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Johnson Noise and Nyquist Theorem

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• Johnson noise

J. B. Johnson discovered the voltage difference across a resistor without any external bias fluctuates at finite temperature [Nature **119**, 50 (1927)]. The characteristic property of Johnson noise is that the mean square voltage is proportional to the resistance R, temperature τ and the bandwidth Δf of the circuit,

$$\langle V^2 \rangle = 4R\tau\Delta f. \tag{1}$$

Making use of the distribution of one-dimensional photons in thermal equilibrium, this interesting phenomena is later explained by H. Nyquist [Phys. Rev. **32**, 110 (1928)].

The non-vanishing mean square voltage $\langle V^2 \rangle$ is caused by thermal fluctuations. Thus, its proportionality to temperature is expected. Loosely speaking, the "channels" of fluctuations are proportional the bandwidth Δf, justifying its appearance in the above relation. But, what about the resistance R? Under the external bias voltage, the current flows through the resistor and the ratio between them is defined as the (linear) resistance. Therefore, resistance R indicates how energy is dissipated when the resistor is slightly out of equilibrium (due to the applied voltage and the presence of flowing current). The relation in Eq. (1) reveals a secret connection between fluctuations (in equilibrium) and dissipation (out of equilibrium) for a resistor. The Nyquist relation is just a special realization of the general fluctuation-dissipation theorem in statistical physics.

• Nyquist's derivation

We now follow Nyquist's original argument to derive Eq. (1). The trick is to connect two identical resistors of resistance R through a one-dimensional transmission wire without dissipation, i.e. zero resistance. The capacitance and inductance of the wire are C and L and the characteristic impedance

$Z_c = \sqrt{L/C} = R$ matches with the resistors. Because the transmission wire is matched on both ends, electromagnetic waves propagating in either directions are absorbed at the ends without any reflection back to the wire. Or, in quantum language, the matching resistors serve as perfect absorbers for photons reaching the ends.

Let us compute the energy density inside the transmission wire in thermal equilibrium first. The frequencies of the standing waves are quantized,

$$\omega_n = \frac{n\pi c'}{L}. \tag{2}$$

For each frequency interval $\delta f = \delta\omega/(2\pi) = c'/(2L)$, there is one photon mode. At high temperatures, the average energy is τ (according to equipartition theorem of energy). The energy density of the wire in thermal equilibrium is

$$\begin{aligned} \rho &= \text{(average energy/length)} \times \text{(number of modes)} \\ &= (\tau/L) \times \Delta f/\delta f = 2\tau\Delta f/c', \end{aligned} \tag{3}$$

proportional to the temperature τ and the bandwidth Δf.

The energy flows can be deduced from the energy density ρ. The right-moving and left-moving energy currents are defined as $J_R = \rho_R c'$ and $J_L = -\rho_L c'$ respectively. Because the total current in thermal equilibrium is zero, $J_R + J_L = 0$, it leads to $\rho_R = \rho_L = \rho/2$. The energy current flowing out of the right end is

$$J_R = \frac{1}{2}\rho c' = \tau\Delta f. \tag{4}$$

Note that, due to the matching condition, the energy flows into the resistor without any reflection.

Because the resistor is in thermal equilibrium, the incoming energy flow J_R equals the dissipating power, $\mathcal{P} = \langle I^2\rangle R$. It is clear from the circuit that $V = 2IR$. The power can also be expressed as voltage fluctuations,

$$\mathcal{P} = \langle I^2\rangle R = \frac{\langle V^2\rangle}{4R}. \tag{5}$$

Equating the powers into the wire in Eq. (4) and that leaving the wire in Eq. (5), we arrived at the relation,

$$\langle V^2\rangle = 4R\tau\Delta f. \tag{6}$$

Note that Nyquist's derivation presented here makes use of equilibrium between the resistors and the transmission wire. Even without knowing the microscopic details in the resistor, one can derive the mean square voltage across the resistor!

• microscopic reasoning

We now turn to the microscopic derivation for Nyquist theorem. Zoom into the microscopic details, there are N conducting electrons in the resistor. The equation of motion governing the center-of-mass dynamics is

$$M\frac{du}{dt} = -M\gamma u + QV(t), \tag{7}$$

where γ arises from friction and $V(t)$ is the fluctuating voltage. The total mass is $M = Nm_e$ and the tool charge is $Q = -Ne$. It is worth emphasizing why we focus on the center of mass. Suppose we try to write down a similar dynamical equation for an individual electron. The noise term then consists of the random forces coming from the voltage fluctuations and also the mutual interactions between electrons. But, by focusing on the dynamics of the center of the mass, the mutual interactions cancel each other and can be ignored. The random force is solely related to the voltage fluctuations.

The equation of motion can be rewritten as,

$$\frac{d}{dt}\left[u(t)e^{\gamma t}\right] = \frac{Q}{ML}V(t)e^{\gamma t}, \tag{8}$$

and the dynamics of the center of mass is obtained by integration,

$$u(t) = u(0)e^{-\gamma t} + \frac{Q}{ML}e^{-\gamma t}\int_0^t V(t')e^{\gamma t'}dt'. \tag{9}$$

The above solution allows us to compute the mean square velocity of the center of mass,

$$\langle u^2(t)\rangle = \langle u^2(0)\rangle e^{-2\gamma t} + \left(\frac{Q}{ML}\right)^2 \int_0^t \int_0^t \langle V(t_1)V(t_2)\rangle e^{\gamma t_1} e^{\gamma t_2} dt_1 dt_2. \tag{10}$$

Assuming the voltage fluctuations at different times are not correlated,

$$\langle V(t_1)V(t_2)\rangle = \langle V^2\rangle \;\; \delta_{t_1,t_2}, \tag{11}$$

where $\langle V^2\rangle = \langle V^2(t)\rangle$ is independent of time in thermal equilibrium. Making use of the integral relation,

$$\begin{aligned}\int_{-\Delta\omega}^{\Delta\omega} d\omega e^{i\omega(t_1-t_2)} \quad &\approx \quad 2\Delta\omega \;\; \delta_{t_1,t_2} \\ &\longrightarrow \quad 2\pi\delta(t_1 - t_2), \;\text{ as } \Delta\omega \text{ goes to infinity.}\end{aligned} \tag{12}$$

Thus, Kronecker delta symbol is related to Dirac's delta function,

$$\delta_{t_1,t_2} \approx \frac{\pi}{\Delta\omega}\delta(t_1 - t_2). \tag{13}$$

Finally, the voltage correlation function can be expressed as,

$$\langle V(t_1)V(t_2)\rangle = \langle V^2\rangle \ \delta_{t_1,t_2} \approx \langle V^2\rangle \times \frac{\pi}{\Delta\omega}\delta(t_1 - t_2). \tag{14}$$

Substituting the voltage correlation function into Eq. (10) and making use of the relation $\langle u^2(t)\rangle = \langle u^2(0)\rangle \equiv \langle u^2\rangle$, the time dependence beautifully cancels out and the relation reads

$$\langle u^2\rangle = \frac{\pi Q^2}{2M^2L^2\gamma\Delta\omega}\langle V^2\rangle. \tag{15}$$

In thermal equilibrium, the mean square velocity of the center of mass can be easily computed,

$$\langle u^2\rangle = \frac{m^2}{M^2}\sum_{ij}\langle v_i v_j\rangle = \frac{\tau}{M}. \tag{16}$$

Therefore, the mean square voltage is proportional to the temperature and the bandwidth,

$$\langle V^2\rangle = 4\left(\frac{M\gamma L^2}{Q^2}\right)\tau\Delta f. \tag{17}$$

We are almost done except the final brush to show that the constant in the bracket is just the resistance R.

In the presence of external voltage V, the center of mass reaches the steady state with a drift velocity,

$$v_d = \frac{Q}{M\gamma L}V \propto V. \tag{18}$$

It is important to emphasize that the proportionality to the driving V is a direct consequence of friction. The current flowing through the resistor is

$$I = \lambda v_d = \frac{Q}{L} \times \frac{QV}{LM\gamma} = \left(\frac{Q^2}{L^2M\gamma}\right)V. \tag{19}$$

According to Ohm's law $V = IR$, the resistance is

$$R = \frac{M\gamma L^2}{Q^2}. \tag{20}$$

Substituting into Eq. (17), Nyquist relation is obtained,

$$\langle V^2 \rangle = 4R\tau\Delta f. \tag{21}$$

Personally, I like the microscopic derivation better but the macroscopic derivation by Nyquist works as well.

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