 \leq x \leq 1$,

Beta(α,β)

$P(X \leq x) = P(Y < \beta)$

Binomial$(\alpha+\beta-1, 1-x)$

$$F(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \boxed{\int_0^x \underbrace{y^{\alpha-1}}_{f}\underbrace{(1-y)^{\beta-1}}_{g'}\,dy} \equiv \frac{B(x;\alpha,\beta)}{B(\alpha,\beta)}.$$

$$= \text{(exercise)} \sum_{i=\alpha}^{\alpha+\beta-1} \frac{(\alpha+\beta-1)!}{i!(\alpha+\beta-1-i)!} x^i (1-x)^{\alpha+\beta-1-i},$$

integration by parts.

for *integer values* of α and β.

$\sum_{u=\beta-1}^{0} \binom{\overset{n}{\alpha+\beta-1}}{\underset{u}{\alpha+\beta-1-i}} \underset{p}{(1-x)}^{\alpha+\beta-1-i} \cdot (1-(1-x))^{(\alpha+\beta-1)-(\alpha+\beta-1-i)}$

➢ Theorem. The mean and variance of a beta distribution with parameters α and β are

$$\mu = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \overset{E(X^2)-[E(X)]^2}{\sigma^2} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Proof.

$$E(X) = \int_0^\infty x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}} \frac{\overset{\alpha\Gamma(\alpha)}{\Gamma(\alpha+1)}\cancel{\Gamma(\beta)}}{\underset{(\alpha+\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)}} \int_0^\infty \underbrace{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)} x^{(\alpha+1)-1}(1-x)^{\beta-1}}_{\text{pdf of beta}(\alpha+1,\beta)}\,dx$$

$$= \frac{\alpha}{\alpha+\beta}.$$

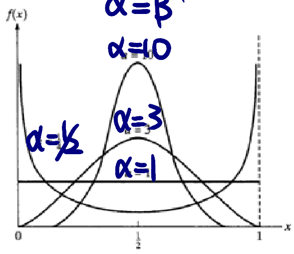
$$E(X^2) = \int_0^\infty x^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \int_0^\infty \underbrace{\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta)} x^{(\alpha+2)-1}(1-x)^{\beta-1}}_{\text{pdf of beta}(\alpha+2,\,\beta)}\, dx$$

$\Gamma(\alpha+2) = (\alpha+1)\alpha\Gamma(\alpha)$, $\Gamma(\alpha+\beta+2) = (\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}.$$

➢ Some properties



- When α=β=1, the beta distribution is the same as the uniform(0, 1).

$x^{\alpha-1}(1-x)^{\alpha-1}$

- Whenever α=β, the beta distribution is symmetric about x=0.5, i.e.,

$(0.5-\Delta)^{\alpha-1}(0.5+\Delta)^{\alpha-1}$

$(0.5+\Delta)^{\alpha-1}(0.5-\Delta)^{\alpha-1}$

$$f(0.5–\Delta)=f(0.5+\Delta).$$

When $\alpha=\beta$

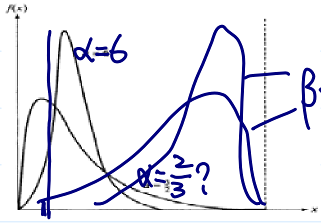
$E(X)=\frac{\alpha}{\beta+\alpha}=\frac{1}{2}$

$Var(X)=\frac{\alpha^2}{4\alpha^2(2\alpha+1)}$

↓ when α↑

- As the common value of α and β increases, the distribution becomes more peaked at x=0.5 and there is less probability outside of the central portion.

$\beta>\alpha$, $\frac{\alpha}{\alpha+\beta}=\frac{1}{20}$



skewed

- When β>α, values close to 0 become more likely than those close to 1; when β<α, values close to 1 are more likely than those close to 0 (**Q**: How to connect it with $E(X)$?) → $\frac{\alpha}{\alpha+\beta}$

➢ Summary for X ~ Beta(α, β)

- Pdf:
$$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & \text{if } 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}$$
- Cdf: $F(x) = B(x; \alpha, \beta)/B(\alpha, \beta)$.
- Parameters: α, β > 0.
- Mean: $E(X)$ = α/(α + β).
- Variance: $Var(X)$ = ~~[α(α + 1)]/[(α + β)(α + β +1)]~~. $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

• Normal Distribution

➢ For μ∈ℝ and σ>0, the function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty,$$

is a pdf since (1) $f(x) \ge 0$ for all $x \in \mathbb{R}$, and (2)

standardization

$\boxed{y=\frac{x-\mu}{\sigma}} \Rightarrow x=\sigma y+\mu$

$\frac{dx}{dy}=\sigma \Rightarrow dx=\sigma dy$

(*)

$$\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \frac{1}{\sqrt{2\pi}} \boxed{\int_{-\infty}^\infty e^{-\frac{y^2}{2}}\, dy} \equiv \frac{I}{\sqrt{2\pi}},$$

and
$$I^2 = \left(\int_{-\infty}^\infty e^{-\frac{x^2}{2}}\, dx\right)\left(\int_{-\infty}^\infty e^{-\frac{y^2}{2}}\, dy\right)$$

$\left.\begin{aligned} x &= r\sin\theta \\ y &= r\cos\theta \end{aligned}\right] \Rightarrow dxdy = r\,dr\,d\theta$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-\frac{x^2+y^2}{2}}\, dxdy = \int_0^\infty \int_0^{2\pi} e^{-\frac{r^2}{2}} r\, d\theta dr$$

$$= 2\pi \int_0^\infty r e^{-\frac{r^2}{2}}\, dr = -2\pi e^{-\frac{r^2}{2}}\Big|_0^\infty = 2\pi.$$

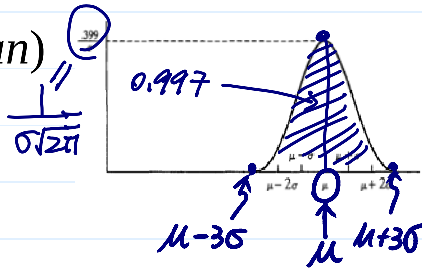
$\begin{vmatrix} dx/dr & dx/d\theta \\ dy/dr & dy/d\theta \end{vmatrix} = r$

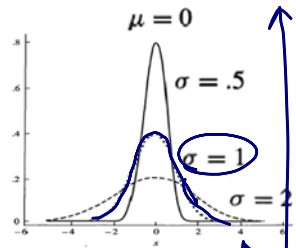
常態

- The distribution of a random variable X with this pdf is called the *normal* (*Gaussian*) distribution with parameters μ and σ, denoted by $N(\mu, \sigma^2)$.
- The normal pdf is a bell-shaped curve.
  - It is symmetric about the point μ, i.e., $f(\mu+\Delta)=f(\mu-\Delta)$ and falls off in the rate determined by σ.
  - The pdf has a maximum at μ (can be shown by differentiation) and the maximum height is $1/(\sigma\sqrt{2\pi})$.

Solve $\frac{df}{dx}=0 \Rightarrow x=\mu$

$\frac{d}{dx}(\ln f)=0 \Rightarrow x=\mu$





➢ The cdf of normal distribution does not have a close form.

➢ Theorem. The mean and variance of a $N(\mu, \sigma^2)$ distribution are μ and σ^2, respectively. $\overset{\shortparallel}{E(X^2)-[E(X)]^2}$

- μ: location parameter; σ^2: scale parameter

Proof. $E(X)=\int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \overset{(*)\text{ in LNp.5-24}}{=} \int_{-\infty}^{\infty}(\sigma y+\mu)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy$

$$= \frac{\sigma}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} ye^{-\frac{y^2}{2}}\,dy}_{-e^{-y^2/2}\big|_{-\infty}^{\infty}} + \mu\int_{-\infty}^{\infty}\underbrace{\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}}_{\text{pdf of } N(0,1)}\,dy$$

$$= \frac{\sigma}{\sqrt{2\pi}}\cdot 0+\mu\cdot 1=\mu.$$

$$E(X^2)=\int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \int_{-\infty}^{\infty}(\sigma y+\mu)^2\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy$$ by (*) in LNp.5-24

$$= \sigma^2\int_{-\infty}^{\infty} y^2\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy+\frac{2\mu\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ye^{-\frac{y^2}{2}}\,dy$$

$$+\mu^2\int_{-\infty}^{\infty}\underbrace{\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}}_{\text{pdf of } N(0,1)}\,dy$$

$$= \sigma^2\cdot 1+\frac{2\mu\sigma}{\sqrt{2\pi}}\cdot 0+\mu^2\cdot 1=\sigma^2+\mu^2.$$

$Var(X)=E(X^2)-[E(X)]^2$

$=\sigma^2+\mu^2-\mu^2=\sigma^2.$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{y}_{f}\underbrace{\left(ye^{-\frac{y^2}{2}}\right)}_{g'}\,dy=\frac{1}{\sqrt{2\pi}}y\left(-e^{-y^2/2}\right)\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\underbrace{\frac{1}{\sqrt{2\pi}}\left(+e^{-\frac{y^2}{2}}\right)}_{\text{pdf of } N(0,1)}\,dy$$

➢ Some properties

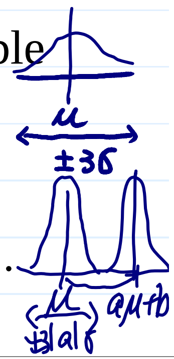
∵ Bell-shaped distribution

- Normal distribution is one of the most widely used distribution. It can be used to model the distribution of many natural phenomena. → e.g. height, weight, error, - -
- Theorem. Suppose that $X\sim N(\mu, \sigma^2)$. The random variable $Y=aX+b$, where $a\neq 0$, is also normally distributed with parameters $a\mu+b$ and $a^2\sigma^2$, i.e., $Y\sim N(a\mu+b, a^2\sigma^2)$. $\overset{\shortparallel}{\mu'}$

Recall
$E(Y)=a\,E(X)+b$
$Var(Y)=a^2\cdot Var(X)$

Proof. Thm in LNp.5-9

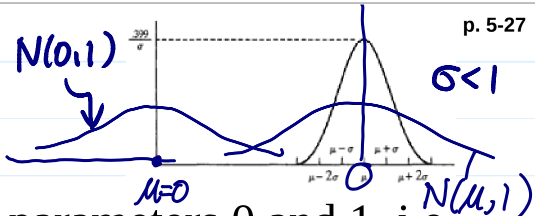
$$f_Y(y)\overset{\downarrow}{=}f_X\left(\frac{y-b}{a}\right)\frac{1}{|a|}=\frac{1}{\sqrt{2\pi}\underbrace{|a|\sigma}_{=\sigma'}}e^{-\frac{[y-(a\mu+b)]^2}{2\sigma^2a^2}}.$$ σ'^2



Standardization (標準化)

- Corollary. If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$$



is a normal random variable with parameters 0 and 1, i.e., $N(0, 1)$, which is called *standard normal distribution*.

Note: $E(Z)=0$, $Var(Z)=1$ for any r.v. X. But, X & Z may not belong to same distribution family in general

- The $N(0, 1)$ distribution is very important since properties of any other normal distributions can be found from those of the standard normal.

- *The cdf of $N(0, 1)$ is usually denoted by Φ.* ← no close form.

- Theorem. Suppose that $X \sim N(\mu, \sigma^2)$. The cdf of X is

$$P(X \le x) = P\left(\boxed{\frac{X-\mu}{\sigma}} \le \frac{x-\mu}{\sigma}\right) \quad F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

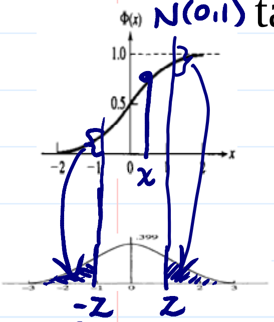
Proof. $F_X(x) = F_Z\left(\frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right).$ (Z: N(0,1))

- Example. Suppose that $X \sim N(\mu, \sigma^2)$. For $-\infty < a < b < \infty$,

$$\begin{aligned} P(a < X < b) &= P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) \\ &= P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right) = P\left(Z < \frac{b-\mu}{\sigma}\right) - P\left(Z < \frac{a-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right). \end{aligned}$$

(Z: N(0,1))

Φ: cdf of N(0,1)



pdf of N(0,1)

- Table 5.1 (textbook, p.201) gives values of Φ. To read the table:
  1. Find the first value of x up to the first place of decimal in the left hand column.
  2. Find the second place of decimal across the top row.
  3. The value of $\Phi(x)$ is where the row from the first step and the column from the second step intersect.

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
• • •										
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0) = 1/2$

$\Phi(0.22) = 0.5871$

$\Phi(3.36) = 0.9996$

- For the values greater than z=3.49, $\Phi(z) \approx 1$.
- For negative values of z, use $\Phi(-z)=1-\Phi(z)$

- Normal distribution plays a central role in the limit theorems of probability (e.g., CLT, chapter 8)

Normal approximation to the Binomial

an example of CLT

- Recall. Poisson approximation to Binomial (LNp.4-28, $n\uparrow$, $p\downarrow 0$)
- Theorem. Suppose that X_n~binomial(n, p). Define (X_n = sum of n independent Bernoulli(p))

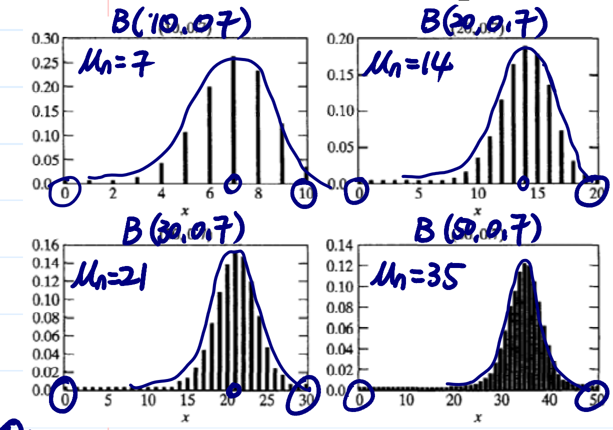
  $\mu_n = E(X_n) = np$, $\sigma_n^2 = Var(X_n) = np(1-p)$

  $$Z_n = (X_n - np)/\sqrt{np(1-p)}.$$

  (standardization)

  Then, as $n\to\infty$, the distribution of Z_n converge to the $N(0, 1)$ distribution, i.e., $F_{Z_n}(z) = P(Z_n \le z) \to \Phi(z)$.

  Proof. It is a special case of the CLT in Chapter 8.



- Plot the pmf of X_n~binomial(n, p)
- Superimpose the pdf of Y_n~N(μ_n, σ_n^2) with $\mu_n = np$ and $\sigma_n^2 = np(1-p)$.
- When n is sufficiently large, the normal pdf approximates the binomial pmf.
- $Z_n \overset{d}{\approx} (Y_n - \mu_n)/\sigma_n$ ~N(0,1)

  c.f. $Z_n \approx (Y_n - \mu_n)/\sigma_n$

- The size of n to achieve a good approximation depends on the value of p.
  - For p near 0.5 $\Rightarrow$ moderate n is enough
  - For p close to zero or one $\Rightarrow$ require much larger n

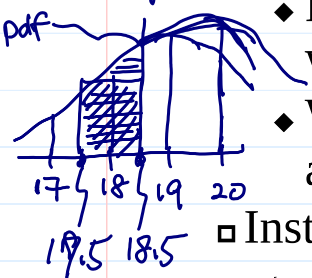
Q: $X_n' = X_n/2$, $Y_n' = Y_n/2$] pdf ≈ pmf? cdf ≈ cdf?

Q: Poisson distribution ≈ Normal distribution

Ans. yes, when λ is large.

- Continuity Correction
  - **Q**: Why need continuity correction? Ans. The binomial(n, p) is a discrete random variable and we are approximating it with a continuous random variable.
    - For example, suppose X~binomial(50, 0.4) and we want to find $P(X=18)$, which is larger than 0.
    - With the normal pdf, however, $P(Y=18)=0$ since we are using a continuous distribution ($Y \sim N(20, 12)$)
  - Instead, we make a continuity correction,

pmf: value → prob.
pdf: area → prob.



$$P(X=18) = P(17.5 < X < 18.5) = P\left(\frac{17.5-(50\cdot 0.4)}{\sqrt{50\cdot 0.4\cdot 0.6}} < Z_n < \frac{18.5-(50\cdot 0.4)}{\sqrt{50\cdot 0.4\cdot 0.6}}\right)$$

$$\overset{\text{by CLT}}{\approx} P\left(\frac{17.5-(50\cdot 0.4)}{\sqrt{50\cdot 0.4\cdot 0.6}} < Z < \frac{18.5-(50\cdot 0.4)}{\sqrt{50\cdot 0.4\cdot 0.6}}\right) \quad (Z \sim N(0,1))$$

$$= P\left(-\frac{2.5}{\sqrt{12}} < Z < -\frac{1.5}{\sqrt{12}}\right) = P\left(Z < -\frac{1.5}{\sqrt{12}}\right) - P\left(Z < -\frac{2.5}{\sqrt{12}}\right)$$

$$= \Phi\left(-\frac{1.5}{\sqrt{12}}\right) - \Phi\left(-\frac{2.5}{\sqrt{12}}\right) = \left(1 - \Phi\left(\frac{1.5}{\sqrt{12}}\right)\right) - \left(1 - \Phi\left(\frac{2.5}{\sqrt{12}}\right)\right)$$

$$= \Phi\left(2.5/\sqrt{12}\right) - \Phi\left(1.5/\sqrt{12}\right).$$

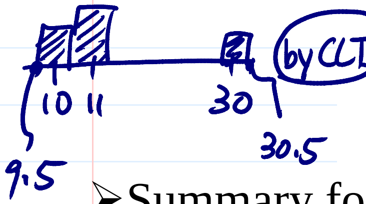
and can obtain the approximate value from Table 5.1.

Similary, $P(X \ge 30) = P(X > 29.5) = P\left(Z_n > \frac{29.5-(50\cdot 0.4)}{\sqrt{12}}\right)$

$$\overset{\text{by CLT}}{\approx} P(Z > 9.5/\sqrt{12}) = 1 - \Phi(9.5/\sqrt{12}). \quad (Z \sim N(0,1))$$

and

$$P(10 \le X \le 30) = P(9.5 < X < 30.5)$$



$$\overset{\text{(by CLT)}}{\approx} \; P\left(\frac{9.5-(50\cdot 0.4)}{\sqrt{12}} < Z_n < \frac{30.5-(50\cdot 0.4)}{\sqrt{12}}\right)$$

$$\approx \; P(-10.5/\sqrt{12} < Z < 10.5/\sqrt{12})$$

$$= \Phi(10.5/\sqrt{12}) - \underbrace{\Phi(-10.5/\sqrt{12})}_{=1-\Phi\left(\frac{10.5}{\sqrt{12}}\right)} = 2\cdot\Phi(10.5/\sqrt{12}) - 1.$$

➢ Summary for $X \sim$ Normal(μ, σ^2)

- Pdf: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty,$
- Cdf: no close form, but usually denoted by $\Phi((x-\mu)/\sigma)$.
- Parameters: $\mu \in \mathbb{R}$ and $\sigma > 0$.
- Mean: $E(X) = \mu$.
- Variance: $Var(X) = \sigma^2$.

(12/1)

• Weibull Distribution

$y = \left(\frac{x-\nu}{\alpha}\right)^\beta \Rightarrow x = \alpha y^{\frac{1}{\beta}} + \nu$

$\frac{dx}{dy} = \frac{\alpha}{\beta} y^{\frac{1}{\beta}-1} \Rightarrow dx = \frac{\alpha}{\beta} y^{\frac{1}{\beta}-1} dy$

(△)

➢ For α, $\beta > 0$ and $\nu \in \mathbb{R}$, the function

$$f(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-\nu}{\alpha}\right)^\beta}, & \text{if } x \ge \nu, \\ 0, & \text{if } x < \nu, \end{cases}$$

is a pdf since (1) $f(x) \ge 0$ for all $x \in \mathbb{R}$, and (2)

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_\nu^\infty \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-\nu}{\alpha}\right)^\beta} dx$$
$$= \int_0^\infty e^{-y}\, dy = -e^{-y}\Big|_0^\infty = 1.$$

- The distribution of a random variable X with this pdf is called the *Weibull* distribution with parameters α, β, and ν .

➢ (exercise) The cdf of Weibull distribution is

by (△) in LNp5-31

$$F(x) = \begin{cases} 1 - e^{-\left(\frac{x-\nu}{\alpha}\right)^\beta}, & \text{if } x \ge \nu, \\ 0, & \text{if } x < \nu. \end{cases}$$

➢ Theorem. The mean and variance of a Weibull distribution with parameter α, β, and ν are

$$\mu = \alpha\Gamma\left(1+\frac{1}{\beta}\right) + \nu \quad \text{and}$$

$$E(X^2)-[E(X)]^2 = \sigma^2 = \alpha^2\left\{\Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\}.$$

Proof. $E(X) = \int_v^\infty x \cdot \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-\nu}{\alpha}\right)^\beta} dx$

$$= \int_0^\infty (\alpha y^{1/\beta} + \mu) e^{-y}\, dy$$
$$= \alpha \int_0^\infty y^{(1/\beta+1)-1} e^{-y}\, dy + \mu \int_0^\infty e^{-y}\, dy = \alpha\Gamma\left(\frac{1}{\beta}+1\right) + \mu$$

$$E(X^2) = \int_v^\infty x^2 \cdot \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-\nu}{\alpha}\right)^\beta} dx$$
$$= \int_0^\infty (\alpha y^{1/\beta} + \mu)^2 e^{-y}\, dy$$
$$= \alpha^2 \int_0^\infty y^{(2/\beta+1)-1} e^{-y}\, dy + 2\alpha\mu \int_0^\infty y^{(1/\beta+1)-1} e^{-y}\, dy + \mu^2 \int_0^\infty e^{-y}\, dy$$
$$= \alpha^2\Gamma\left(\frac{2}{\beta}+1\right) + 2\alpha\mu\Gamma\left(\frac{1}{\beta}+1\right) + \mu^2$$

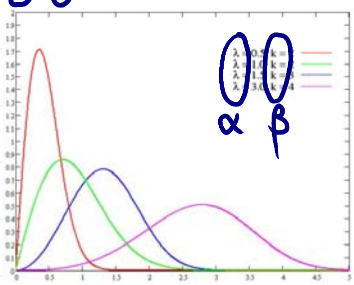
- Some properties
  - Weibull distribution is widely used to model lifetime $\xleftrightarrow{\text{Cf.}}$ exponential.
  - α: scale parameter; β: shape parameter; ν: location parameter
  - Theorem. If X~exponential(λ), then

$$Y = \alpha\,(\lambda X)^{1/\beta} + \nu$$

, Note: $\lambda X \sim$ exponential(1)

is distributed as Weibull with parameter α, β, and ν (exercise).

$\nu=0$



α β

- Cauchy Distribution
  - For $\mu \in \mathbb{R}$ and $\sigma>0$, the function

$$f(x) = \frac{\sigma}{\pi}\frac{1}{\sigma^2+(x-\mu)^2}, \quad -\infty < x < \infty,$$

is a pdf since (1) $f(x) \geq 0$ for all $x \in \mathbb{R}$, and (2)

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{\sigma}{\pi}\frac{1}{\sigma^2+(x-\mu)^2}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{1+y^2}\,dy = \frac{1}{\pi}\tan^{-1}(y)\Big|_{-\infty}^{\infty} = 1.$$

$y=\frac{x-\mu}{\sigma}$

$\Rightarrow x=\sigma y+\mu$

$\frac{dx}{dy}=\sigma \Rightarrow dx=\sigma dy$

  - The distribution of a random variable X with this pdf is called the *Cauchy* distribution with parameters μ and σ, denoted by Cauchy(μ, σ).

- The cdf of Cauchy distribution is

$$F(x) = \int_{-\infty}^{x} \frac{\sigma}{\pi}\frac{1}{\sigma^2+(y-\mu)^2}\,dy = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-\mu}{\sigma}\right)$$

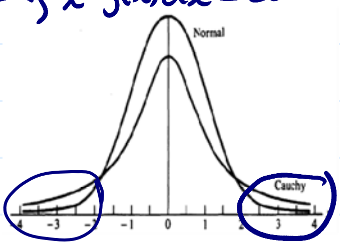
for $-\infty<x<\infty$. (exercise)

- The mean and variance of Cauchy distribution do not exist because the integral does not converge absolutely.

$\int |x| f(x)dx = \infty$

$\int x^2 f(x)dx = \infty$

- Some properties
  - Cauchy is a $\boxed{\text{heavy tail}}$ distribution
  - μ: location parameter; σ: scale parameter
  - Theorem. If X~Cauchy(μ, σ), then $aX+b$~Cauchy($a\mu+b$, $|a|\sigma$). (exercise)



Note: a pdf $f(x) \downarrow 0$, when $x \to \infty$ or $x \to -\infty$

But $f(x) \downarrow 0$ how fast?

Cf. Normal distribution

❖ **Reading**: textbook, Sec 5.4, 5.5, 5.6