2024 Fall PHYS2310 電磁學（Electromagnetism) Quiz #1

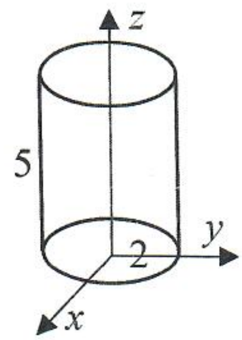
[Griffiths Ch. 1] 2024/09/24, 11:20am–12:00am, 教師：張存續

0. 寫上學號:______________及姓名:__________________即得 10%

1. Let $\vec{\mathscr{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that
   (a) $\nabla(1/\mathscr{r}) = ?$ (20%)
   (b) $\nabla\times\nabla(1/\mathscr{r}) = ?$ (20%)

Ans:

2. The divergence theorem
   (a) Find the divergence of the function
   $\mathbf{v} = s\,(2+\sin^2\phi)\,\hat{\mathbf{s}} + s\,\sin\phi\,\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}$ (25%)
   (b) Test the divergence of this function, using the cylinder (radius 2, height 5) shown in the figure. (25%)



Ans:

向量請以粗體 **A** 表示，或者戴箭頭 $\vec{A}$ 或者 both $\vec{\mathbf{A}}$，單位向量請戴尖帽子 $\hat{\mathbf{A}}$

基本上法向量 $\hat{\mathbf{n}}$ 不同於 $\hat{\mathscr{r}}$，並且有時候，$\hat{\mathscr{r}}$ 終究不等同球座標的單位向量 $\hat{\mathbf{r}}$，請同學們注意，習慣請儘早培養

隨著以後題目複雜度的提升，有時候一份答案的完整回答會分成許多部份的計算或者是證明，所以請養成習慣 Highlight 自己計算過程中符合題目要求的答案

別隨意更改題目定義，麻煩請跟助教或者是老師確認

注意積分撰寫的格式

注意積分因子的寫法

1.

a.20%

$$\vec{\boldsymbol{\mathscr{r}}}=(x,y,z)-(x',y',z') \qquad \mathscr{r}=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right)=-\frac{1}{\mathscr{r}^2}\nabla\mathscr{r}=-\frac{1}{\mathscr{r}^2}\left(\hat{x}\partial_x+\hat{y}\partial_y+\hat{z}\partial_z\right)\left(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)$$

$$=-\frac{1}{\mathscr{r}^2}\frac{2(x-x')\hat{x}+2(y-y')\hat{y}+2(z-z')\hat{z}}{2\left(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)}$$

$$=-\frac{1}{\mathscr{r}^2}\frac{1}{\mathscr{r}}\vec{\boldsymbol{\mathscr{r}}}=-\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

b.20%

$$\nabla\times\nabla\left(\frac{1}{\mathscr{r}}\right)=0$$

$$\nabla\times\left(-\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)=-\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial_x & \partial_y & \partial_z\\ \dfrac{x-x'}{\mathscr{r}^3} & \dfrac{y-y'}{\mathscr{r}^3} & \dfrac{z-z'}{\mathscr{r}^3}\end{vmatrix}$$

$$=-\left\{\hat{x}\left[\partial_y\left(\frac{z-z'}{\mathscr{r}^3}\right)-\partial_z\left(\frac{y-y'}{\mathscr{r}^3}\right)\right]-\hat{y}\left[\partial_x\left(\frac{z-z'}{\mathscr{r}^3}\right)-\partial_z\left(\frac{x-x'}{\mathscr{r}^3}\right)\right]+\hat{z}\left[\partial_x\left(\frac{y-y'}{\mathscr{r}^3}\right)-\partial_y\left(\frac{x-x'}{\mathscr{r}^3}\right)\right]\right\}$$

$$=-\left\{\begin{aligned}&\hat{x}\left[\frac{-2(z-z')\partial_y(\mathscr{r})}{\mathscr{r}^4}-\frac{-2(y-y')\partial_z(\mathscr{r})}{\mathscr{r}^4}\right]-\hat{y}\left[\frac{-2(z-z')\partial_x(\mathscr{r})}{\mathscr{r}^4}-\frac{-2(x-x')\partial_z(\mathscr{r})}{\mathscr{r}^4}\right]\\&+\hat{z}\left[\frac{-2(y-y')\partial_x(\mathscr{r})}{\mathscr{r}^4}-\frac{-2(x-x')\partial_y(\mathscr{r})}{\mathscr{r}^4}\right]\end{aligned}\right\}$$

$$=-\left\{\begin{aligned}&\hat{x}\left[\frac{-2(z-z')\times\frac{-2(y-y')}{2\mathscr{r}}}{\mathscr{r}^4}-\frac{-2(y-y')\frac{-2(z-z')}{2\mathscr{r}}}{\mathscr{r}^4}\right]-\hat{y}\left[\frac{-2(z-z')\frac{-2(x-x')}{2\mathscr{r}}}{\mathscr{r}^4}-\frac{-2(x-x')\frac{-2(z-z')}{2\mathscr{r}}}{\mathscr{r}^4}\right]\\&+\hat{z}\left[\frac{-2(y-y')\frac{-2(x-x')}{2\mathscr{r}}}{\mathscr{r}^4}-\frac{-2(x-x')\frac{-2(y-y')}{2\mathscr{r}}}{\mathscr{r}^4}\right]\end{aligned}\right\}$$

$$=0$$

2.

a.25%

$$\nabla\cdot\mathbf{v}=\frac{1}{s}\frac{\partial}{\partial s}(sv_s)+\frac{1}{s}\frac{\partial v_\phi}{\partial\phi}+\frac{\partial v_z}{\partial z}=2\left(2+\sin^2\phi\right)+\left(\cos^2\phi-\sin^2\phi\right)+3=8$$

b.25%

$$\int_V \nabla\cdot\mathbf{v}\,d\tau = \oint_A \mathbf{v}\cdot d\mathbf{a} \ldots 5\%$$

$$\int_V \nabla\cdot\mathbf{v}\,d\tau = \int_0^2 ds \int_0^{2\pi} s\,d\phi \int_0^5 dz\,(8) = \left(\frac{1}{2}\times 2^2 \times 2\pi \times 5\right)\times 8 = 160\pi \ldots 10\%$$

$$\oint_A \mathbf{v}\cdot d\mathbf{a} = \overbrace{\int_0^5 dz \int_0^{2\pi} s\,d\phi \left[s\left(2+\sin^2\phi\right)\right]\Big|_{s=2}}^{\text{side face}} + \overbrace{\int_0^2 ds \int_0^{2\pi} s\,d\phi\,(3z)\Big|_{z=5}}^{\text{top face}} - \overbrace{\int_0^2 ds \int_0^{2\pi} s\,d\phi\,(3z)\Big|_{z=0}}^{\text{botton face}}$$

$$= \left(80\pi + 20 \underbrace{\int_0^{2\pi} \sin^2\phi\,d\phi}_{\sin^2\phi=\frac{1}{2}(1-\cos 2\phi)}\right) + 60\pi - 0\pi = (80\pi + 20\pi) + 60\pi - 0\pi = 160\pi \ldots 10\%$$