Conditional Probability

- **Q**: Should the following probabilities be different?
  - ➢Event = 王建民本季戰績至少獲得6次勝投
    - P=?? in the beginning of the season
    - P=?? in the middle of the season
    - P=?? now
  - ➢Event = rain tomorrow
    - P=?? if no information about where you are staying
    - P=?? if you are staying in a desert
    - P=?? if a typhoon will hit the place you stay tomorrow
- **Q**: What causes the differences?
  - ➢For an event, *new information* (i.e., some other event has occurred) could change its probability
  - ➢We call the altered probability a conditional probability
- Mathematical Definition: If A and B are two events in a sample space Ω and $P(A)>0$, then

$$P(B|A) \equiv \frac{P(A \cap B)}{P(A)}$$

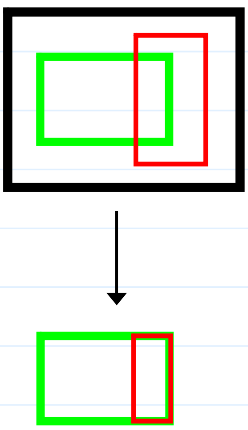
is called the *conditional probability* of B given A.

➢In the classical model,

$$P(A) = \#A/\#\Omega \quad \text{and} \quad P(A \cap B)=\#(A\cap B)/\#\Omega$$

$$\Rightarrow P(B|A) = \frac{\#(A \cap B)/\#\Omega}{\#A/\#\Omega} = \frac{\#(A \cap B)}{\#A}.$$

- Example: A family is known to have 2 children, *at least one of whom is a girl*. **Q**: Probability that the other is a boy=??
  - $\Omega = \{bb, bg, gb, gg\}$
  - $A=\{bg, gb, gg\}$ and $B=\{bb, bg, gb\}$
  - $P(B|A) = \#(A\cap B)/\#A = 2/3$
- Note: $\#\Omega$ is reduced to $\#A$.

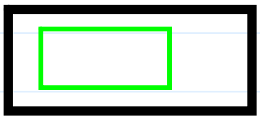


➢In effect by conditioning,

- we are restricting the sample space from Ω to A, i.e., $\Omega \rightarrow A$,
- and, for an arbitrary event B in Ω to occur when A has occurred, we need that both A and B occur together, i.e., $B \rightarrow B \cap A$.

➢The division by $P(A)$ in the definition above rescales all probabilities from the entire sample space Ω to being relative to the new sample space A

- $P(B|A)$ is a probability measure defined on B, but not A.
  - $P(\cdot|A)$ satisfies the 3 axioms of probability (exercise)

$P(\cdot)$  $P(\cdot|A)$ 

  - Any propositions developed in Chapter 2 for probability measures can be applied on $P(\cdot|A)$. (For example, $P(B^c|A)=1-P(B|A)$.)

- 3 Useful Formulas for Calculating Probabilities (for 2-events case)

1. If $P(A)>0$, then $P(A \cap B) = P(A)P(B|A)$.

2. If $0<P(A)<1$, then

$$P(B) = P(A)P(B|A) + P(A^c)P(B|A^c).$$

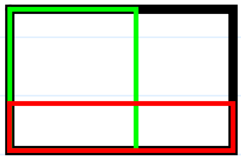
3. If $0<P(A)<1$ and $P(B)>0$, then

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}.$$

- Example (Urn Problem)
  - The Story. n balls sequentially and randomly chosen, *without replacement*, from an urn containing R red and $N-R$ white balls ($n \le N$). **Q**: Given that k of the n balls are red ($k \le R$), probability that the 1st ball chosen is red = ??
  - Let A={k of the n balls are red}
    B={1st ball chosen is red}
  - Method 1: $P(B|A) = \binom{n-1}{k-1} \Big/ \binom{n}{k} = k/n$
  - Method 2:
    - $P(A) = \left[\binom{R}{k} \times \binom{N-R}{n-k}\right] \Big/ \binom{N}{n}$
    - $P(A \cap B) = P(B)P(A|B) = \frac{R}{N} \times \frac{\binom{R-1}{k-1} \times \binom{N-R}{n-k}}{\binom{N-1}{n-1}}$
    - $P(B|A) = P(A \cap B)/P(A) = k/n$

➢Example (Diagnostic Tests)

- The Story. A diagnostic test for a *rare* disease (e.g., an Xray for lung cancer) is part of a routine physical exam.



- Let Ω = {the whole population of Taiwan}

  D={disease is present}

  E={test indicates disease present}

- Suppose that $P(D)=0.001$, $P(E|D)=0.98$, $P(E|D^c)=0.01$

  Q: Do you think the test is effective?

- Let us examine it from an alternative viewpoint

$$\begin{aligned} P(E) &= P(D)P(E|D) + P(D^c)P(E|D^c) \\ &= 0.001 \times 0.98 + 0.999 \times 0.01 = 0.01097. \end{aligned}$$

$$P(D|E) = \frac{P(D)P(E|D)}{P(D)P(E|D) + P(D^c)P(E|D^c)} = \frac{0.00098}{0.01097} = 0.0893.$$

$$P(D^c|E) = 1 - P(D|E) = 0.9107$$

Q: Now, do you still think the test is effective?

Q: But, why the 2 interpretations so different? What causes it?

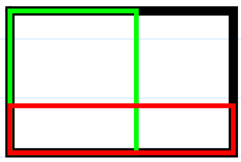
- The probability of D increased by a factor of roughly 90 ($0.001 \to 0.0893$) when E occurs, but 0.0893 is still **small**

- The $P(E|D^c)$ (=0.01) and $P(E^c|D)$ (=1−$P(E|D)$=0.02) are called the *false positive* and *false negative rates*, respectively.

➢Example (Sampling Experiments): An urn contains R red balls and N–R white balls. Sample 2 balls from the urn.

- A = {red on the first draw}

  B = {red on the second draw}



- Sampling *Without Replacement*:

$$P(A) = \frac{R(N-1)}{N(N-1)} = \frac{R}{N}, \qquad P(A \cap B) = \frac{R(R-1)}{N(N-1)},$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{(R(R-1))/(N(N-1))}{R/N} = \frac{R-1}{N-1}.$$

Similarly, $P(B|A^c) = \dfrac{R}{N-1}$. (exercise)

$$\begin{aligned} P(B) &= P(A)P(B|A) + P(A^c)P(B|A^c) \\ &= \frac{R}{N} \cdot \frac{R-1}{N-1} + \frac{N-R}{N} \cdot \frac{R}{N-1} \\ &= \frac{R^2 - R + NR - R^2}{N(N-1)} = \frac{R(N-1)}{N(N-1)} = \frac{R}{N}. \end{aligned}$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{(R(R-1))/(N(N-1))}{R/N} = \frac{R-1}{N-1}.$$

- Notes:
  - The probabilities are proportional to # of red balls left
  - $P(A|B) = P(B|A) \Rightarrow$ Symmetry.
- Sampling *With Replacement*:

$$P(A) = \frac{R}{N}\cdot\frac{N}{N} = \frac{R}{N}, \quad P(B) = \frac{N}{N}\cdot\frac{R}{N} = \frac{R}{N},$$

$$P(A\cap B) = \frac{R}{N}\cdot\frac{R}{N} = \frac{R^2}{N^2}, \quad P(B|A) = \frac{R^2/N^2}{R/N} = \frac{R}{N} = P(B).$$

- Extensions of the 3 Useful Formulas (for m-events case)

1.(Multiplication Law) If $A_1, \ldots, A_m$ are events for which $P(A_1\cap\cdots\cap A_{m-1}) > 0$, then

$$P(A_1\cap\cdots\cap A_m) = P(A_1)P(A_2|A_1)P(A_3|A_1\cap A_2)\cdots P(A_m|A_1\cap\cdots\cap A_{m-1}).$$

- Example (Birthday Problem, LNp.2-3). Let A_k be the event that the k^{th} birthday differs from the first $k-1$. Then, $P(A_1)=1$,
  - $P(A_k|A_1\cap\cdots\cap A_{k-1}) = \frac{365-k+1}{365}$
  - $P(A_1\cap\cdots\cap A_n) = P(A_1)P(A_2|A_1)\cdots P(A_n|A_1\cap\cdots\cap A_{n-1})$

$$= \prod_{k=1}^{n}\frac{365-k+1}{365} = \frac{365!}{365^n\cdot(365-n)!} = \frac{(365)_n}{365^n}.$$

- Example (Matching Problem, LNp.2-11). **Q**: Probability that exactly k of n members have matches = ??
  - Let Ω be all permutations $\omega = (i_1, \ldots, i_n)$ of 1, 2, …, n.
  - Let $A_j = \{\omega: i_j = j\}$ and

$$A = \bigcup_{1\le j_1<\cdots<j_k\le n}(A_1^c\cap\cdots\cap A_{j_1-1}^c\cap A_{j_1}\cap A_{j_1+1}^c\cap\cdots\cap A_n^c)$$

  - By symmetry, $P(A) = \binom{n}{k}\times P(A_1\cap\cdots\cap A_k\cap A_{k+1}^c\cap\cdots\cap A_n^c)$
  - Let $E = A_1\cap\cdots\cap A_k$ and $G = A_{k+1}^c\cap\cdots\cap A_n^c$
  - Then, $P(E\cap G) = P(E)P(G|E)$, where

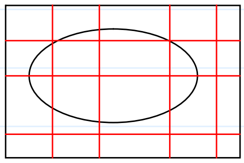
$$P(E) = P(A_1)P(A_2|A_1)\cdots P(A_k|A_1\cap\cdots\cap A_{k-1})$$
$$= \frac{1}{n}\times\frac{1}{n-1}\times\cdots\times\frac{1}{n-k+1} = \frac{(n-k)!}{n!} = \frac{1}{(n)_k}$$

and, $P(G|E) = \sum_{i=0}^{n-k}(-1)^i\frac{1}{i!} \equiv p_{n-k}$

- $P(A) = \binom{n}{k}\frac{1}{(n)_k}p_{n-k} = \frac{p_{n-k}}{k!} \approx \frac{e^{-1}}{k!}$, when n is large

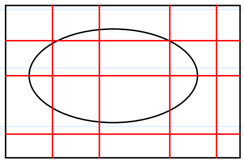
2.(Law of Total Probability) Let $A_1, \ldots, A_m$ be a partition of Ω and $P(A_i)>0$, $i=1, \ldots, m$, then for any event $B\subset\Omega$,

$$P(B) = \sum_{i=1}^m P(A_i)P(B|A_i).$$



3.(Bayes' Rule) Let $A_1, \ldots, A_m$ be a partition of Ω and $P(A_i)>0$, $i=1, \ldots, m$. If B is an event such that $P(B)>0$, then for $1\leq j\leq m$,

$$P(A_j|B) = \frac{P(A_j)P(B|A_j)}{\sum_{i=1}^m P(A_i)P(B|A_i)}.$$



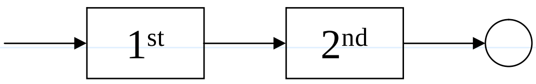
- From Bayesians' viewpoint,

  $P(A_j)$=probability of A_j *before* B occurs $\rightarrow$ *prior* prob.

  $P(A_j|B)$=probability of A_j *after* B occurs $\rightarrow$ *posterior* prob.

  $\Rightarrow$ The Bayes' rule tells how to update the probabilities of A_j in light of the new information (i.e., B occurs)

- An Application of Bayes' Rule. Suppose that a random experiment consists of two random stages



  - The probabilities of the 2nd-stage results depend on what happened in the 1st stage
  - We never see the result of the 1st stage, only the final result
  - We may be interested in finding the probability for outcomes in the 1st stage given the final result

➢ Example (Gold Coins):

- The Story.
  - Box 1 contains 2 silver coins.
  - Box 2 contains 1 gold and 1 silver coin.
  - Box 3 contains 2 gold coins.
  - Experiment: (i) Select a box at random and, (ii) Examine the 2 coins in order (assuming all choices are equally likely at each stage)
- **Q**: Given that 1st coin is gold, what is the probability that Box k is selected, k=1, 2, 3?
  - Let A_k= {Box k is selected}, B = {1st coin is gold},
  - $$P(B|A_k) = \begin{cases} 0, & \text{if } k=1, \\ 1/2, & \text{if } k=2, \\ 1, & \text{if } k=3. \end{cases}$$
    $$P(B) = \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot 1 = \frac{1}{2}.$$
  - $$P(A_1|B) = \frac{(1/3)\cdot 0}{1/2} = 0.$$
    Similarly, $P(A_2|B) = 1/3$, $P(A_3|B) = 2/3$.

- **Q**: Given that 1st coin is gold, what is the probability that 2nd coin is gold?
  - Let C = {2nd coin is gold}. $P(B\cap C|A_k) = \begin{cases} 0, & \text{if } k=1, \\ 0, & \text{if } k=2, \\ 1, & \text{if } k=3. \end{cases}$
  - $$P(B\cap C) = \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 = \frac{1}{3}.$$
    $$P(C|B) = \frac{P(B\cap C)}{P(B)} = \frac{1/3}{1/2} = \frac{2}{3}.$$

➢ **Q**: Why are the 3 formulas useful in calculating probabilities? (Note: They all benefit from conditional probabilities.)

Ans: (i) 繁→簡&簡&···&簡;

(ii) 簡=conditioning because the sample space is reduced from Ω to a smaller set. (For example, in many cases, $P(B|A_j)$'s are known or are easier to find)

- Odds and Conditional Odds

➢ The odds of an event A:

$$o(A) \equiv \frac{P(A)}{P(A^c)} = \frac{P(A)}{1-P(A)}$$

➢The odd of event B given A:

$$o(B|A) \equiv \frac{P(B|A)}{P(B^c|A)}$$

and

$$o(B|A) = o(B) \times \frac{P(A|B)}{P(A|B^c)}$$

❖ **Reading**: textbook, Sec 3.1, 3.2, 3.3, 3.5

Independence

- Definition (independence for 2-events case): Two events A and B are said to be *independent* if and only if

$$P(A \cap B) = P(A)P(B).$$

Otherwise, they are said to be *dependent*.

➢Notes. For independent events A and B, if $P(A) > 0$, then

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B),$$

similarly, if $P(B) > 0$, $P(A|B) = P(A)$.

Q: How to interpret the equality of the conditional and unconditional probabilities?

➢Example (Sampling 2 balls, LNp.3-6) A and B were independent for sampling with replacement, but dependent for sampling without replacement.

➢Example (Cards): If a card is selected from a standard deck, let

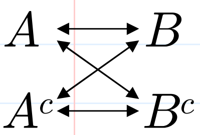
- A = {ace} and B={spade}. Then,
- $P(A) = \frac{4}{52} = \frac{1}{13}$, $P(B) = \frac{13}{52} = \frac{1}{4}$,

  $P(A \cap B) = \frac{1}{52} = P(A)P(B)$

  $\Rightarrow$ Face and Suit are independent

➢Theorem (Independence and Complements, 2-events case).

If A and B are independent, then so are A^c and B.



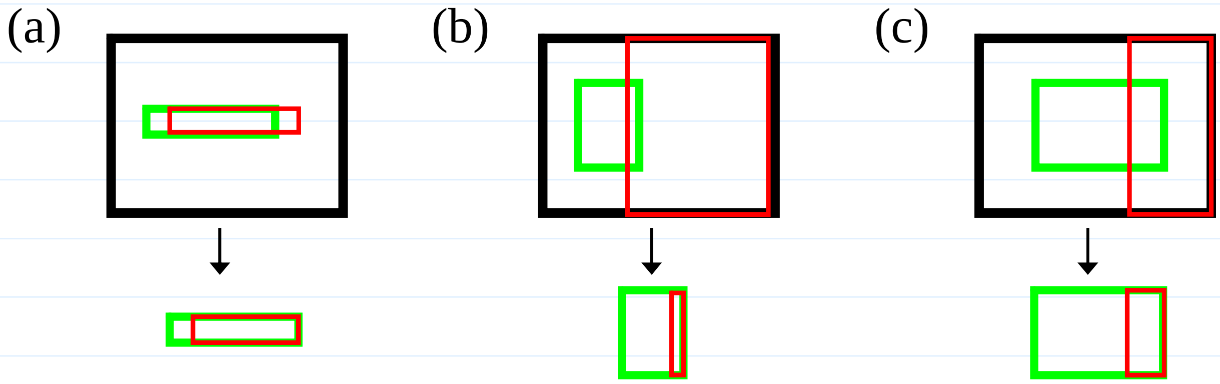
- Corollary: If A and B^c are independent, as are A^c and B^c

- Corellary: If A and B are independent and $0<P(A)<1$, $0<P(B)<1$, then

$$P(B)=P(B|A)=P(B|A^c),$$
$$P(B^c)=P(B^c|A)=P(B^c|A^c),$$
$$P(A)=P(A|B)=P(A|B^c),$$
$$P(A^c)=P(A^c|B)=P(A^c|B^c).$$

Q: What do these equalities say?

➢ **Q**: Which of the following graphs represents "green and red events are independent"?

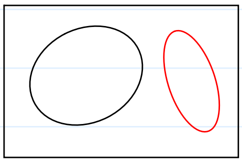


Q: Let green event={graduate from Tsing-Hua University}, red event={your future dream will come true}.

Which of the graphs would you prefer?

➢Theorem (Independence and Mutually Exclusive). If A and B are mutually exclusive and $P(A)>0$, $P(B)>0$, then A and B are dependent since

$$P(B|A)=0\neq P(B).$$



- Definition (independence for n-events case). Events $A_1, \ldots, A_n$ are said to be *pairwise independent* iff

$$P(A_i\cap A_j)=P(A_i)P(A_j),$$

for all $1\le i<j\le n$; $A_1, \ldots, A_n$ are said to be *mutually independent* iff

$$P(A_i\cap A_j)=P(A_i)P(A_j),\quad \text{for } 1\le i<j\le n,$$
$$P(A_i\cap A_j\cap A_k)=P(A_i)P(A_j)P(A_k),\quad \text{for } 1\le i<j<k\le n,$$
$$\ldots,$$
$$P(A_{i_1}\cap\cdots\cap A_{i_k})=P(A_{i_1})\cdots P(A_{i_k}),\quad \text{for } 1\le i_1<\cdots<i_k\le n,$$

where $k=2, \ldots, n$.

➢Note:

- Mutual independence implies pairwise independence; but, the converse statement is usually not true.
- "n events are independent" means "mutually independent"

➢ Example (Sampling With Replacement)

- A sample of n balls is drawn with replacement from an urn containing R red and N–R white balls
- Let A_k={red on the k^{th} draw}, then $P(A_k)=R/N$, k=1, …, n.
- For all $1\le i_1<\cdots<i_k\le n$, where k=2, …, n,

$$P(A_{i_1}\cap\cdots\cap A_{i_k})=\frac{R^k N^{n-k}}{N^n}=\left(\frac{R}{N}\right)^k=P(A_{i_1})\cdots P(A_{i_k}),$$

$\Rightarrow A_1, \ldots, A_n$ are mutually independent

➢ Example. Draw one card from a standard deck.

- Let A={Spades or Clubs},
  B={Hearts or Clubs},
  C={Diamonds or Clubs}.
- $P(A)=26/52=1/2,$ similarly, $P(B)=P(C)=1/2.$
- $P(A\cap B)=P(\{\text{Clubs}\})=\frac{13}{52}=\frac{1}{4}=P(A)P(B),$ similarly,

$$P(A\cap C)=1/4=P(A)P(C),\quad P(B\cap C)=1/4=P(B)P(C).$$

$\Rightarrow A$, B, and C are pairwise independent

- However,

$$P(A\cap B\cap C)=P(\{\text{Clubs}\})=\frac{1}{4}\neq\frac{1}{8}=P(A)P(B)P(C),$$

$\Rightarrow$ A, B, and C are *not* mutually independent

➢ Theorem (Independence and Complements, n-events case). $A_1, \ldots, A_n$ are mutually independent if and only if

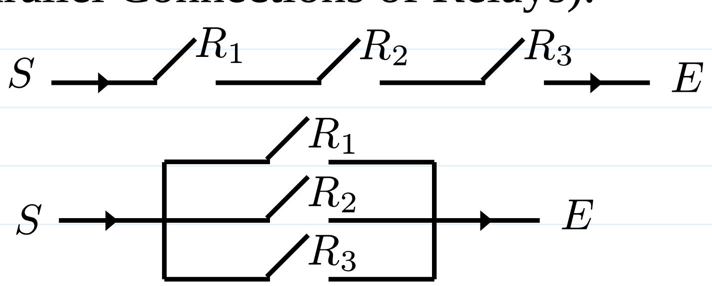
$$P(B_1\cap\cdots\cap B_n)=P(B_1)\cdots P(B_n),$$

where B_i is either A_i or A_i^c, for i=1, …, n.

- Example (Series and Parallel Connections of Relays).

Series Connection

Parallel Connection



- The Story. For n electrical relays R_1, …, R_n, let

$$A_k = \{R_k \text{ works properly}\},$$

k=1,…, n, and suppose that A_1, …, A_n are independent.

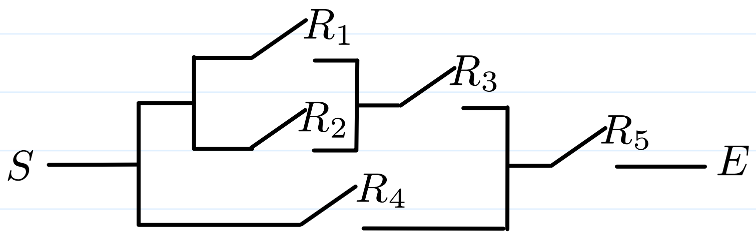
- Series Connection. The probability that current can flow from S to E (which corresponds to the event $A_1 \cap \cdots \cap A_n$) is

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdots P(A_n).$$

- Parallel Connection. The probability that current can flow from S to E (which corresponds to the event $A_1 \cup \cdots \cup A_n$) is

$$P(A_1 \cup \cdots \cup A_n) = 1 - P(A_1^c \cap \cdots \cap A_n^c)$$
$$= 1 - P(A_1^c) \cdots P(A_n^c) = 1 - \prod_{k=1}^{n} [1 - P(A_k)]$$

- Combination of Series and Parallel Connections



➢ Theorem. If A_1, …, A_n are mutually independent and B_1, …, B_m, $m \le n$, are formed by taking unions or intersections of mutually exclusive subgroups of A_1, …, A_n, then B_1, …, B_m are independent.

• Definition (conditional independence): Events B_1, …, B_n are (*pairwise* or *mutually*) *independent* under the probability measure $P(\cdot|A)$

- e.g., B_1 and B_2 are conditionally independent given A iff

$$P(B_1 \cap B_2|A) = P(B_1|A)P(B_2|A),$$

or, equivalently,

$$P(B_1|B_2 \cap A) = P(B_1|A).$$

- Example (Gold Coins):
  - The Story.
    - Box i contains i gold coins and $k-i$ silver coins, i=0,1,…,k.
    - Experiment: (i) Select a box at random and, (ii) draw coins *with replacement* from the box
  - **Q**: Given that first n draws are all gold, what is the probability that $(n+1)^{st}$ draw is gold?
    - Let A_i= {Box i is selected}, B = {first n draws are gold}, C = {$(n+1)^{st}$ draw is gold}
    - By law of total probability,

$$P(C|B) = \sum_{i=0}^{k} P(A_i|B)P(C|A_i \cap B)$$

    - Because B and C are conditionally independent given A_i,

$$P(C|A_i \cap B) = P(C|A_i) = i/k$$

    - By Bayes' rule,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{k} P(A_j)P(B|A_j)} = \frac{[1/(k+1)](i/k)^n}{\sum_{j=1}^{k}[1/(k+1)](j/k)^n}$$

    - Hence, $P(C|B) = \sum_{i=0}^{k}(i/k)^{n+1} \Big/ \sum_{i=0}^{k}(i/k)^n$

  - **Q**: Are the two events B and C independent?

❖ **Reading**: textbook, Sec 3.4, 3.5