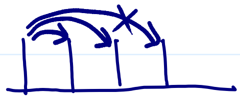
Introduction to Probability

- Uncertainty/Randomness (不確定性/隨機性) in our life
  - ➢Many events are random in that their result is unknowable before the event happens.
    - Will it rain tomorrow?
    - How many wins can 王建民 achieve this season?
    - What numbers will I roll on two dice?
    - **Q**: Is your height/weight measure random?
  - ➢We often want to assess *how likely* are the outcomes of such events. Probability is that measurement.
- Distinction between Discovery (發現) and Invention (發明), e.g.,
  - ➢哥倫布 "發現" 新大陸 → 原本就有
  - ➢愛迪生 "發明" 電燈泡 → 無中生有
  - ➢**Q**: 相對論是發明還是發現?
- 機率論是人類 "發明" 來處理生活中的不確定性之理論
  - ➢愛因斯坦: "上帝永遠不會擲骰子"

Combinatorial Analysis

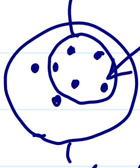


- An example:
  - ➢A communication system is to consist of n *seemingly identical* antennas that are to be lined up in a linear order
  - ➢A resulting system will be functional as long as no two consecutive antennas are defective
  - ➢If it turns out m (=2) of the n (=4) antennas are defective, what is the probability that the resulting system will be functional?

□ □ □ □
1 1 0 0 ×
1 0 1 0 ✓
1 0 0 1 ×
0 1 1 0 ✓
0 1 0 1 ✓
0 0 1 1 ×

Prob $= \frac{3}{6}$

A (subset)



Ω (set)

Prob $= \frac{\#A}{\#\Omega}$

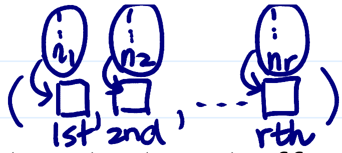
each element with equal chance to happen

$\#\Omega$ 0 0 1 1 $\binom{4}{2}=6$

$\#A$ ^ 1 ^ 1 ^ $\binom{3}{2}=3$

- Many problems in probability theory can be solved simply by counting the number of different ways that a certain event can occur
- The mathematical theory of counting is formally known as *combinatorial analysis*
- What to Count? (i) Lists, (ii) Permutations, (iii) Combination, (iv) Partition, (v) Number of integer solutions.

- Lists
  - Definition



order matter

- Order Pairs: $(x, y)=(w, z)$ iff $w=x$ and $z=y$.
- Ordered Triples: $(x, y, z)=(u, v, w)$ iff $u=x$, $v=y$, and $w=z$.
- *List of Length r*: $(x_1, \ldots, x_r)=(y_1, \ldots, y_s)$ iff $s=r$ and $x_i=y_i$ for $i=1, \ldots, r$.

- Example (License Plates): A license plate has the form $LMNwxyz$, where

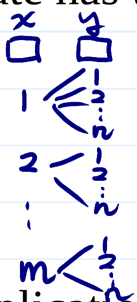
1st 2nd 3rd 4th 7th

BAD0001
BAD1000

$$L, M, N \in \{A, B, \ldots, Z\},$$

$$w, x, y, z \in \{0, 1, \ldots, 9\},$$

and, so, is a list of length seven.



- The basic principle of counting - multiplication principle
  - For Two: If there are m choices for x and for each choice of x, n choice for y, then there are mn choices for (x, y).
  - For several: If there are n_i choices for x_i, $i=1, \ldots, r$, then there are

$$n_1 n_2 \cdots n_r$$

choices for $(x_1, \ldots, x_r)$.

- Example:

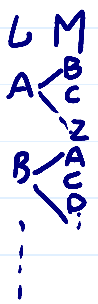
As I was going to St. Ives, I met a man with seven wives
Every wife had seven sacks, Every sack had seven cats
Every cat had seven kits, Kits, cats, sacks, wives
How many were going to St. Ives?

- Ans: none
- However, how many were going the other way?

7 Wives, $7\times7=49$ sacks, $49\times7=343$ cats, $343\times7=2401$ kits

Total=7+49+343+2401=2800

by multiplication principle



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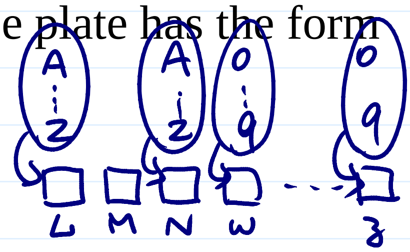
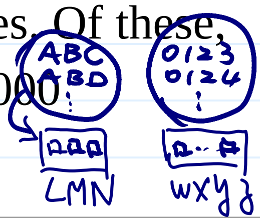


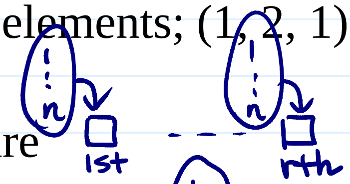
There are $26^3\times10^4=175{,}760{,}000$ license plates. Of these,

$$26\times25\times24\times10\times9\times8\times7 = 78{,}624{,}000$$

have distinct letter and digits (no repetition).



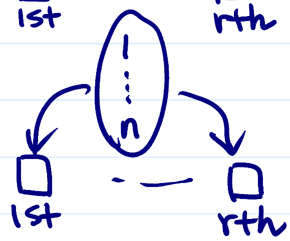
- Permutation
  - Definition: A *permutation* of length r is a list $(x_1, \ldots, x_r)$ with distinct components (no repetition); that is $x_i \neq x_j$ when $i \neq j$
  - Example: (1, 2, 3) is a permutation of three elements; (1, 2, 1) is not a permutation



  - Counting Formulas. From n objects, there are

$$n^r = n \times \cdots \times n \quad (r \text{ factors})$$

lists of length r and



$$(n)_r \equiv n \times (n-1) \times \cdots \times (n-r+1)$$

↳ by multiplication principle.

permutations of length r may be formed.

  - Example: There are 10^3=1000 three digit numbers, of which $(10)_3$=10×9×8=720 lists with distinct digits.

lists

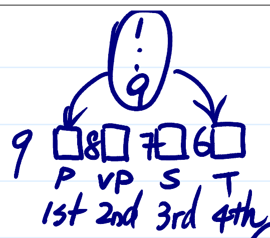
permutations.

  - Some notations
    - Factorials: For positive integers n and r, when r=n, write

$$n! \equiv (n)_n = n \times (n-1) \times \cdots \times 2 \times 1$$

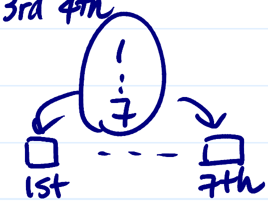
    - Conventions: $(n)_0$=1 and 0!=1

    - Some Notes
      - The textbook only consider r=n.
      - $(n)_r$=0 if r>n.
      - If r<n, then $n!$=$(n)_r(n-r)!$



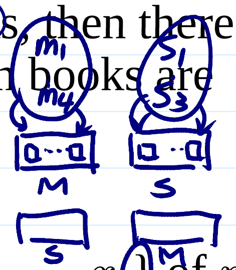
  - (*) Example: A group of 9 people may choose officers (P, VP, S, T) in $(9)_4$=3024 ways.

$n=9,\ r=4,\ (9)_4 = 9!/\cancel{5}!$ permutations



  - Example:
    - 7 books may be arranged in 7!=5040 ways
    - If there are 4 math books and 3 science books, then there are 2×4!×3!=288 arrangements in which the math books are together and the science books are together

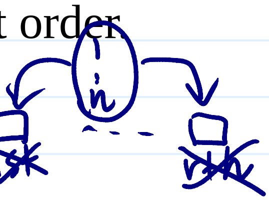
$m_1\ m_2\ m_3\ m_4$ $s_1\ s_2\ s_3$



- Combinations
  - Definition: A *combination* of size r is a set $\{x_1, \ldots, x_r\}$ of r distinct elements. Two combinations are equal if they have the same elements, possibly written in different order
  - Example: {1, 2, 3}={3, 2, 1}, but (1, 2, 3) ≠ (3, 2, 1)

={1,3,2}={2,1,3}, ={2,3,1}={3,1,2}

1st 2nd 3rd



ignore order.

➢ Example: How many committees of size 4 may be chosen from 9 people? Choose officers in two steps:

c.f. the Example(*) in LN p1-6.

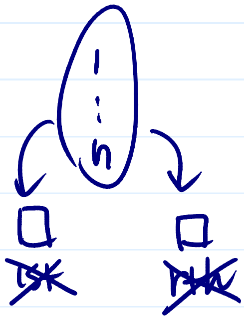
Choose a committee in ?? Ways.

Choose officers from the committee in 4! Ways

From the Basic principle

- $(9)_4 = 4! \times ??$
- So, $?? = (9)_4/4! = 126 = \frac{9!}{5!}\cdot\frac{1}{4!}$ combinations.

➢ Combinations Formula



- From $n \geq 1$ objects,

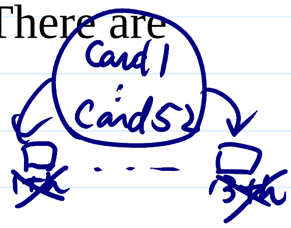
$$\binom{n}{r} = \frac{1}{r!}(n)_r$$

combinations of size $r \leq n$ may be formed

- Example (bridge): A bridge hand is a combination of r=13 cards drawn from a standard deck of n=52. There are

$$\binom{52}{13} = 635,013,559,600$$

such hands. — combinations.



• Binomial coefficients

➢ Alternatively,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

➢ **The Binomial Theorem**: For all $-\infty < x,\ y < \infty$

alternative proof in textbook by induction

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

- Proof: If

$$(x+y)^n = (x+y) \times \cdots \times (x+y).$$

is expanded, then $x^r y^{n-r}$ will appear as often as x can be chosen from r of the n factors; i,e., in $\binom{n}{r}$ ways

- Example. When n=3, $(x+y)^3 = \underset{\binom{3}{0}}{1}x^3 + \underset{\binom{3}{1}}{3}x^2y + \underset{\binom{3}{2}}{3}xy^2 + \underset{\binom{3}{3}}{1}y^3.$

➢ Binomial identities

- Setting x=y=1, we get $2^n = \sum_{r=0}^{n}\binom{n}{r}$
  - ◆ Example: how many subsets are there of a set consisting of n elements?
- Letting x=–1 and y=1, we get $0 = \sum_{r=0}^{n}\binom{n}{r}\cdot(-1)^r$

➢A useful identity:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

1 2 ⋯ n

contain 1: $\binom{n-1}{r-1}$

not contain 1: $\binom{n-1}{r}$

• Partitions

➢Example: How many distinct arrangements formed from the letters

$$\text{M I}_1 \text{ S}_1 \text{ S}_2 \text{ I}_2 \text{ S}_3 \text{ S}_4 \text{ I}_3 \text{ P}_1 \text{ P}_2 \text{ I}_4 ?$$

- There are 11 letters which can be arranged in 11! Ways
- But, this leads to double counting. If the 4 "S" are permuted, then nothing is changed. Similarly, for the 4 "I"s and 2 "P"s.
- So, the each configuration of letters

$$4! \times 4! \times 2! = 1{,}152$$

times and the answer is $\dfrac{11!}{4!4!2!} = 34,650.$

➢Definition: Let Z be a set with n elements. If $r \geq 2$ is an integer, then, an *ordered partition* of Z into r subsets is a list

$$(Z_1, \ldots, Z_r)$$

where $Z_1, \ldots, Z_r$ are mutually exclusive subsets of Z whose union is Z; i.e.,

- $Z_i \cap Z_j = \emptyset$, if $i \neq j$, and
- $Z_1 \cup \cdots \cup Z_r = Z$.

1 2 ⋯ n

n_1 seats

Combinations

n_1 n_2 n_r

1st, 2nd, ⋯, rth

➢Let $n_i = \#Z_i$, the number of elements in Z_i. Then, $n_1, \ldots, n_r \geq 0$, and $n_1 + \cdots + n_r = n$.

3467 S, 2 5 8 11 I, 9 10 P, 1 M

- Example: In the "MISSISSIPPI" example, 11 positions,

$$Z = \{1, 2, \ldots, 11\}$$

1 2 ⋯ 11

S I P M

Card 1 ⋯ Card 52

13 13 13

S W N E

were partitioned into four groups of size

n_1=4 "I"s, n_2=1 "M"s, n_3=2 "P"s, n_4=4 "S"s

- In a bridge game, a deck of 52 cards is partitioned into four hands of size 13 each, one for each of South, West, North, and East.

➢The Partitions Formula. Let $n, r \geq 1$, and $n_1, \ldots, n_r \geq 0$ be integers s.t. $n_1 + \cdots + n_r = n$. If Z is a set of n elements, then there are

$$\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \cdots = \binom{n}{n_1, \cdots, n_r} \equiv \frac{n!}{n_1! \times \cdots \times n_r!}$$

c.f. binomial coefficient, (LNp.1-8), r=2

(called *multinomial coefficients*) ways to partition Z into r subsets $(Z_1, \ldots, Z_r)$ for which $\#Z_i = n_i$ for $i=1, \ldots, r$.

➢The multinomial theorem ← c.f. binomial Thm (LNp.1-8)

$$(x_1+\cdots+x_r)^n = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1,\cdots,n_r} x_1^{n_1}\cdots x_r^{n_r}.$$

➢ Examples:

- 9 children are to be divided into $\boxed{\text{A, B, C}}$ 3 teams of 3 each. How many different divisions? diff. (123 | 456 | 789, 456 | 123 | 789)
  Ans: $\binom{9}{3\ 3\ 3} = 9!/3!3!3!$
- 9 children are to be divided into 3 groups of 3 each, to play a game. How many different divisions? same
  order can be ignored only when $n_1=n_2=\cdots=n_r$
  Ans: $\binom{9}{3\ 3\ 3}/3!$
- a knockout tournament involving n=2^m players
  - n player divided into n/2 pairs
  - losers of each pair eliminated; winner go next round
  - the process repeated until a single player remains
  - Q: How many possible $\boxed{\text{outcomes}}$ for the 1$^{\text{st}}$ round?
    # of different pairing (order matter) $n!/2!\cdots2! = \frac{n!}{2^{n/2}}$ | $\frac{n!}{(\frac{n}{2})!} = \frac{n!}{2^{n/2}} \times \frac{1}{(\frac{n}{2})!} (2^{n/2})$
    1 beat 2, 3 : 5, 8 : 4, 6 : 7
  - Q: How many possible outcomes of the tournament?
    $\frac{n!}{(\frac{n}{2})!} \times \frac{(\frac{n}{2})!}{(\frac{n}{4})!} \times \cdots \times \frac{2}{1} = n!$

champion, 2nd round, 1st round

alternative argument: 1st, 2nd, 3rd, 4th, ..., nth — champion, 2nd place — (1 ⋯ n) — permutation $n!$

Q: how many terms in multinomial Thm LNp.1-11?

- The Number of Integer Solutions

➢If n and r are positive integers, how many integer solutions are there to the equations: $n_1, \ldots, n_r \geq 0$ and $n_1+\cdots+n_r=n$?

➢Example: How many arrangements from a A's and b B's, for example, ABAAB? There are $\binom{a+b}{a} = \binom{a+b}{b}$ such arrangements, since an arrangement is determined by the a places occupied by A.

$a=3$, $b=2$

➢Example: Suppose n=8 and r=4. Represent solutions by "o" and "+" by "|".

- For example, $ooo|oo||ooo$ means n_1=3, n_2=2, n_3=0, n_4=3.
- Note: only r–1 (=3) "|"s are needed.
- There are as many solutions as there are ways to arrange "o" and "|". By the last example, there are
  $$\binom{8+3}{3} = \binom{11}{3} = 165$$
  solutions. (n = 8, $r-1$ = 3)

- A general formula. For positive integers n and r, there are

$$\binom{n+r-1}{r-1} = \binom{n+r-1}{n}$$

a, b

integer solutions to $\boxed{n_1, \ldots, n_r \geq 0}$ and $n_1 + \cdots + n_r = n$.

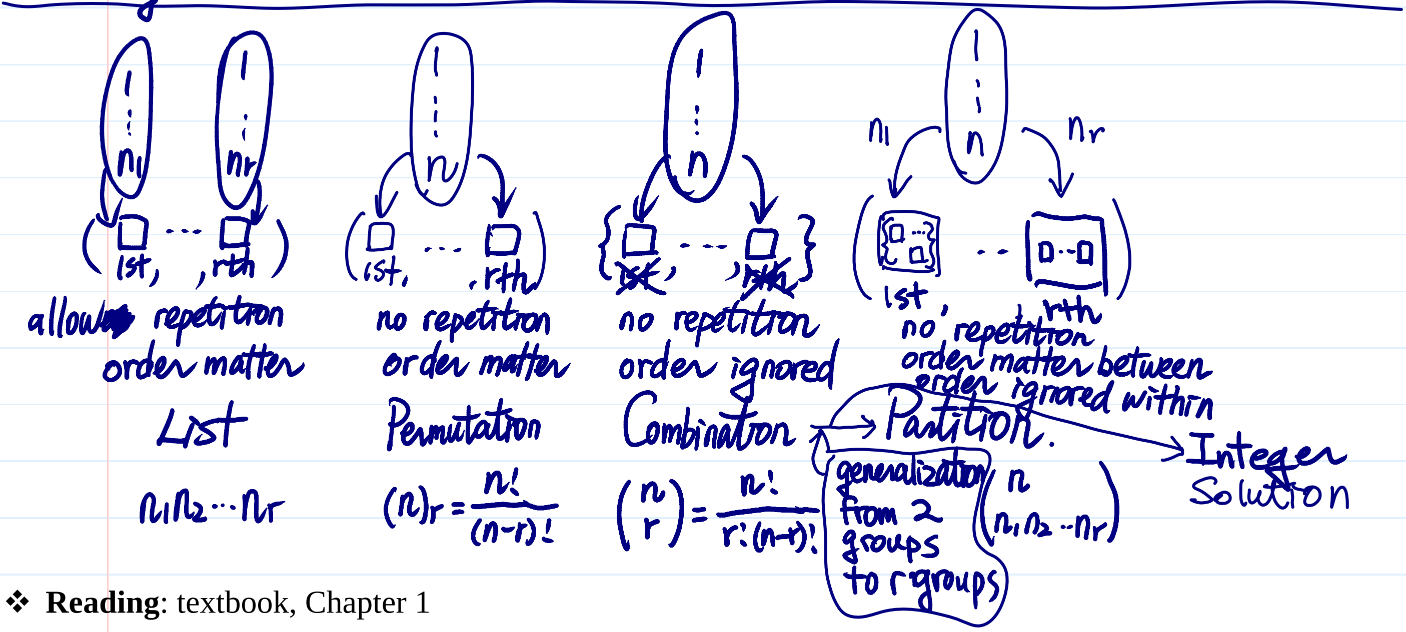
- If $n \geq r$, then there are

$$\binom{n-1}{r-1}$$

solutions with $\boxed{n_i \geq 1}$, for i=1, …, r.

$n_i' = n_i - 1$
$n_i' \geq 0$
$n_1' + \cdots + n_r' = n_1 + \cdots + n_r - r = n - r$

Summary



❖ **Reading**: textbook, Chapter 1