Quiz 2

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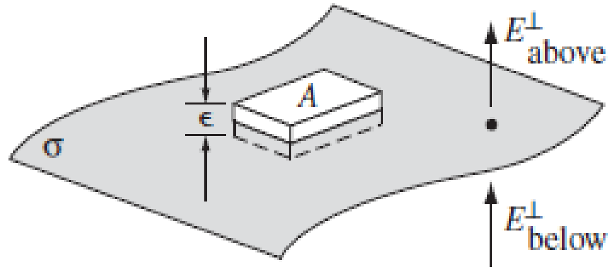
Electrostatic Boundary Conditions: **Normal**

The electric field is not continuous at a surface with charge density σ. Why?

Consider a Gaussian pillbox.



Gauss's law states that $$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

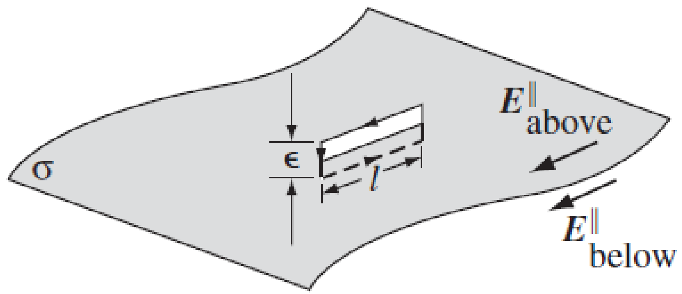
The sides of the pillbox contribute nothing to the flux, in the limit as the thickness ε goes to zero.

$$(E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}})A = \frac{\sigma A}{\varepsilon_0} \Rightarrow (E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}}) = \frac{\sigma}{\varepsilon_0}$$

Electrostatic Boundary Conditions: **Tangential**

The tangential component of **E**, by contract,
is always continuous.

Consider a thin rectangular loop.



The curl of the electric field states that $\oint_P \mathbf{E} \cdot d\vec{\ell} = 0$

The ends give nothing (as $\epsilon \to 0$), and the sides give

$$(E^{//}_{\text{above}} - E^{//}_{\text{below}})\ell = 0 \quad \Rightarrow \quad E^{//}_{\text{above}} = E^{//}_{\text{below}}$$

In short, $\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \dfrac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$

Boundary Conditions in terms of potential

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}} \;\;\Rightarrow\;\; (\nabla V_{\text{above}} - \nabla V_{\text{below}}) = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

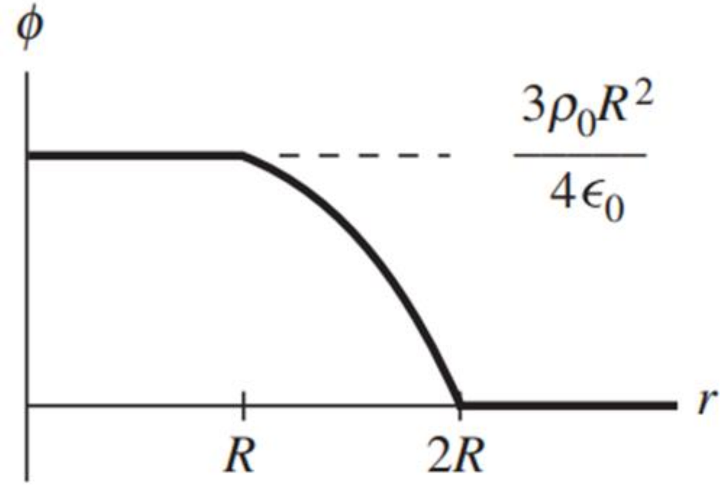
or $$\left(\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n}\right) = -\frac{\sigma}{\varepsilon_0}$$

where $$\frac{\partial V_{\text{above}}}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$$

denotes the normal derivative of V.

A distribution of charge has cylindrical symmetry. As a function of the distance r from the symmetry axis, the electric potential is

$$\phi(r)=\begin{cases}\dfrac{3\rho_0 R^2}{4\varepsilon_0} & (\text{for } r\le R)\\[2mm] \dfrac{\rho_0}{4\varepsilon_0}(4R^2-r^2) & (\text{for } R<r<2R)\\[2mm] 0 & (\text{for } r\ge 2R)\end{cases},$$



where ρ_0 is a quantity with the dimensions of volume charge density.

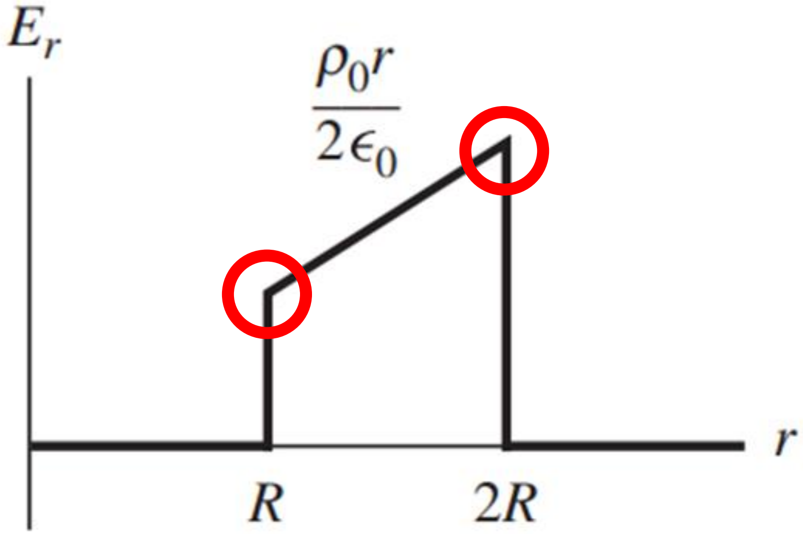
(a) Find and make rough plots of the electric field, for all values of r.

(b) Determine the charge distribution and explain the reasons for the discontinuities in the electric field.

$$\mathbf{E}=\begin{cases}-\nabla\phi=-\hat{\mathbf{r}}\dfrac{\partial}{\partial r}\left[\dfrac{\rho_0}{4\varepsilon_0}(4R^2-r^2)\right]=\dfrac{\rho_0 r}{2\varepsilon_0}\hat{\mathbf{r}} \text{ for } R<r<2R\\ 0,\ \text{otherwise}\end{cases}$$

$$\Rightarrow \rho=\varepsilon_0\nabla\cdot\mathbf{E}=\varepsilon_0\frac{1}{r}\frac{\partial}{\partial r}\left[r\cdot\frac{\rho_0 r}{2\varepsilon_0}\right]=\rho_0$$

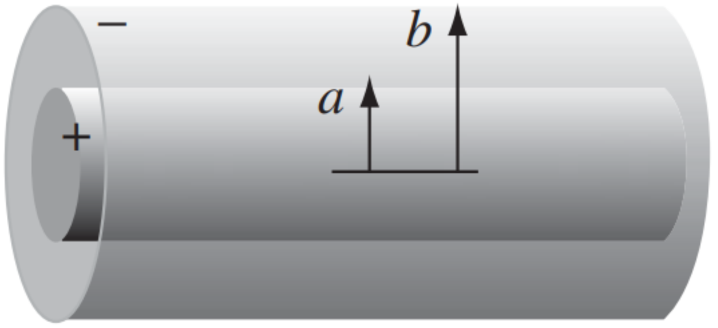
$$\begin{cases}\sigma_R=\dfrac{\rho_0 R}{2}\\ \sigma_{2R}=-\rho_0 R\end{cases}$$



Discontinuities → Surface charges

From your charge distribution, calculate the total charge per unit length along the cylinder. Explain the result.

The cross-sectional area of the $R < r < 2R$ region is $\pi(2R)^2 - \pi R^2 = 3\pi R^2$. So the volume density ρ_0 in this region yields a charge in a length ℓ of the cylinder equal to



$$\rho_0 \ell (3\pi R^2) = 3\pi R^2 \rho_0 \ell.$$

The surface at $r = R$ yields a charge in length ℓ equal to

$$\sigma_R (2\pi R)\ell = (\rho_0 R/2)(2\pi R)\ell = \pi R^2 \rho_0 \ell.$$

And the surface at $r = 2R$ yields a charge in length ℓ equal to

$$\sigma_{2R}(2\pi \cdot 2R)\ell = (-\rho_0 R)(4\pi R)\ell = -4\pi R^2 \rho_0 \ell.$$

Adding up the above three charges and dividing by , we see that the total charge per unit length is zero.

Homework Exercises (Chap.3)

Griffiths:
11, 13, 16, 20, 27, 43, 54, 56,
7, 8, 12, 19, 23, 28, 29, 32, 44, 49