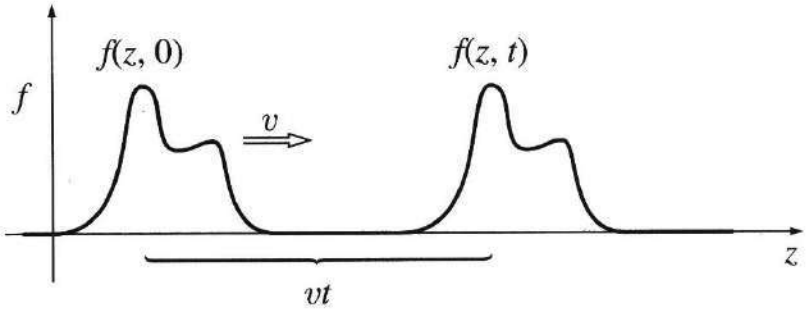
Chapter 9: Electromagnetic Waves

9.1 Waves in One Dimension 9.1.1 The Wave Equation

What is a “wave”?



A start: A wave is a disturbance of a continuous medium that propagates with a fixed shape at constant velocity.

- In the presence of absorption, the wave will diminish in size as it moves;
- If the medium is dispersive, different frequencies travel at different speeds;
- Standing waves do not propagate;
- Light wave can propagate in vacuum;…

The Wave Equation

How to represent such a “wave” mathematically?

Hint: The wave at different times, once at $t = 0$, and again at some later time t --- each point on the wave form simply shifts to the right by an amount vt, where v is the velocity.

initial shape $\quad f(z,0) = g(z)$

subsequent form $\quad f(z,t) = ?$

$$f(z,t) = f(z - vt, 0) = g(z - vt)$$

(capture (mathematically) the essence of wave motion.)

The function $f(z, t)$ depends on them only in the very special combination $z - vt$;
When that is true, the function $f(z, t)$ represents a wave of fixed shape traveling in the z direction at speed v.

The Wave Equation (II)

Examples:

$$\begin{cases} f_1(z,t) = Ae^{-b(z-vt)^2} \\ f_2(z,t) = A\sin[b(z-vt)] \\ f_3(z,t) = \dfrac{A}{b(z-vt)^2+1} \end{cases}$$

How about these functions?

$$f_4(z,t) = Ae^{-b(z^2+vt)}$$

$$f_5(z,t) = A\sin(bz)\cos(bvt)$$

$$= \frac{A}{2}[\sin(b(z+vt)) + \sin(b(z-vt))]$$ ← a standing wave

The Wave Equation of a String

From Newton's second law we have

$$F[\sin(\theta+\Delta\theta)-\sin(\theta)] = (\mu\Delta x)\frac{\partial^2 y}{\partial t^2}$$

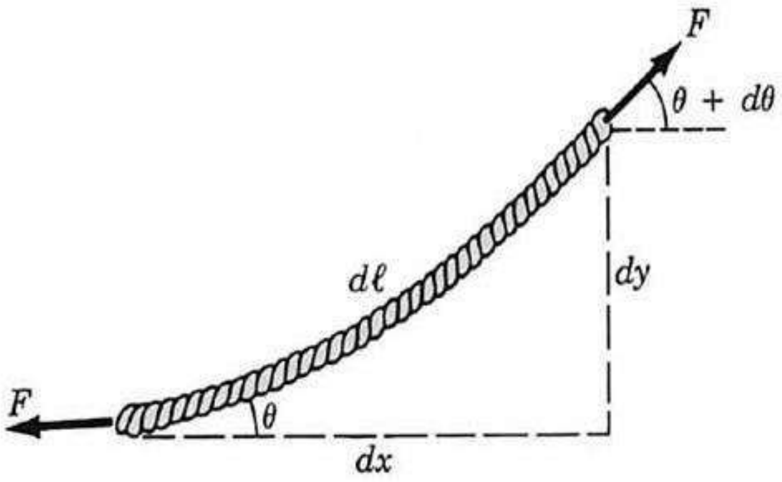
the force on the segment

the mass per unit length

Small angle approximation:

$$\sin\theta \approx \tan\theta = \frac{\partial y}{\partial x}$$

$$\frac{\partial^2 y}{\partial x^2} = (\mu / F)\frac{\partial^2 y}{\partial t^2}$$



The Wave Equation

Derive the wave equation that a disturbance propagates without changing its shape.

$$f(z,t) = g(z - vt); \qquad \text{Let } u \equiv z - vt$$

$$\frac{\partial f}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du} \quad \Rightarrow \quad \frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{dg}{du}\right) = v^2\frac{d^2 g}{du^2}$$

$$\frac{\partial f}{\partial z} = \frac{df}{du}\frac{\partial u}{\partial z} = \frac{dg}{du} \quad \Rightarrow \quad \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}$$

$$\frac{d^2 g}{du^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial z^2} \quad \Rightarrow \quad \frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 \qquad \text{qed}$$

$+v$ or $-v$

$$f(z,t) = g(z - vt) + h(z + vt)$$ the wave equation is linear.

9.1.2 Sinusoidal Waves

(i) Terminology

wave speed

$$f(z,t) = A\cos[k(z - vt) + \delta]$$

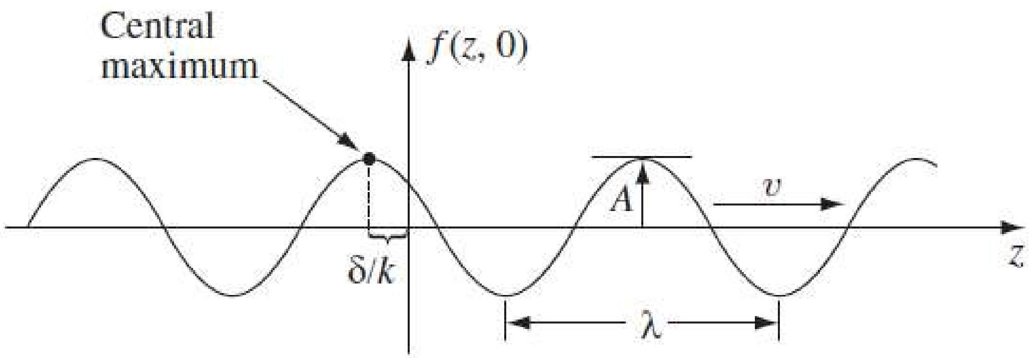
amplitude wave number phase constant

$$f(z,t) = A\cos[k(z - vt) + \delta] = A\cos(kz - \omega t + \delta)$$

$$k = \frac{2\pi}{\lambda}, \quad \lambda\text{: wave length}$$

$$\omega = kv = 2\pi \frac{v}{\lambda} = 2\pi f$$

$$\begin{cases} \omega: \text{ angular frequency} \\ f: \text{ frequency} \end{cases}$$



Sinusoidal Waves

(ii) Complex notation

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

[oi-ler; German]

$$f(z,t) = A\cos[k(z - vt) + \delta] = \text{Re}[Ae^{i(kz-\omega t+\delta)}]$$

$$= \text{Re}[Ae^{i\delta}e^{i(kz-\omega t)}] = \text{Re}[\underline{\tilde{A}e^{i(kz-\omega t)}}]$$

$\tilde{f} \equiv \tilde{A}e^{i(kz-\omega t)}$ complex wave function

$\tilde{A} \equiv Ae^{i\delta}$ complex amplitude; phasor

$f(z,t) = \text{Re}[\tilde{f}(z,t)]$ ˜ : tilde

The advantage of the complex notation is that exponentials are *much easier to manipulate* than sines and cosines.

Example 9.1

The advantage of the complex notation.
Suppose we want to combine two sinusoidal waves:

$$f_3 = f_1 + f_2 = \mathrm{Re}[\tilde{f}_1] + \mathrm{Re}[\tilde{f}_2] = \mathrm{Re}[\tilde{f}_1 + \tilde{f}_2] = \mathrm{Re}[\tilde{f}_3]$$

Simply add the corresponding complex wave functions, and take the real part.
In particular, when they have the same frequency and wave number

$$\tilde{f}_3 = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)} = \tilde{A}_3 e^{i(kz-\omega t)}$$

$$\text{where } \tilde{A}_3 = A_3 e^{i\delta_3} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2}$$

Try doing this without using the complex notation.

Sinusoidal Waves (III)

(iii) Linear combinations of sinusoidal waves

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{i(kz-\omega t)} dk, \quad \text{where} \quad \omega = \omega(k)$$

$\tilde{A}(k)$ can be obtained in terms of the initial conditions.
$f(z,0)$ and $\dot{f}(z,0)$ from the theory of Fourier transforms.

Any wave can be written as a linear combination of sinusoidal waves.

So from now on we shall confine our attention to sinusoidal waves.

9.1.3 Boundary Conditions:
Reflection and Transmission

Incident wave: $\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)}$

Reflected wave: $\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)}$

Transmitted wave: $\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)}$

∗ *All parts of the system are oscillating at the same frequency ω.*

The wave velocities are different in two regimes, which means the wave lengths and wave numbers are also different.

$$\frac{v_1}{v_2} = \frac{k_2}{k_1} = \frac{\lambda_1}{\lambda_2}$$

The waves in the two regions:

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)} & \text{for } z < 0 \\ \tilde{A}_T e^{i(k_2 z - \omega t)} & \text{for } z > 0 \end{cases}$$

Boundary Conditions

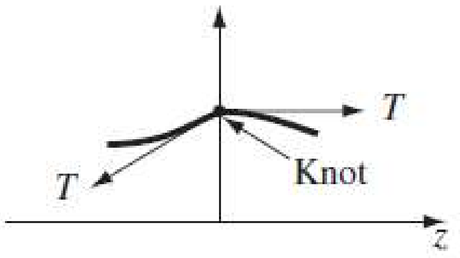
Mathematically, $f(z, t)$ is continuous at $z = 0$.

$$f(0^-, t) = f(0^+, t)$$

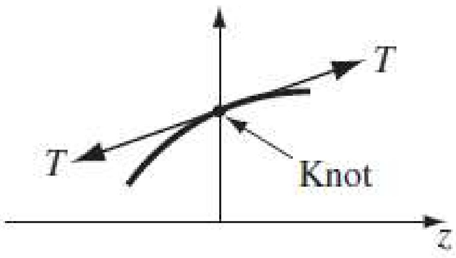
The derivative of $f(z, t)$ must also be continuous at $z = 0$.

$$\left.\frac{df}{dz}\right|_{0^-} = \left.\frac{df}{dz}\right|_{0^+}$$

Why?



(a) Discontinuous slope; force on knot



(b) Continuous slope; no force on knot

The complex wave function obeys the same rules:

$$\tilde{f}(0^-, t) = \tilde{f}(0^+, t); \qquad \left.\frac{d\tilde{f}}{dz}\right|_{0^-} = \left.\frac{d\tilde{f}}{dz}\right|_{0^+}$$

Boundary Conditions Determine the Complex Amplitudes

$$\tilde{f}(0^-,t) = \tilde{f}(0^+,t) \quad \Rightarrow \quad \tilde{A}_I + \tilde{A}_R = \tilde{A}_T$$

$$\left.\frac{d\tilde{f}}{dz}\right|_{0^-} = \left.\frac{d\tilde{f}}{dz}\right|_{0^+} \quad \Rightarrow \quad k_1(\tilde{A}_I - \tilde{A}_R) = k_2\tilde{A}_T$$

$$\left.\begin{array}{l} \tilde{A}_I + \tilde{A}_R = \tilde{A}_T \\ k_1(\tilde{A}_I - \tilde{A}_R) = k_2\tilde{A}_T \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \tilde{A}_R = \left(\dfrac{k_1 - k_2}{k_1 + k_2}\right)\tilde{A}_I = \left(\dfrac{v_2 - v_1}{v_2 + v_1}\right)\tilde{A}_I \\ \tilde{A}_T = \left(\dfrac{2k_1}{k_1 + k_2}\right)\tilde{A}_I = \left(\dfrac{2v_2}{v_2 + v_1}\right)\tilde{A}_I \end{array}\right.$$

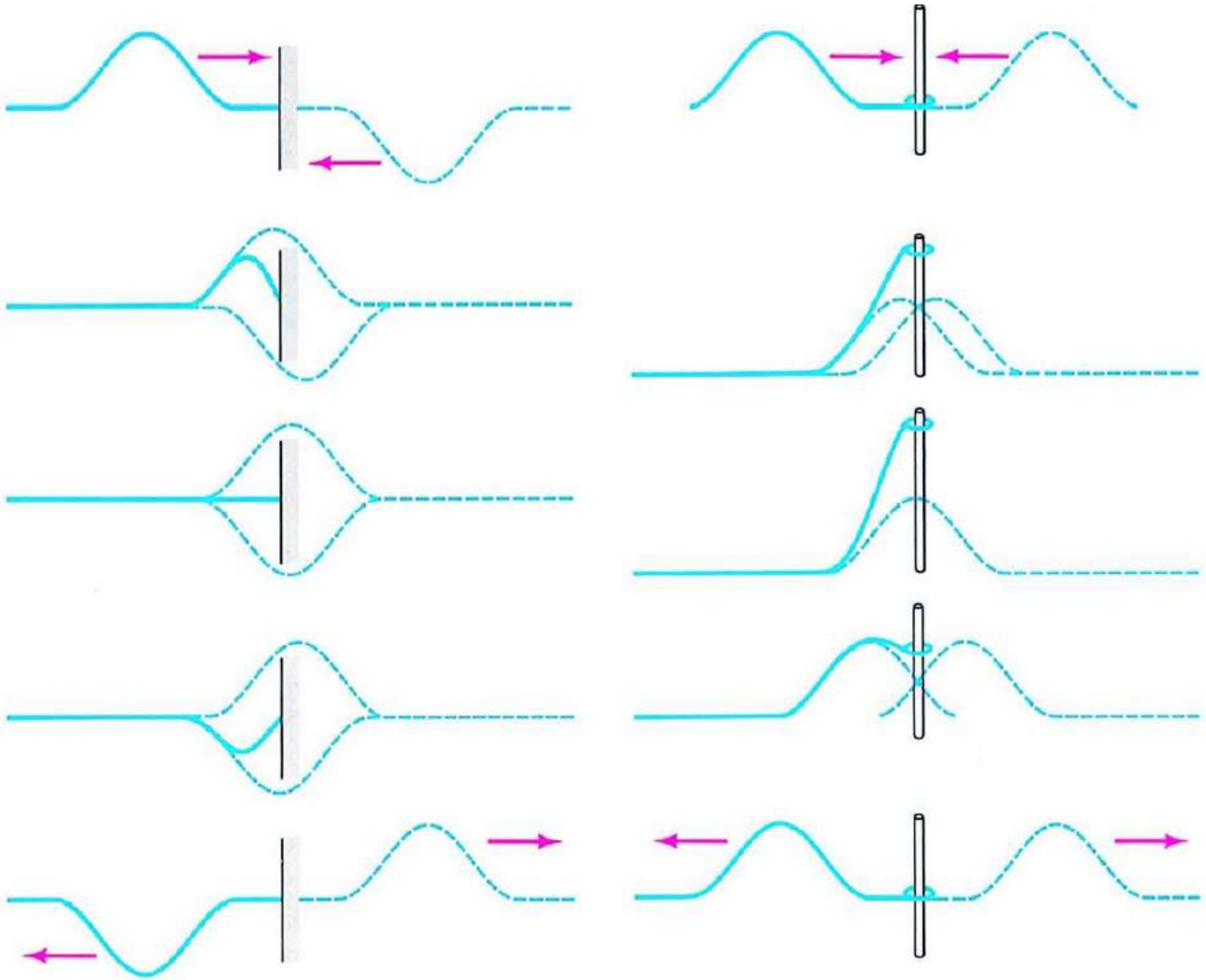
When $v_2 > v_1$, all three waves have the same phase angle.

When $v_2 < v_1$ the reflected wave is out of phase by 180°.

Consider two extreme cases, fixed end and open end.

The Fixed End and Open End

Superposition of the actual pulse and an imaginary pulse.



(a) (b)

9.1.4 Polarization

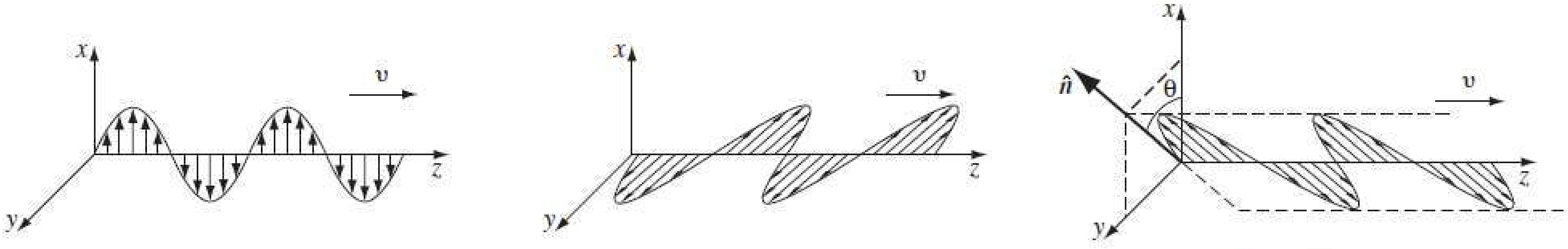
Transverse waves: the displacement of the wave is perpendicular to the direction of propagation, e.g., EM waves.

Longitudinal waves: the displacement of the wave is along the direction of propagation, e.g., sound waves.

Transverse waves occur in two independent states of polarization:

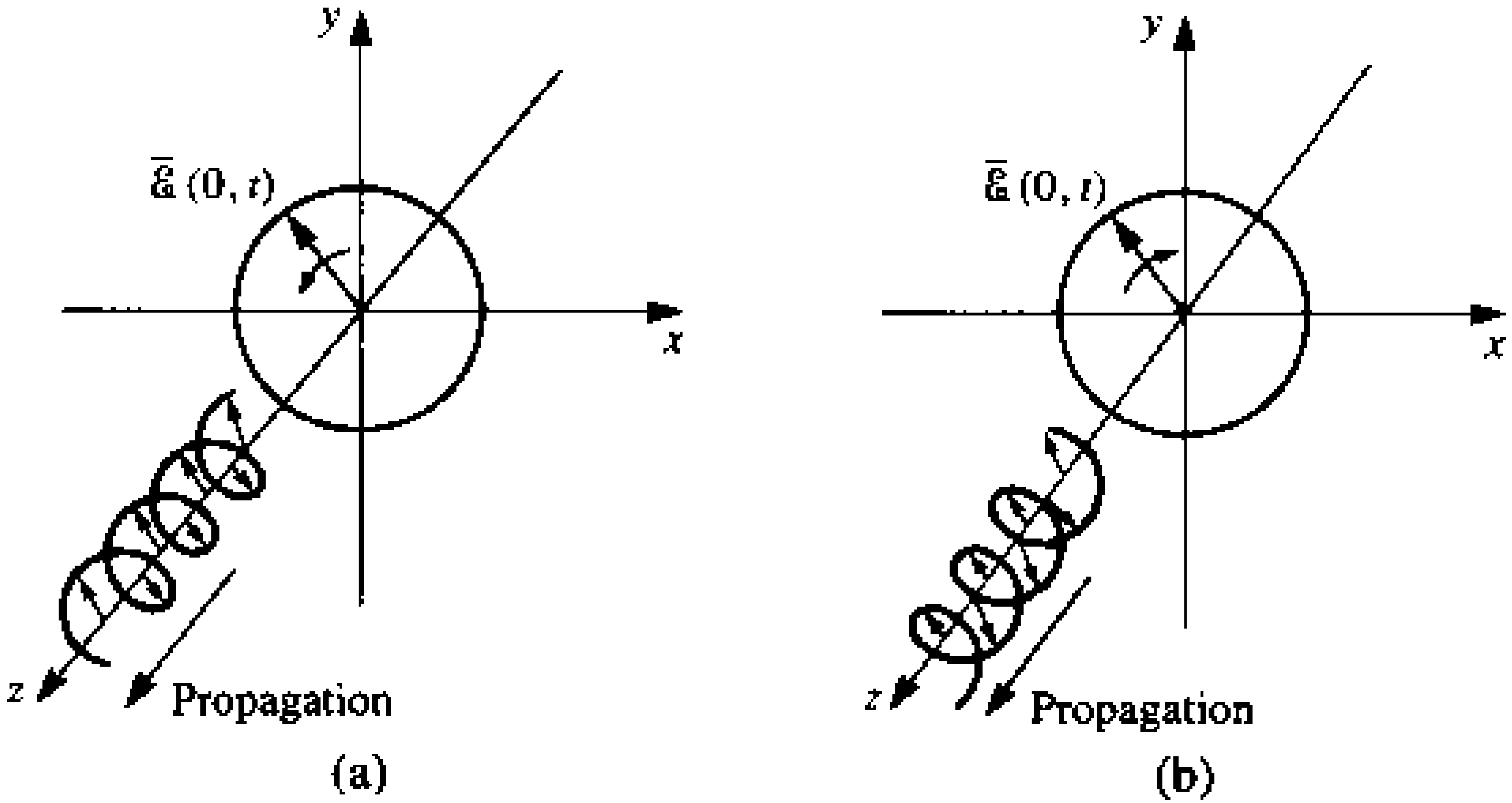
$$\tilde{\mathbf{f}}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\mathbf{x}} \qquad \tilde{\mathbf{f}}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\mathbf{y}}$$

General form: $\tilde{\mathbf{f}}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\mathbf{n}}$, where $\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}$



(a) Vertical polarization (b) Horizontal polarization (c) Polarization vector

Right and Left Hand Circular Polarizations



Electric field polarization for (a) RHCP and (b) LHCP plane waves.

9.2 Electromagnetic Waves in Vacuum
9.2.1 The Wave Equation for E and B

In regions of space where there is no charge or current, Maxwell's equations read

$$\text{(i) } \nabla\cdot\mathbf{E}=0 \qquad \text{(iii) } \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\text{(ii) } \nabla\cdot\mathbf{B}=0 \qquad \text{(iV) } \nabla\times\mathbf{B}=\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla\times(\nabla\times\mathbf{E})=-\frac{\partial(\nabla\times\mathbf{B})}{\partial t} \quad\Rightarrow\quad \nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}=-\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

$$\nabla\times(\nabla\times\mathbf{B})=\mu_0\varepsilon_0\frac{\partial(\nabla\times\mathbf{E})}{\partial t} \quad\Rightarrow\quad \nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}=-\mu_0\varepsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}$$

$$\text{since }\begin{cases}\nabla\cdot\mathbf{E}=0\\ \nabla\cdot\mathbf{B}=0\end{cases} \quad\Longrightarrow\quad \begin{cases}\nabla^2\mathbf{E}=\mu_0\varepsilon_0\dfrac{\partial^2\mathbf{E}}{\partial t^2}\\[2ex] \nabla^2\mathbf{B}=\mu_0\varepsilon_0\dfrac{\partial^2\mathbf{B}}{\partial t^2}\end{cases}$$

The Wave Equation for **E** and **B**

In vacuum, each Cartesian component of **E** and **B** satisfies the three-dimensional wave equation

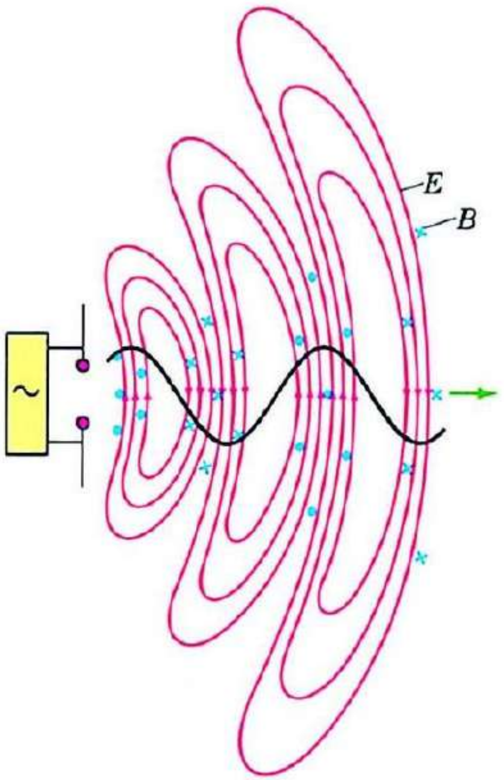
$$\begin{cases} \nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \mathbf{E}}{\partial t^2} \\ \nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \mathbf{B}}{\partial t^2} \end{cases} \Rightarrow \quad \nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

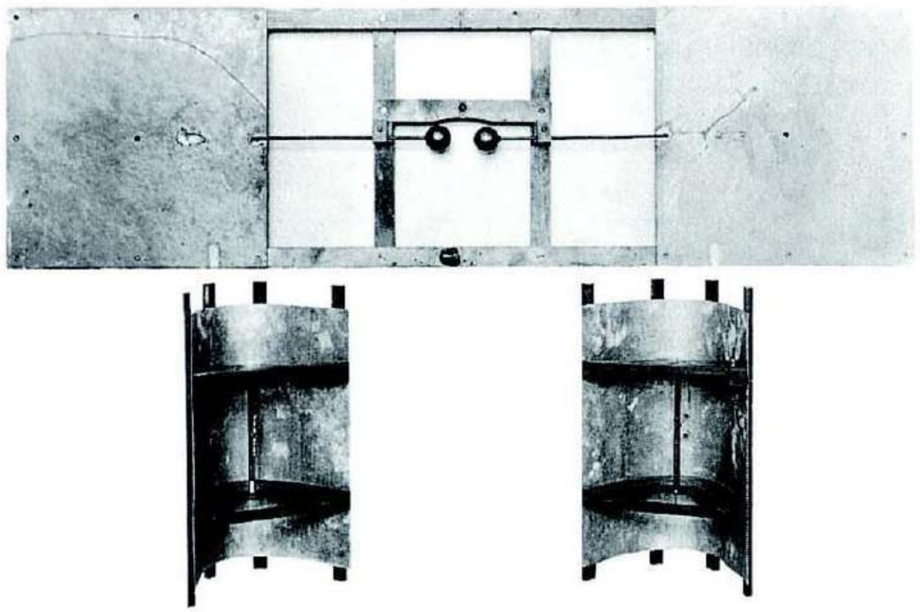
Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, traveling at a speed

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s} \longleftarrow$$ the speed of light

Hertz's Experiment

When Maxwell's work was published in 1867 it did not receive immediate acceptance. It is Hertz who conclusively demonstrated the existence of the electromagnetic wave.





9.2.2 Monochromatic Plane Waves

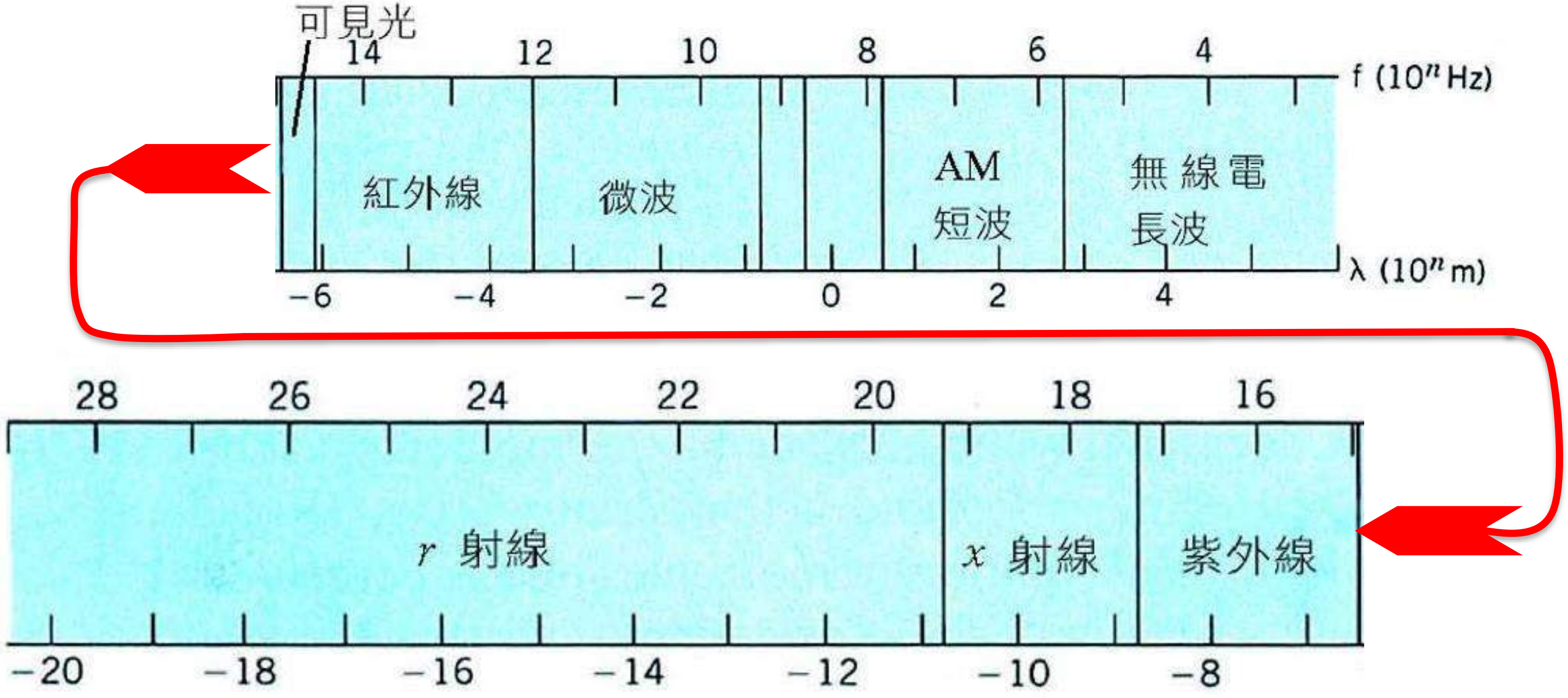
Since different frequencies in the visible range correspond to different colors, such waves are called monochromatic. This definition can be applied to the whole spectrum. A wave of single frequency is called a monochromatic wave.

The Visible Range		
Frequency (Hz)	Color	Wavelength (m)
1.0×10^{15}	near ultraviolet	3.0×10^{-7}
7.5×10^{14}	shortest visible blue	4.0×10^{-7}
6.5×10^{14}	blue	4.6×10^{-7}
5.6×10^{14}	green	5.4×10^{-7}
5.1×10^{14}	yellow	5.9×10^{-7}
4.9×10^{14}	orange	6.1×10^{-7}
3.9×10^{14}	longest visible red	7.6×10^{-7}
3.0×10^{14}	near infrared	1.0×10^{-6}

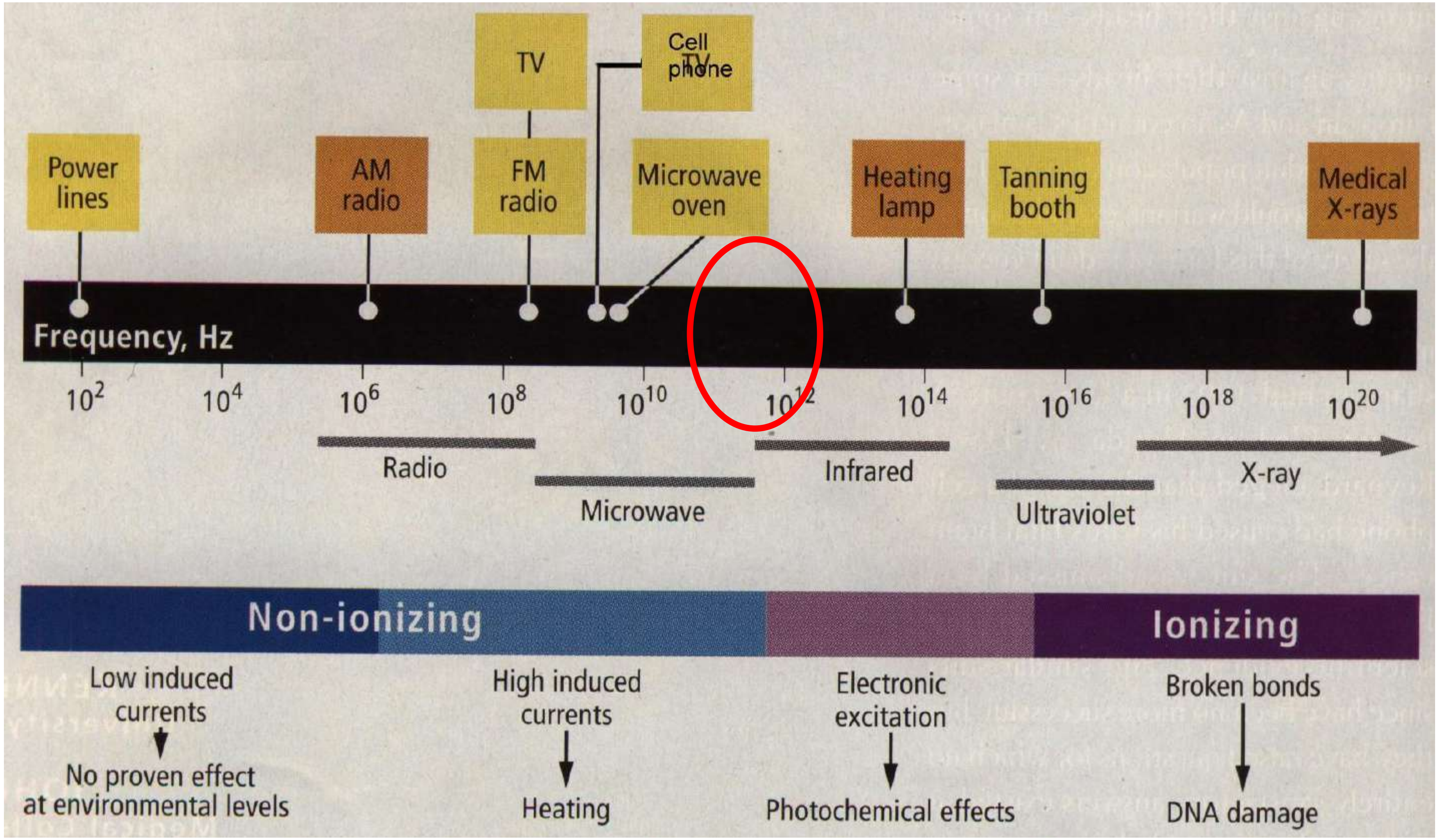
TABLE 9.1

The Electromagnetic Spectrum

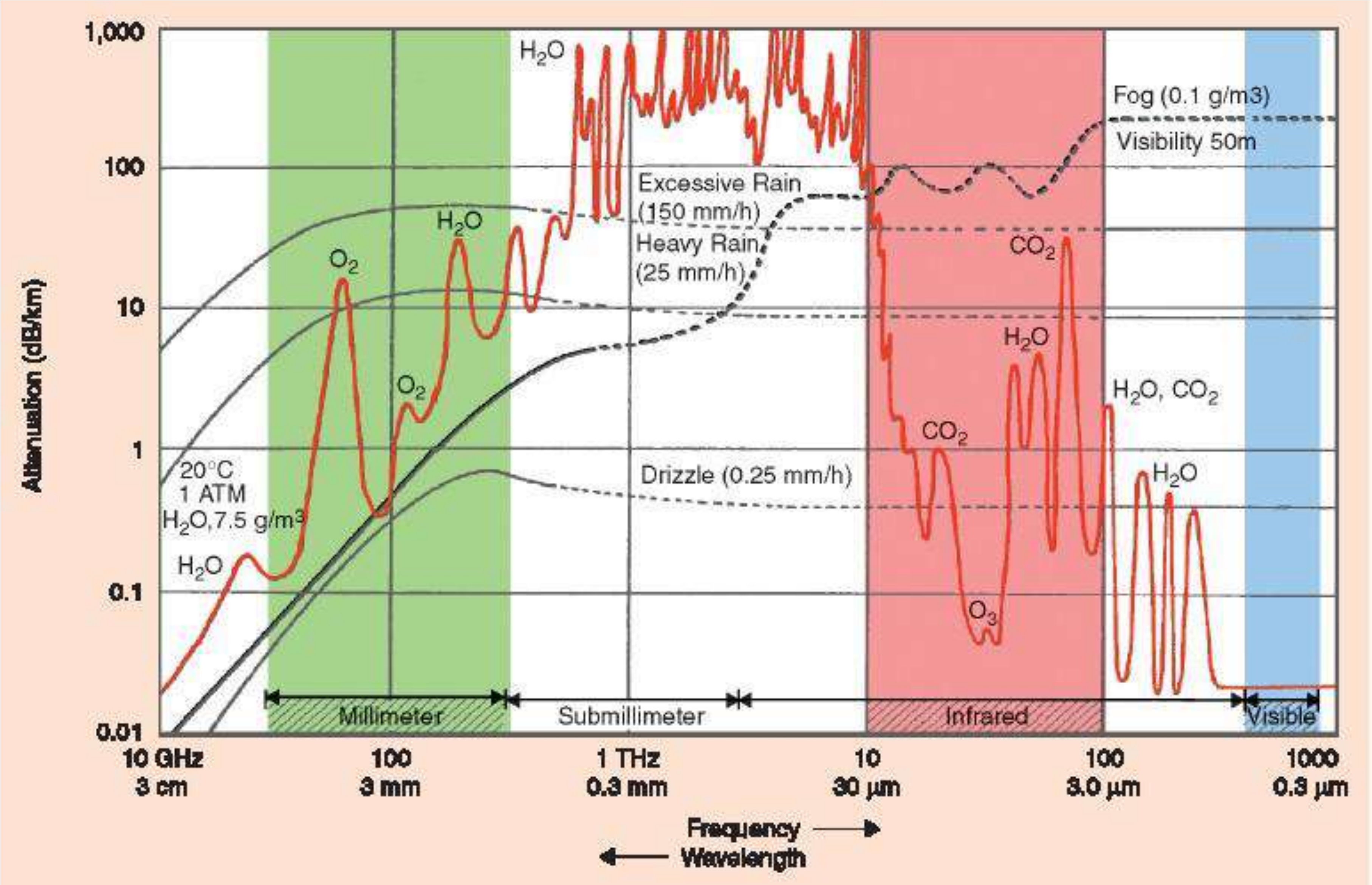
Electromagnetic waves span an immerse range of frequencies, from very long wavelength to extremely high energy with frequency 10^{23} Hz. There is no theoretical limit to the high end.



Mainly Heating Effect in Micro/mm-Wave Spectrum



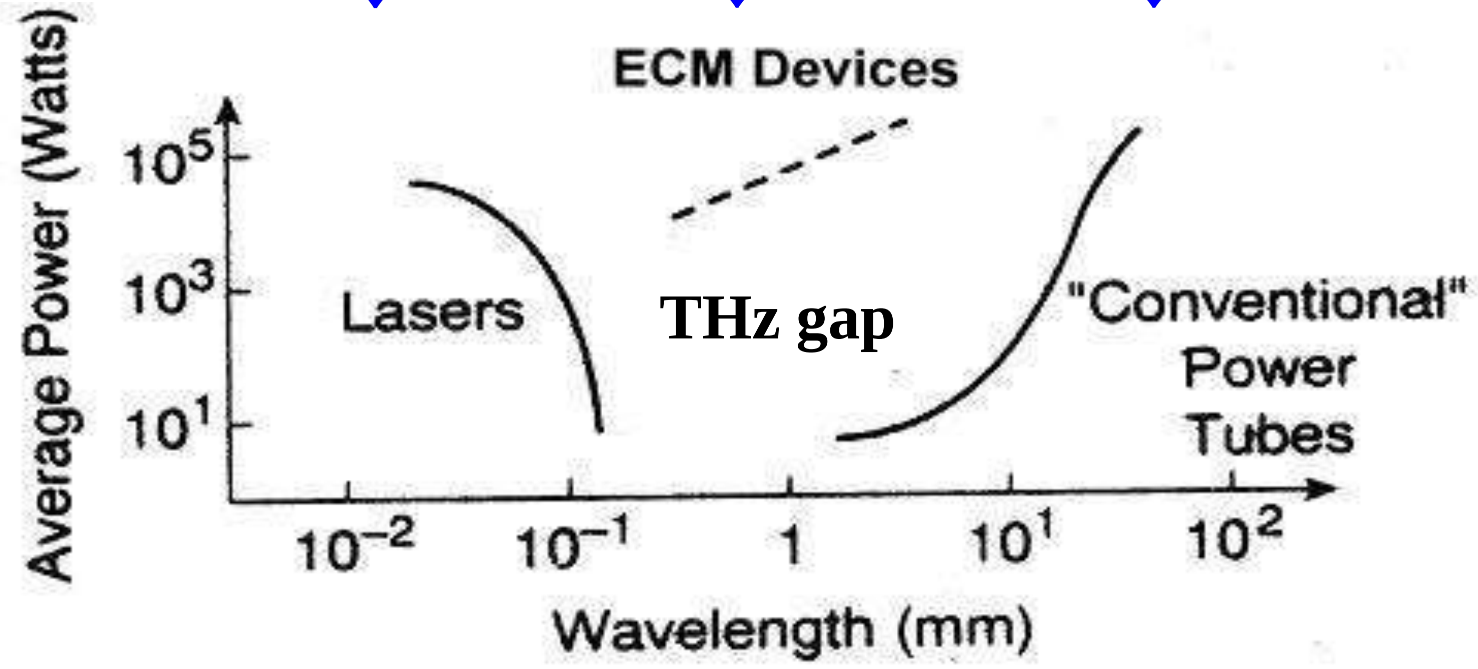
Windows for Research and Application Opportunities



Spectrum to Be Exploited

--- Significance of the Electron Cyclotron Maser

one photon per excitation,	multiple-photon per electron,	multiple photon per electron,
large interaction space	large interaction space	interaction space ~ wavelength
↓	↓	↓



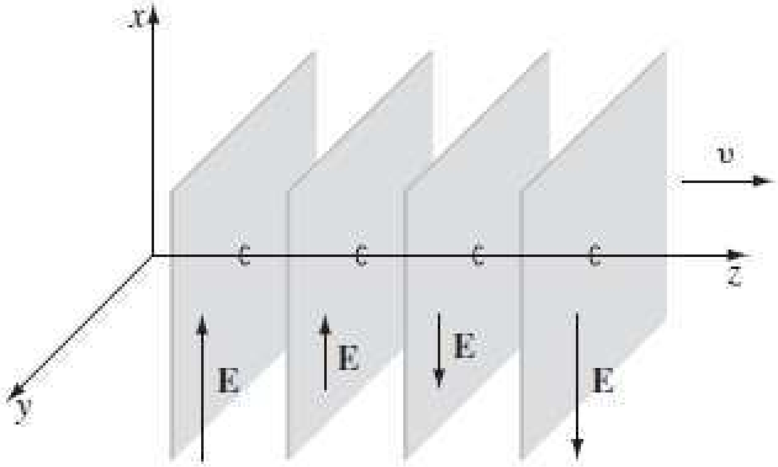
Monochromatic Plane Waves

Consider a monochromatic wave of frequency ω and the wave is traveling in the z direction and has no x or y dependence, called plane waves.
Plane waves: the fields are uniform over every plane perpendicular to the direction of propagation.

Are these waves common? Yes, very common.

$$\begin{cases} \tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)} \\ \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)} \end{cases}$$

where $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are the complex amplitudes.



Transverse Electromagnetic Waves

Q: What is the relation between $\mathbf{E}$ and $\mathbf{B}$?

$$\nabla \cdot \mathbf{E} = 0 \qquad \frac{\partial \tilde{E}_z}{\partial z} = (\tilde{E}_0)_z ike^{i(kz-\omega t)} = 0 \qquad \Rightarrow (\tilde{E}_0)_z = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \frac{\partial \tilde{B}_z}{\partial z} = (\tilde{B}_0)_z ike^{i(kz-\omega t)} = 0 \qquad \Rightarrow (\tilde{B}_0)_z = 0$$

That is, electromagnetic waves are transverse: the electric and magnetic fields are perpendicular to the direction of propagation. Moreover, Faraday's law $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

$$\hat{\mathbf{x}}: \quad \frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z} = -\frac{\partial \tilde{B}_x}{\partial t} \quad \Rightarrow \quad k(\tilde{E}_0)_y = -\omega(\tilde{B}_0)_x$$

$$\hat{\mathbf{y}}: \quad \frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x} = -\frac{\partial \tilde{B}_y}{\partial t} \quad \Rightarrow \quad k(\tilde{E}_0)_x = \omega(\tilde{B}_0)_y$$

$$\hat{\mathbf{z}}: \quad \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} = -\frac{\partial \tilde{B}_z}{\partial t} \quad \Rightarrow \quad 0 = 0$$

Transverse Electromagnetic Waves (II)

Ampere's law with Maxwell's correction: $\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$

$$\hat{\mathbf{x}}: \quad \frac{\partial \tilde{B}_z}{\partial y} - \frac{\partial \tilde{B}_y}{\partial z} = \mu_0 \varepsilon_0 \frac{\partial \tilde{E}_x}{\partial t} \quad \Rightarrow \quad k(\tilde{B}_0)_y = \mu_0 \varepsilon_0 \omega (\tilde{E}_0)_x$$

$$\hat{\mathbf{y}}: \quad \frac{\partial \tilde{B}_x}{\partial z} - \frac{\partial \tilde{B}_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial \tilde{E}_y}{\partial t} \quad \Rightarrow \quad k(\tilde{B}_0)_x = -\mu_0 \varepsilon_0 \omega (\tilde{E}_0)_y$$

$$\hat{\mathbf{z}}: \quad \frac{\partial \tilde{B}_y}{\partial x} - \frac{\partial \tilde{B}_x}{\partial y} = \mu_0 \varepsilon_0 \frac{\partial \tilde{E}_z}{\partial t} \quad \Rightarrow \quad 0 = 0$$

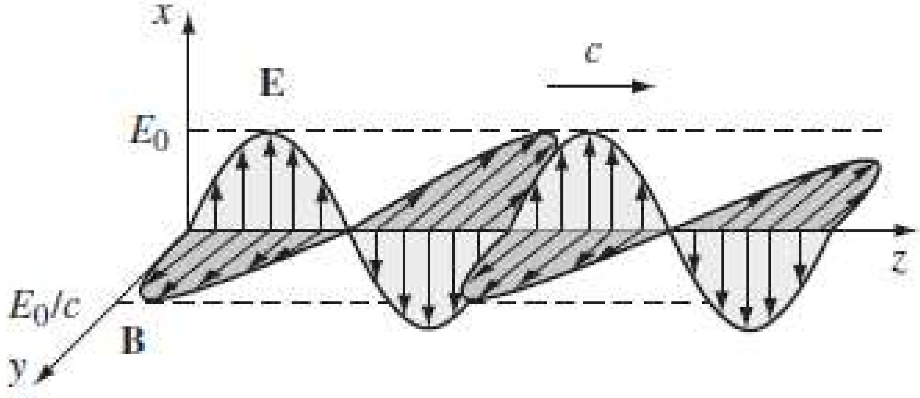
In free space, the speed of light is $c = \dfrac{\omega}{k} = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$

More compactly, $\tilde{\mathbf{B}}_0 = \dfrac{k}{\omega}(\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_0) = \dfrac{1}{c}(\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_0) \Rightarrow \mathbf{E} \perp \mathbf{B}$

amplitude relation: $B_0 = \dfrac{1}{c} E_0$

Example 9.2

Prove: If **E** points in the x direction, then **B** points in the y direction.



Sol:

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}\hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_0 = \frac{k}{\omega}(\hat{\mathbf{z}}\times\tilde{\mathbf{E}}_0) = \frac{1}{c}\tilde{E}_0 e^{i(kz-\omega t)}(\hat{\mathbf{z}}\times\hat{\mathbf{x}}) = \frac{1}{c}\tilde{E}_0 e^{i(kz-\omega t)}\hat{\mathbf{y}}$$

Take the real part:

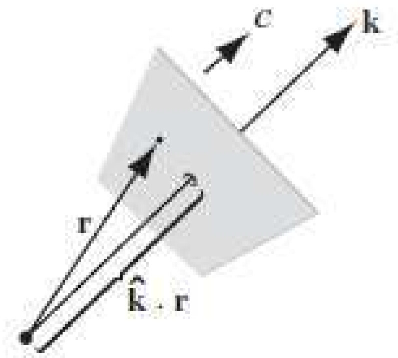
$$\mathbf{E}(z,t) = E_0\cos(kz-\omega t+\delta)\hat{\mathbf{x}}$$

$$\mathbf{B}(z,t) = \frac{1}{c}E_0\cos(kz-\omega t+\delta)\hat{\mathbf{y}}$$

Q: Why not use sine function?

Plane Waves Traveling in an Arbitrary Direction

There is nothing special about the z direction---we can generalize to monochromatic plane waves traveling in an arbitrary direction.



The **propagation** (or **wave**) **vector**, $\mathbf{k}$: pointing in the direction of propagation.

Generalization of kz: using the scalar product $\mathbf{k}\cdot\mathbf{r}$.

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{n}} \leftarrow \text{ the polarization vector}$$

$$\tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{c}\tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\hat{\mathbf{k}}\times\hat{\mathbf{n}}) = \frac{1}{c}\hat{\mathbf{k}}\times\tilde{\mathbf{E}}$$

Q: Can you write down the real electric and magnetic fields?

9.2.3 Energy and Momentum in Electromagnetic Waves

The energy per unit volume stored in the electromagnetic field is

$$u=\frac{1}{2}(\varepsilon_0 E^2+\frac{1}{\mu_0}B^2)$$

Monochromatic plane wave: $B^2=\frac{1}{c^2}E^2=\mu_0\varepsilon_0 E^2$

$$u=\frac{1}{2}(\varepsilon_0 E^2+\frac{1}{\mu_0}B^2)=\frac{1}{2}(\varepsilon_0 E^2+\varepsilon_0 E^2)=\varepsilon_0 E^2$$

Their contributions are equal.

$$u=\varepsilon_0 E^2=\varepsilon_0 E_0{}^2\cos^2(kz-\omega t+\delta)$$

As the wave travels, it carries this energy along with it.

Q: How about the momentum? See next slide.

Energy Transport and the Poynting Vector

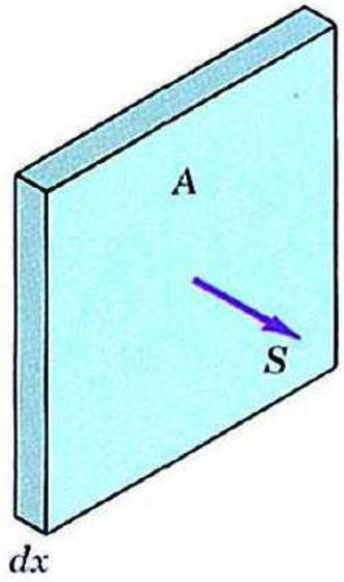
Consider two planes, each of area A, a distance dx apart, and normal to the direction of propagation of the wave. The total energy in the volume between the planes is $dU = uAdx$.

The rate at which this energy passes through a unit area normal to the direction of propagation is

$$S = \frac{1}{A}\frac{dU}{dt} = \frac{1}{A} uA \frac{dx}{dt} = uc$$

$$S = uc = \frac{EB}{\mu_0}$$

$$\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}$$ (the vector form)



$$\mathbf{g} = \frac{1}{c^2}\mathbf{S} = \frac{1}{c}u\hat{\mathbf{z}} = \frac{1}{c}\varepsilon_0 E_0{}^2 \cos^2(kz - \omega t + \delta)\hat{\mathbf{z}}$$

Average Effect

In the case of light, the period is so brief, that any macroscopic measurement will encompass many cycles.

All we want is the average value.

$$\langle u \rangle = \frac{1}{2}\varepsilon_0 {E_0}^2$$

$$\langle \mathbf{S} \rangle = \frac{1}{2} c\varepsilon_0 E_0^2 \hat{\mathbf{z}} = \langle u \rangle c\hat{\mathbf{z}}$$

$$\langle \mathbf{g} \rangle = \frac{1}{c^2}\langle \mathbf{S} \rangle = \frac{1}{2c}\varepsilon_0 E_0^2 \hat{\mathbf{z}}$$

The average power per unit area transported by an electromagnetic wave is called the **intensity**:

$$I \equiv \langle S \rangle = \frac{1}{2} c\varepsilon_0 E_0^2 = \frac{E_0^2}{2\mu_0 c}$$

$$S_{av}$$

Example

A radio station transmits a 10-kW signal at a frequency of 100 MHz. For simplicity, assume that it radiates as a point source. At a distance of 1 km from the antenna, find: (a) the amplitude of the electric and magnetic field strengths, and (b) the energy incident normally on a square plate of side 10 cm in 5 min.

Solution:

$$(a)\quad S_{av} = \frac{\text{Average power}}{4\pi r^2} = \frac{E_0^2}{2\mu_0 c}$$

$$\Rightarrow \frac{10000}{4\pi 1000^2} \times 2 \times 4\pi \times 10^{-7} \times 3 \times 10^8 = E_0^2$$

$$\begin{cases} E_0 = 0.775 \text{ V/m} \\ B_0 = \dfrac{E_0}{c} = 2.58 \times 10^{-9} \text{ T} \end{cases}$$

$$(b)\quad \Delta U = S_{av} A \Delta t = 2.4 \times 10^{-3} \text{ J}$$

Momentum and Radiation Pressure

An electromagnetic wave transports linear momentum.

The linear momentum carried by an electromagnetic wave is related to the energy it transports according to

$$p = \frac{U}{c}$$

If surface is perfectly reflecting, the momentum change of the wave is doubled, consequently, the momentum imparted to the surface is also doubled.

The force exerted by an electromagnetic wave on a surface may be related to the Poynting vector

$$\frac{F}{A} = \frac{\Delta p}{A\Delta t} = \frac{\Delta U}{Ac\Delta t} = \frac{SA}{Ac} = \frac{S}{c} = u$$

Momentum and Radiation Pressure (II)

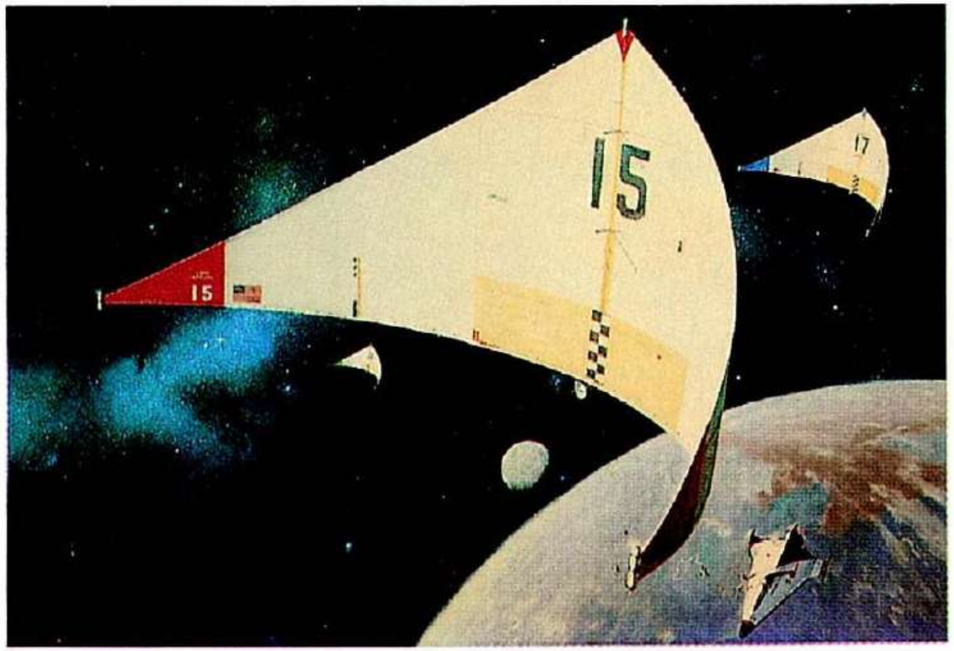
The radiation pressure at normal incident is

$$\frac{F}{A} = \frac{S}{c} = u$$

Examples: (a) the tail of comet, (b) A “solar sail”

(a)

(b)



Homework of Chap.9 (I)

Problem 9.2 Show that the **standing wave** $f(z,t) = A\sin(kz)\cos(kvt)$ satisfies the wave equation, and express it as the sum of a wave traveling to the left and a wave traveling to the right (Eq. 9.6).

Problem 9.6

(a) Formulate an appropriate boundary condition, to replace Eq. 9.27, for the case of two strings under tension T joined by a knot of mass *m*.

(b) Find the amplitude and phase of the reflected and transmitted waves for the case where the knot has a mass *m* and the second string is massless.

Problem 9.10 The intensity of sunlight hitting the earth is about 1300 $\mathrm{W/m^2}$. If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?

Homework of Chap.9 (I)

Problem 9.8 Equation 9.36 describes the most general **linearly** polarized wave on a string. Linear (or "plane") polarization (so called because the displacement is parallel to a fixed vector $\hat{\mathbf{n}}$) results from the combination of horizontally and vertically polarized waves of the *same phase* (Eq. 9.39). If the two components are of equal amplitude, but *out of phase* by 90° (say, $\delta_v = 0$, $\delta_h = 90^\circ$), the result is a *circularly* polarized wave. In that case:

(a) At a fixed point z, show that the string moves in a circle about the z axis. Does it go *clockwise* or *counterclockwise*, as you look down the axis toward the origin? How would you construct a wave circling the *other* way? (In optics, the clockwise case is called **right circular polarization**, and the counterclockwise, **left** circular polarization.)[3]

(b) Sketch the string at time t = 0.

(c) How would you shake the string in order to produce a circularly polarized wave?

Problem 9.12 In the complex notation there is a clever device for finding the time average of a product. Suppose $f(\mathbf{r},t) = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_a)$ and $g(\mathbf{r}, t) = B\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_b)$. Show that $\langle fg \rangle = (1/2)\,\mathrm{Re}(\tilde{f}\tilde{g}^*)$, where the star denotes complex conjugation. [Note that this only works if the two waves have the same **k** and ω, but they need not have the same amplitude or phase.] For example,

$$\langle u \rangle = \frac{1}{4}\mathrm{Re}\left(\varepsilon_0 \tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right) \quad \text{and} \quad \langle \mathbf{S} \rangle = \frac{1}{2\mu_0}\mathrm{Re}\left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*\right).$$

9.3 Electromagnetic Waves in Matter

9.3.1 Propagation in Linear Media

In regions where there is no free charge and free current, Maxwell's equations become

$$\nabla \cdot \mathbf{D} = 0 \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

If the medium is *linear*, $\mathbf{D} = \varepsilon \mathbf{E} \text{ and } \mathbf{H} = \frac{1}{\mu}\mathbf{B}$

If the medium is *linear* and *homogeneous* (ε and μ do not vary from point to point),

$$\nabla \cdot \mathbf{E} = 0 \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

The Index of Refraction

Electromagnetic waves propagate through a linear homogeneous medium at a speed

$$\begin{cases} \nabla^2 \mathbf{E} = \mu\varepsilon \dfrac{\partial^2 \mathbf{E}}{\partial t^2} \\ \nabla^2 \mathbf{B} = \mu\varepsilon \dfrac{\partial^2 \mathbf{B}}{\partial t^2} \end{cases} \Rightarrow \nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$$

$$n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

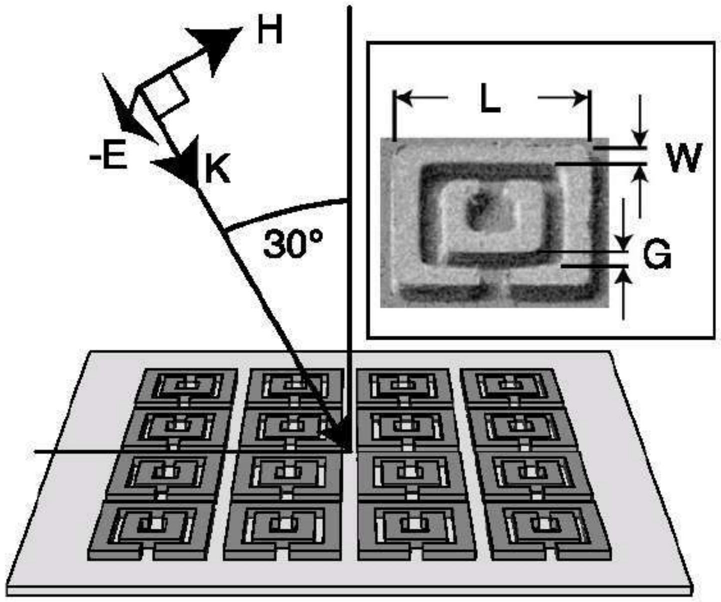
The index of refraction of the material

For most materials, μ is very close to μ_0, so $n \cong \sqrt{\dfrac{\varepsilon}{\varepsilon_0}} = \sqrt{\varepsilon_r}$

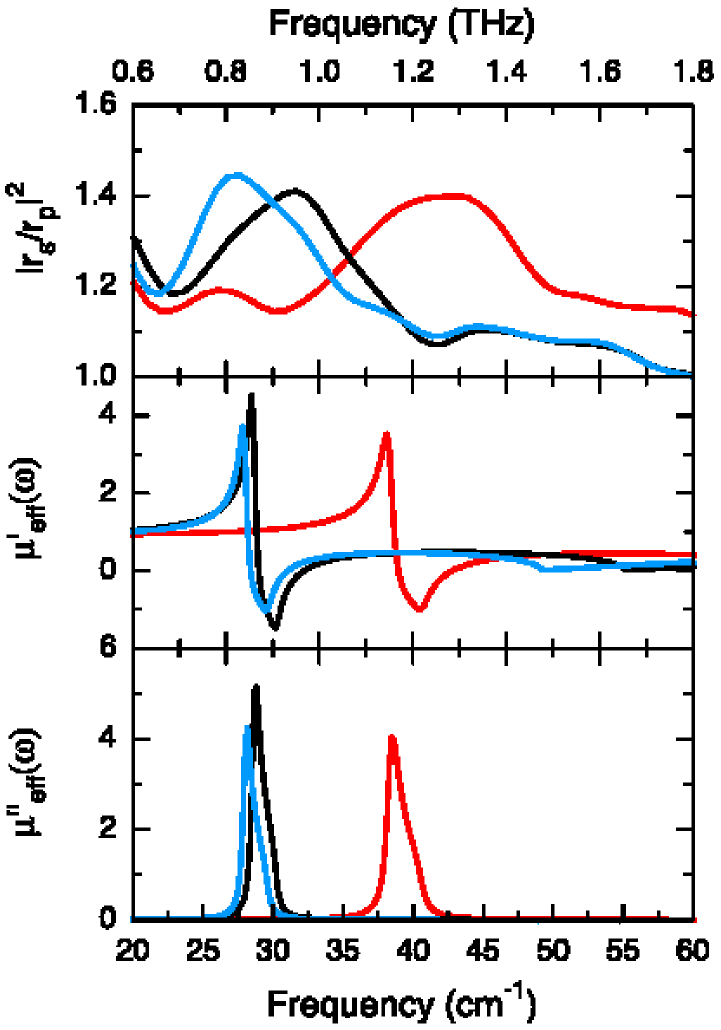
Since ε_r is almost always greater than 1, light travels more slowly through matter.

Q: What happens when ε_r is less than 1 or negative?

THz Meta-materials



$$\mu_{\text{eff}}(\omega) = 1 - \frac{F\omega^2}{\omega^2 - \omega_0^2 + i\Gamma\omega}$$
$$= \mu'_{\text{eff}}(\omega) + i\mu''_{\text{eff}}(\omega)$$



T. J. Yen, *et. al*., "Terahertz magnetic response from artificial materials", Science, **303**, 1494 (2004).

Energy Density, Poynting Vector, and Intensity in Linear Media

All of our previous results carry over, with the simple transcription

$$\varepsilon_0 \to \varepsilon$$
$$\mu_0 \to \mu$$
$$c \to v$$

$$u = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu} B^2\right) \qquad \mathbf{g} = \frac{1}{v} u \hat{\mathbf{z}}$$

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu} \qquad I \equiv \langle S \rangle = \frac{1}{2} v \varepsilon E_0^2$$

Q: What happens when a wave passes from one transparent medium into another? Boundary conditions.

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \qquad \mathbf{E}_1^{//} - \mathbf{E}_2^{//} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \mathbf{H}_1^{//} - \mathbf{H}_2^{//} = (\mathbf{K}_f \times \hat{\mathbf{n}})$$

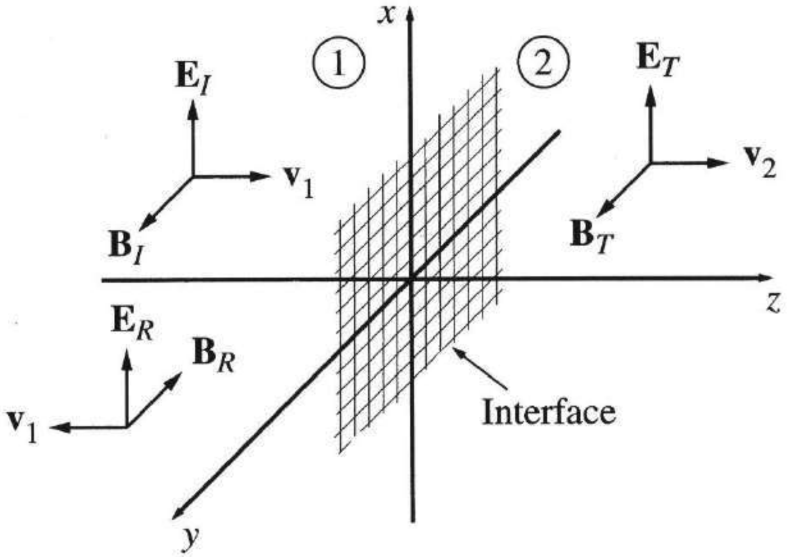
9.3.2 Reflection and Transmission at Normal Incidence

A plane wave of frequency ω, traveling in the z direction and polarized in the x direction, approaches the interface from the left.

Incident wave:

$$\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_I(z,t) = \frac{1}{v_1} \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{\mathbf{y}}$$



Reflected wave:

$$\tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_R(z,t) = -\frac{1}{v_1} \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{\mathbf{y}}$$

Transmitted wave:

$$\tilde{\mathbf{E}}_T(z,t) = \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_T(z,t) = \frac{1}{v_2} \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{\mathbf{y}}$$

The Boundary Conditions

Normal incident: no components perpendicular to the surface.

$$\mathbf{E}_1^{//} - \mathbf{E}_2^{//} = 0$$

$$\mathbf{H}_1^{//} - \mathbf{H}_2^{//} = (\mathbf{K}_f \times \hat{\mathbf{n}}) \Rightarrow \frac{1}{\mu_1}\mathbf{B}_1^{//} - \frac{1}{\mu_2}\mathbf{B}_2^{//} = 0$$

$$\tilde{\mathbf{B}}_0 = \frac{k}{\omega}(\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_0) = \frac{1}{v}(\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_0)$$

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}$$

$$\Rightarrow (\tilde{E}_{0I} - \tilde{E}_{0R}) = \beta \tilde{E}_{0T}, \quad \text{where } \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

In terms of the incident amplitude:

$$\tilde{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0I}$$

$$\tilde{E}_{0T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0I}$$

if $\mu \simeq \mu_0 \quad \Rightarrow$

$$\tilde{E}_{0R} = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)\tilde{E}_{0I}$$

$$\tilde{E}_{0T} = \left(\frac{2v_2}{v_1 + v_2}\right)\tilde{E}_{0I}$$

Comparison: The Complex Amplitudes of a String

Incident wave: $\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)}$

Reflected wave: $\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)}$

Transmitted wave: $\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)}$

Boundary conditions: $\tilde{f}(0^-,t) = \tilde{f}(0^+,t)$ $\quad \left.\frac{d\tilde{f}}{dz}\right|_{0^-} = \left.\frac{d\tilde{f}}{dz}\right|_{0^+}$

$$\left.\begin{array}{l} \tilde{A}_I + \tilde{A}_R = \tilde{A}_T \\ k_1(\tilde{A}_I - \tilde{A}_R) = k_2 \tilde{A}_T \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \tilde{A}_R = \left(\dfrac{k_1 - k_2}{k_1 + k_2}\right)\tilde{A}_I = \left(\dfrac{v_2 - v_1}{v_2 + v_1}\right)\tilde{A}_I \\ \tilde{A}_T = \left(\dfrac{2k_1}{k_1 + k_2}\right)\tilde{A}_I = \left(\dfrac{2v_2}{v_2 + v_1}\right)\tilde{A}_I \end{array}\right.$$

When $v_2 > v_1$, all three waves have the same phase angle.

When $v_2 < v_1$, the reflected wave is out of phase by 180°.

Reflection and Transmission Coefficients

The reflected wave is in phase if $v_2 > v_1$
and is out of phase if $v_2 < v_1$

$$\tilde{E}_{0R} = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)\tilde{E}_{0I} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)\tilde{E}_{0I}$$

$$\tilde{E}_{0T} = \left(\frac{2v_2}{v_1 + v_2}\right)\tilde{E}_{0I} = \left(\frac{2n_1}{n_1 + n_2}\right)\tilde{E}_{0I}$$

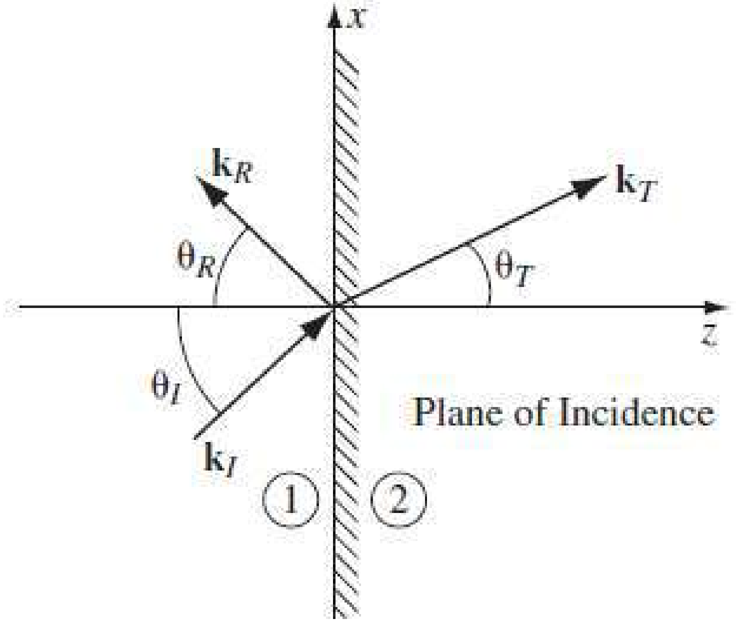
The intensity (average power per unit area) is: $I \equiv \langle S \rangle = \frac{1}{2} v\varepsilon E_0^2$

Reflection coefficient $R \equiv \frac{I_R}{I_I} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$

Transmission coefficient $T \equiv \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}$

9.3.2 Reflection and Transmission at Oblique Incidence

Suppose that a monochromatic plane wave of frequency ω, traveling in the $\mathbf{k}_I$ direction



Incident wave:

$$\tilde{\mathbf{E}}_I(\mathbf{r},t)=\tilde{\mathbf{E}}_{0I}e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}$$

$$\tilde{\mathbf{B}}_I(\mathbf{r},t)=\frac{1}{v_1}(\hat{\mathbf{k}}_I\times\tilde{\mathbf{E}}_I)$$

Reflected wave:

$$\tilde{\mathbf{E}}_R(\mathbf{r},t)=\tilde{\mathbf{E}}_{0R}e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}$$

$$\tilde{\mathbf{B}}_R(\mathbf{r},t)=\frac{1}{v_1}(\hat{\mathbf{k}}_R\times\tilde{\mathbf{E}}_R)$$

Transmitted wave:

$$\tilde{\mathbf{E}}_T(\mathbf{r},t)=\tilde{\mathbf{E}}_{0T}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}$$

$$\tilde{\mathbf{B}}_T(\mathbf{r},t)=\frac{1}{v_2}(\hat{\mathbf{k}}_T\times\tilde{\mathbf{E}}_T)$$

Boundary Conditions

All three waves have the same frequency ω.

$$\omega = k_I v_1 = k_R v_1 = k_T v_2 \quad \text{or} \quad k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T$$

Using the boundary conditions

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = 0 \qquad \mathbf{E}_1^{//} - \mathbf{E}_2^{//} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\mathbf{B}_1^{//} - \frac{1}{\mu_2}\mathbf{B}_2^{//} = 0$$

A generic structure for the four boundary conditions.

$$\underbrace{(\ )_{0I}\, e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)} + (\ )_{0R}\, e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}}_{①} - \underbrace{(\ )_{0T}\, e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}}_{②} = 0$$

Laws of Reflection and Refraction

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r}$$

$$k_{Ix} x + k_{Iz} z = k_{Rx} x + k_{Rz} z = k_{Tx} x + k_{Tz} z$$

$$\text{at } z = 0 \;\Rightarrow\; k_{Ix} x = k_{Rx} x = k_{Tx} x$$

$$\therefore k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

θ_I, θ_R, and θ_T are angles of incidence, reflection, and refraction, respectively.

$p = \hbar k$ momentum conservation

The law of reflection: $\theta_I = \theta_R$

The law of refraction: (Snell's law)

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{k_I}{k_T} = \frac{v_2}{v_1} = \frac{n_1}{n_2}$$

Common properties of waves: These equations are obtained from their generic form.

Boundary Conditions (ii)

$$\underbrace{(\ )_{0I}\, e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)} + (\ )_{0R}\, e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}}_{①} - \underbrace{(\ )_{0T}\, e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}}_{②} = 0$$

We have taken care of the exponential factors—they cancel.
The boundary conditions become:

(i) $\varepsilon_1(\tilde{\mathbf{E}}_{0I} + \tilde{\mathbf{E}}_{0R})_z = \varepsilon_2(\tilde{E}_{0T})_z$ Normal D

(ii) $(\tilde{\mathbf{B}}_{0I} + \tilde{\mathbf{B}}_{0R})_z = (\tilde{\mathbf{B}}_{0T})_z$ Normal B

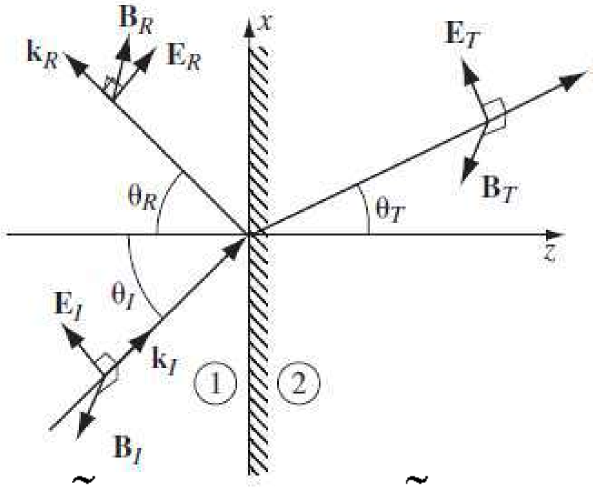
(iii) $(\tilde{\mathbf{E}}_{0I} + \tilde{\mathbf{E}}_{0R})_{x,y} = (\tilde{E}_{0T})_{x,y}$ Tangential E

(iv) $\dfrac{1}{\mu_1}(\tilde{\mathbf{B}}_{0I} + \tilde{\mathbf{B}}_{0R})_{x,y} = \dfrac{1}{\mu_2}(\tilde{\mathbf{B}}_{0T})_{x,y}$ Tangential H

where $\tilde{\mathbf{B}}_0(\mathbf{r},t) = \dfrac{1}{v}(\hat{\mathbf{k}} \times \tilde{\mathbf{E}}_0)$

E-Parallel to the Plane of Incidence

Q: If the polarization of the incident wave is parallel to the plane of incidence, are the reflected and transmitted waves also polarized in this plane? Yes.



Normal D (i) $\varepsilon_1(-\tilde{E}_{0I}\sin\theta_I + \tilde{E}_{0R}\sin\theta_R) = \varepsilon_2(-\tilde{E}_{0T}\sin\theta_T)$

Tangential E (iii) $(\tilde{E}_{0I}\cos\theta_I + \tilde{E}_{0R}\cos\theta_R) = (\tilde{E}_{0T}\cos\theta_T)$

Normal B (ii) $0 = 0$

Tangential H (iv) $\frac{1}{\mu_1 v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{1}{\mu_2 v_2}(\tilde{E}_{0T})$

E-Parallel to the Plane of Incidence (ii)

$$\text{(iii)} \quad (\tilde{E}_{0I} + \tilde{E}_{0R}) = \alpha(\tilde{E}_{0T}) \qquad \alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}$$

$$\text{(iv)} \quad (\tilde{E}_{0I} - \tilde{E}_{0R}) = \beta(\tilde{E}_{0T}) \qquad \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}$$

$$\Rightarrow \tilde{E}_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0I} \qquad \tilde{E}_{0T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0I}$$ Fresnel's equations

How about the first boundary condition?
Does this condition contribute anything new?

$$\text{(i)} \quad (\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{\varepsilon_2 \sin\theta_T}{\varepsilon_1 \sin\theta_I}(\tilde{E}_{0T}) \qquad \frac{\varepsilon_2 \sin\theta_T}{\varepsilon_1 \sin\theta_I} = \frac{\mu_1 v_1}{\mu_2 v_2}\ ?$$

Brewster's Angle

$$\begin{cases} \tilde{E}_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0I} \\ \tilde{E}_{0T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0I} \end{cases} \quad \text{where } \alpha \equiv \frac{\cos\theta_T}{\cos\theta_I} \text{ and } \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}$$

When $\alpha = \beta$, there is no reflected wave. $\tilde{E}_{0R} = 0$

$$\frac{\cos\theta_T}{\cos\theta_I} = \frac{\mu_1 v_1}{\mu_2 v_2}, \text{ then } \theta_I \text{ is called Brewster's angle, } \theta_B.$$

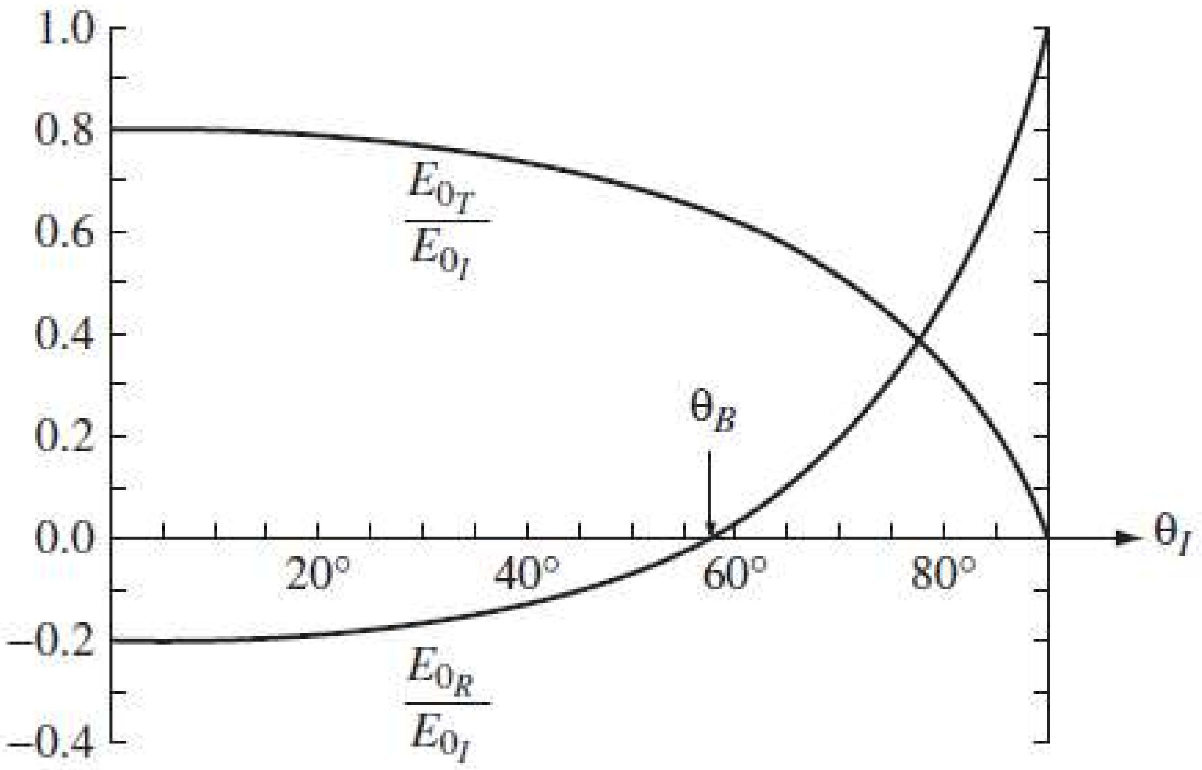
$$\frac{1-\sin^2\theta_T}{1-\sin^2\theta_B} = \beta^2 \text{ and } \sin^2\theta_T = \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B \text{ (Snell's law)}$$

$$1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B = \beta^2 - \beta^2 \sin^2\theta_B \Rightarrow \sin^2\theta_B = \frac{1-\beta^2}{(n_1/n_2)^2-\beta^2}$$

Brewster's Angle (II)

If $\mu_1 \cong \mu_2$, $\beta \cong n_2 / n_1$ and $\sin^2 \theta_B \cong \beta^2 / (1 + \beta^2)$

$$\Rightarrow \qquad \tan \theta_B \cong \frac{n_2}{n_1}$$



Transmission and Reflection

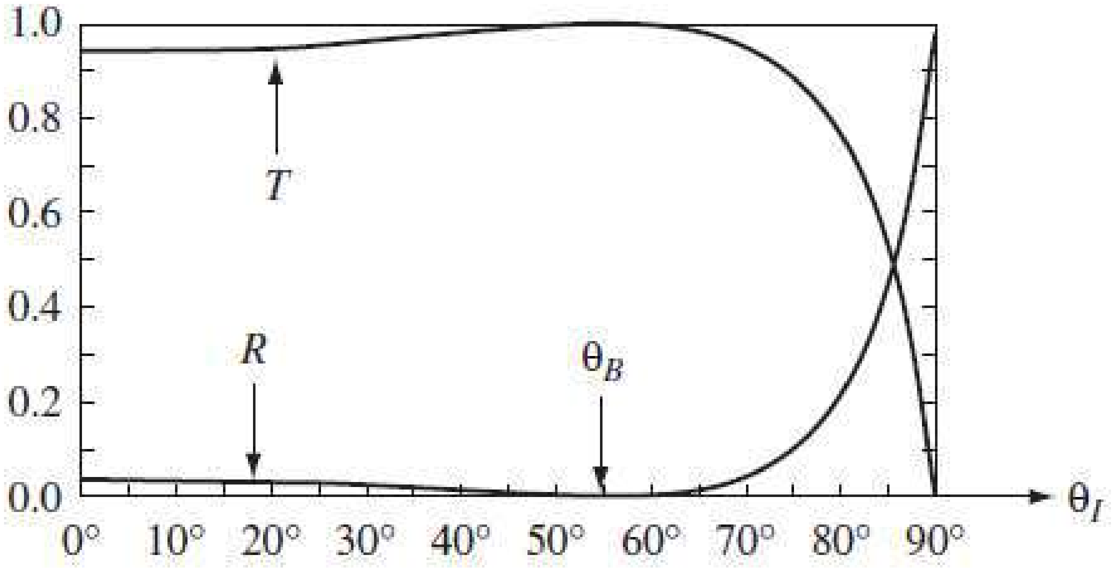
$$I_I \equiv \langle \mathbf{S} \cdot \hat{\mathbf{z}} \rangle = \frac{1}{2} v_1 \varepsilon_1 E_{0I}^2 \cos\theta_I$$

$$I_R = \frac{1}{2} v_1 \varepsilon_1 E_{0R}^2 \cos\theta_R = (\frac{\alpha - \beta}{\alpha + \beta})^2 I_I$$

$$I_T = \frac{1}{2} v_2 \varepsilon_2 E_{0T}^2 \cos\theta_T = \alpha\beta(\frac{2}{\alpha + \beta})^2 I_I$$

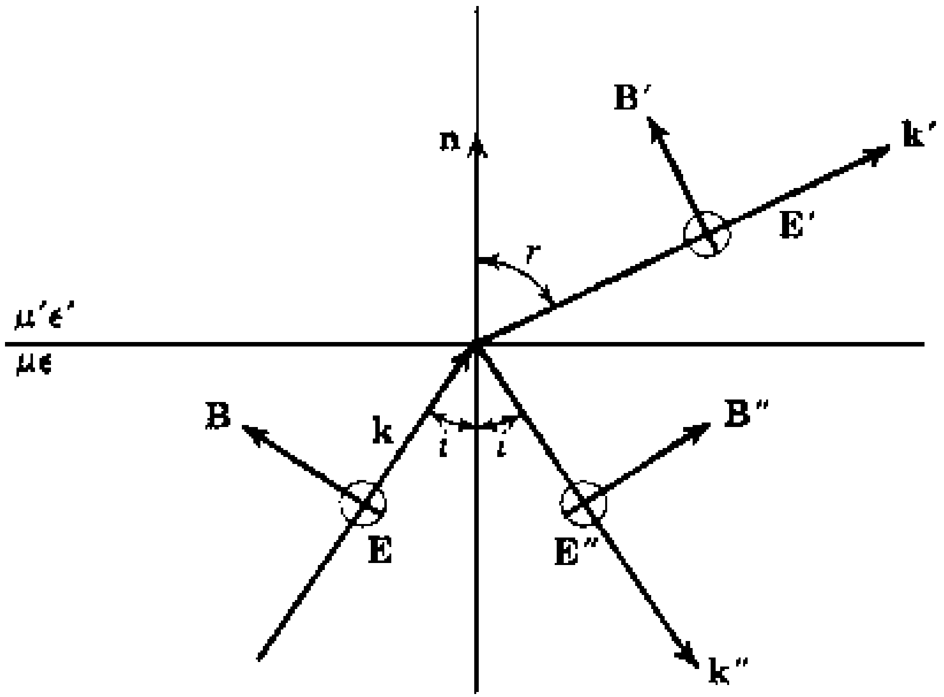
$$R \equiv \frac{I_R}{I_I} = (\frac{\alpha - \beta}{\alpha + \beta})^2$$

$$T \equiv \frac{I_T}{I_I} = \alpha\beta(\frac{2}{\alpha + \beta})^2$$



E-Perpendicular to the Plane of Incidence

Q: If the polarization of the incident wave is **perpendicular** to the plane of incidence, are the reflected and transmitted waves also polarized in this plane? Yes.



See Problem 9.16

9.4 Absorption and Dispersion
9.4.1 Electromagnetic Waves in Conductors

When wave propagates through vacuum or insulating materials such as glass or teflon, assuming no free charge and no free current is reasonable.

But in conductive media such as metal or plasma, the free charge and free current are generally not zero.
The free current is proportional to the electric field: Ohm's law

conductivity

$$\mathbf{J}_f = \sigma \mathbf{E}$$

Maxwell's equations for *linear* media assume the form

$$\nabla \cdot \mathbf{E} = \frac{\rho_f}{\varepsilon} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = \mu\sigma\mathbf{E}$$

Electromagnetic Waves in Conductors (II)

The continuity equation for free charge: $\frac{\partial \rho_f}{\partial t} = -\nabla \cdot \mathbf{J}_f$

conductivity

$\mathbf{J}_f = \sigma \mathbf{E}$

$\rho_f = \varepsilon \nabla \cdot \mathbf{E}$

$$\Rightarrow \quad \frac{\partial \rho_f}{\partial t} = -\sigma(\nabla \cdot \mathbf{E}) = -\sigma \frac{\rho_f}{\varepsilon} = -\frac{\sigma}{\varepsilon}\rho_f$$

For a homogeneous linear medium: $\rho_f(t) = e^{-\frac{t}{\tau}} \rho_f(0)$

where $\tau = \frac{\varepsilon}{\sigma}$

Classification of conductors:

superconductor $\sigma = \infty, \ \tau = 0$

perfect conductor $\sigma = \infty, \ \tau = 0$

good conductor $\tau << \frac{1}{\omega}$

poor conductor $\tau >> \frac{1}{\omega}$

What's the difference?

See Prob. 7.42

$\tau \approx 10^{-19}$s for copper

$\tau_c \sim 10^{-14}$s collision time

Electromagnetic Waves in Conductors (III)
Omitting Transient Effect

Omit the transient behavior.

Assume no charges accumulation: $\rho_f = 0$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mu\sigma\mathbf{E}$$

$$\Rightarrow \nabla \times (\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}) = \nabla \times (\nabla \times \mathbf{E}) + \frac{\partial (\nabla \times \mathbf{B})}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$$

=0

$$\frac{\partial (\nabla \times \mathbf{B})}{\partial t} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t}, \quad \nabla^2 \mathbf{B} = \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t} \ \text{(likewise)}$$

Electromagnetic Waves in Conductors (IV)
Complex Wave Number

These equations still admit plane-wave solutions,

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla^2 \mathbf{B} = \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t}$$

$$\Rightarrow \begin{cases} \tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z - \omega t)} \\ \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z - \omega t)} \end{cases}$$

Note this time the "wave number" $\tilde{k}$ is complex:

$$\tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$$

$$\tilde{k} = k + i\kappa, \quad \text{where} \begin{cases} k \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1\right]^{1/2} \\ \kappa \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{1/2} \end{cases}$$

The Real Parts of The Fields

$$\tilde{\mathbf{E}}(z,t)=\tilde{\mathbf{E}}_0 e^{-\kappa z}e^{i(kz-\omega t)} \xrightarrow[\text{Faraday's law}]{} \tilde{\mathbf{B}}(z,t)=\frac{\tilde{k}}{\omega}\tilde{\mathbf{E}}_0 e^{-\kappa z}e^{i(kz-\omega t)}$$

$$\tilde{k}=k+i\kappa=Ke^{i\phi}$$

$$K\equiv\sqrt{k^2+\kappa^2}=\omega\sqrt{\varepsilon\mu\sqrt{1+(\frac{\sigma}{\varepsilon\omega})^2}} \quad \text{and} \quad \phi\equiv\tan^{-1}(\kappa/k)$$

$$\tilde{\mathbf{B}}(z,t)=\frac{\tilde{k}}{\omega}\tilde{\mathbf{E}} \quad \Rightarrow \quad B_0 e^{i\delta_B}=\frac{Ke^{i\phi}}{\omega}E_0 e^{i\delta_E}$$

$$\delta_B-\delta_E=\phi \quad \text{and} \quad \frac{B_0}{E_0}=\frac{K}{\omega}=\sqrt{\varepsilon\mu\sqrt{1+(\frac{\sigma}{\varepsilon\omega})^2}}$$

$$\mathbf{E}(z,t)=E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\hat{\mathbf{x}}$$

$$\mathbf{B}(z,t)=\frac{K}{\omega}E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\hat{\mathbf{y}}$$

skin depth

$$d\equiv\frac{1}{\kappa}\simeq\frac{1}{\omega}\sqrt{\frac{2}{\varepsilon\mu}\frac{\varepsilon\omega}{\sigma}}=\sqrt{\frac{2}{\mu\sigma\omega}}$$

9.4.2 Reflection at a Conducting Surface

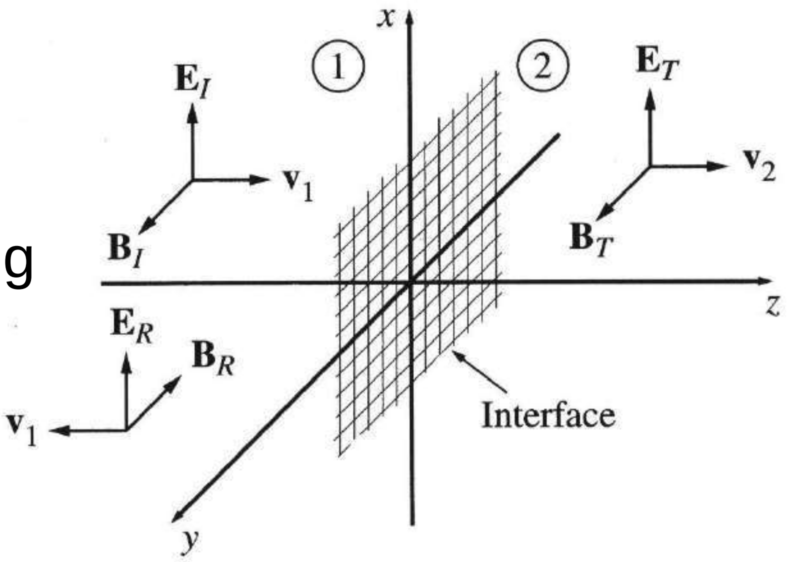
$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f \qquad \mathbf{E}_1^{//} - \mathbf{E}_2^{//} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\mathbf{B}_1^{//} - \frac{1}{\mu_2}\mathbf{B}_2^{//} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

Where σ_f is the free surface charge, $\mathbf{K}_f$ is the free surface current, and $\hat{\mathbf{n}}$ is a unit vector perpendicular to the surface, pointing from medium (2) into medium (1).

Normal incident

(1) nonconducting linear medium



(2) conductor

Reflection at a Conducting Surface (II)

Incident wave:

$$\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_I(z,t) = \frac{1}{v_1} \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{\mathbf{y}} \infty$$

Reflected wave:

$$\tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_R(z,t) = -\frac{1}{v_1} \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{\mathbf{y}}$$

Transmitted wave:

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_{0T} e^{-\kappa z} e^{i(k_2 z - \omega t)} \hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}(z,t) = \frac{\tilde{k}_2}{\omega} \tilde{E}_{0T} e^{-\kappa z} e^{i(k_2 z - \omega t)} \hat{\mathbf{y}}$$

Normal components of the fields

$$\begin{aligned} \varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} &= \sigma_f \\ B_1^{\perp} - B_2^{\perp} &= 0 \end{aligned} \Rightarrow \sigma_f = 0 \left(\begin{array}{l} \text{because electric field polarized} \\ \text{in } x \text{ direction, so } E_1^{\perp} = E_2^{\perp} = 0. \end{array} \right)$$

Reflection at a Conducting Surface (III)

Tangential components of the fields at $z = 0$:

$$\mathbf{E}_1^{//} - \mathbf{E}_2^{//} = 0$$

$$\frac{1}{\mu_1}\mathbf{B}_1^{//} - \frac{1}{\mu_2}\mathbf{B}_2^{//} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

$\Longrightarrow$

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) - \frac{\tilde{k}_2}{\mu_2 \omega}\tilde{E}_{0T} = K_f$$

Case (i) $K_f = 0$, the simplest case.

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$

$$(\tilde{E}_{0I} - \tilde{E}_{0R}) = \tilde{\beta}\tilde{E}_{0T}, \text{ where } \beta \equiv \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$$

$\Longrightarrow$

$$\tilde{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0I}$$

$$\tilde{E}_{0T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0I}$$

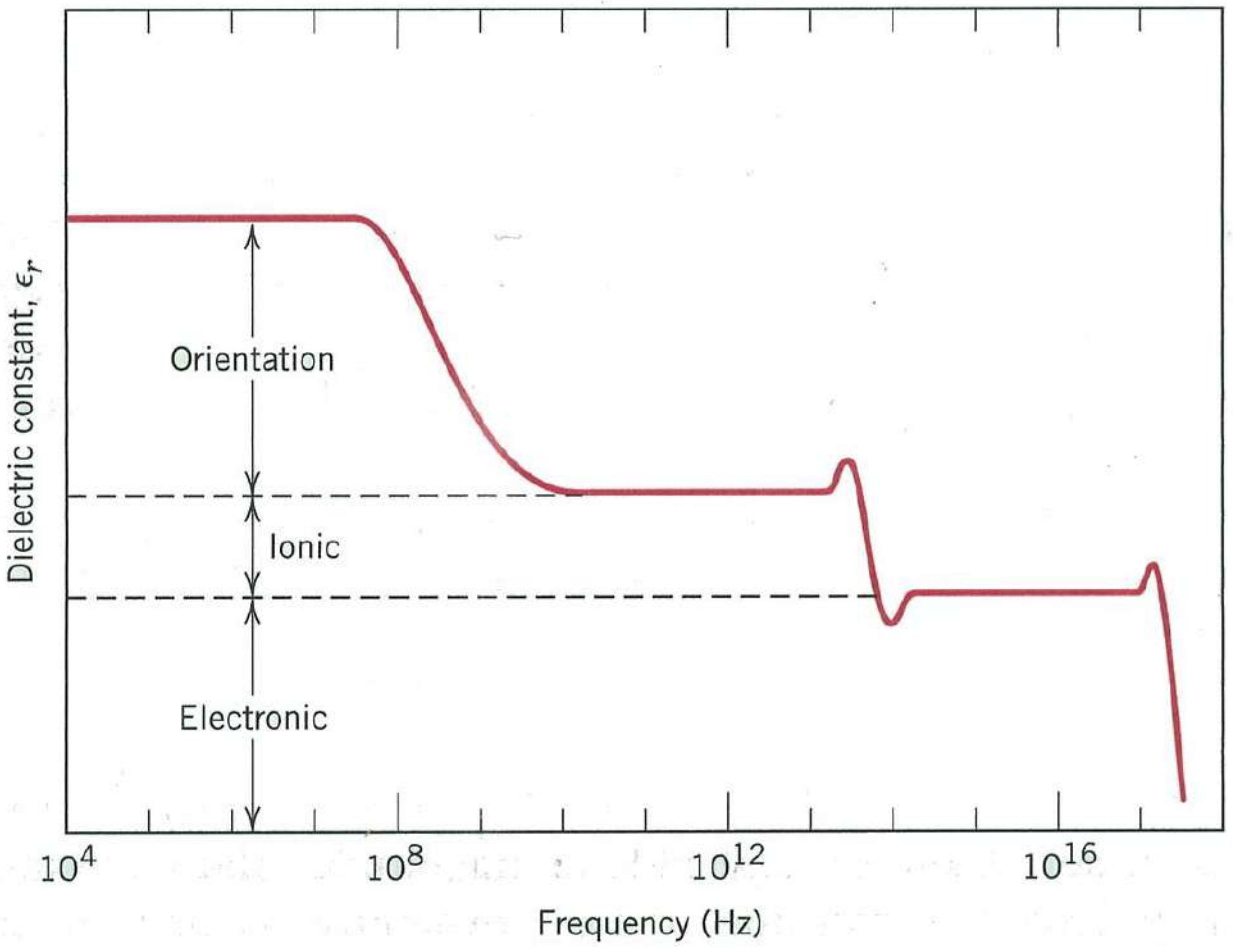
For a perfect conductor ($\sigma = \infty$), $k_2 = \infty$ ➔ $\tilde{E}_{0R} = -\tilde{E}_{0I}$ and $\tilde{E}_{0T} = 0$

That's why excellent conductors make good mirrors.

Case (ii) $K_f \neq 0$, EM field manipulation (interesting subject.)

9.4.3 The Frequency Dependence of Permittivity

When the speed of a wave depends on its frequency, the supporting medium is called **dispersive**.



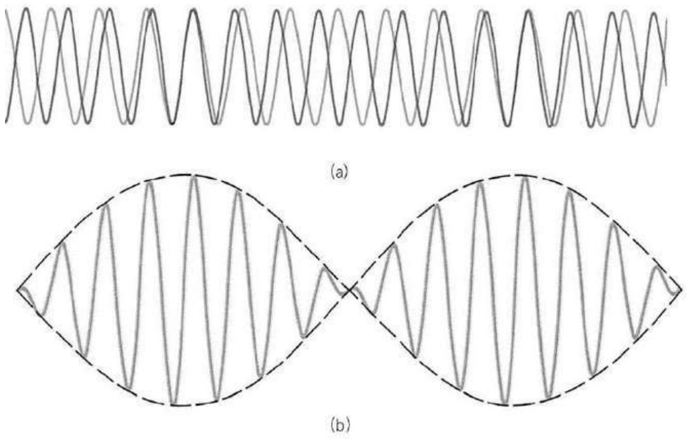
The Group Velocity and Phase Velocity

When two waves of slightly different frequencies are superposed, the resulting disturbance varies *periodically in amplitude*.

$$A\sin((k_0+\Delta k)z-(\omega_0+\Delta\omega)t)+A\sin((k_0-\Delta k)z-(\omega_0-\Delta\omega)t)$$

$$=A\sin((k_0 z-\omega_0 t)+(\Delta kz-\Delta\omega t))+A\sin((k_0 z-\omega_0 t)-(\Delta kz-\Delta\omega t))$$

$$=2A\cos[(\Delta kz-\Delta\omega t)]\sin[(k_0 z-\omega_0 t)]$$



Phase velocity $v_p=\dfrac{\omega_0}{k_0}$

Group velocity $v_g=\dfrac{\Delta\omega}{\Delta k}=\dfrac{d\omega}{dk}$

Phase delay $\tau_p=\int\dfrac{1}{v_p}dz=\int\dfrac{k_0}{\omega_0}dz=\dfrac{\Delta\phi}{\omega_0}$

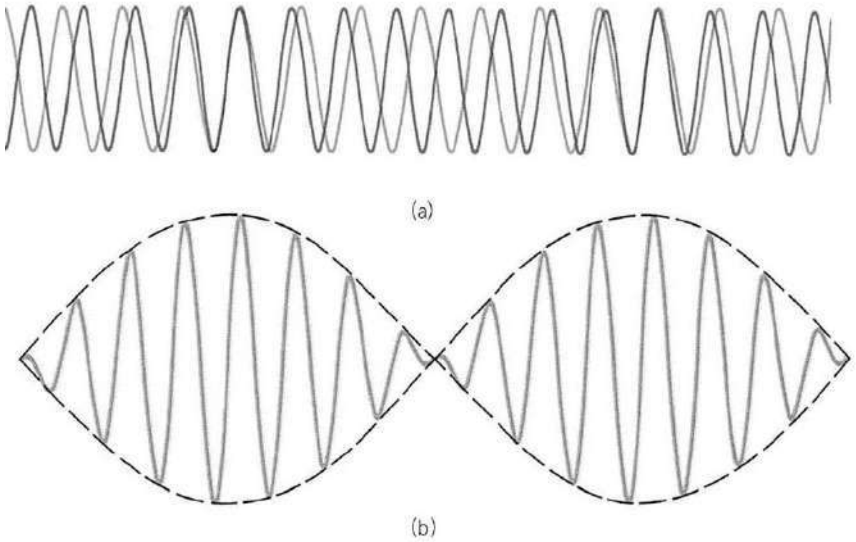
Group delay $\tau_g=\int\dfrac{1}{v_g}dz=\int\dfrac{dk}{d\omega}dz=\dfrac{d(\Delta\phi)}{d\omega}$

Interference in Time: Beats

When two waves of slightly different frequencies are superposed, the resulting disturbance varies *periodically in amplitude*.

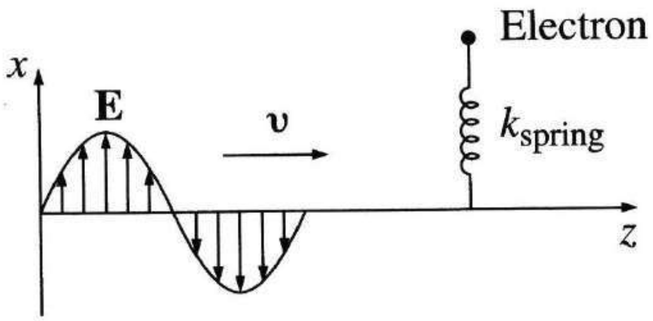
$$y = y_1 + y_2 = A\sin(2\pi f_1 t) + A\sin(2\pi f_2 t)$$
$$= 2A\cos[2\pi(\frac{f_1 - f_2}{2})t]\sin[2\pi(\frac{f_1 + f_2}{2})t]$$

Beat frequency ($|f_1 - f_2|$): frequency of the amplitude envelope



Simplified Model for the Frequency Dependence of Permittivity in Nonconductors

The electrons in a nonconductor are bound to specific molecule or atom.



The simplified binding force: $F_{\text{binding}} = -k_{\text{spring}} x = -m\omega_0^2 x$

The damping force on the electron: $F_{\text{damping}} = -m\gamma \dfrac{dx}{dt}$

(rate of change of electron momentum due to collision)

The driving force on the electron: $F_{\text{driving}} = qE = qE_0 \cos(\omega t)$

Newton's law: $m\dfrac{d^2x}{dt^2} = F_{\text{tot}} = F_{\text{binding}} + F_{\text{damping}} + F_{\text{driving}}$

Permittivity in Nonconductors

The equation of motion

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = qE_0\cos(\omega t)$$

$$\text{Re}\left\{m\frac{d^2\tilde{x}}{dt^2} + m\gamma\frac{d\tilde{x}}{dt} + m\omega_0^2\tilde{x} = qE_0e^{-i\omega t}\right\}$$

Let the system oscillates at the driving frequency ω

$$\tilde{x} = \tilde{x}_0 e^{-i\omega t},\ \text{ where } \tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0$$

The dipole moment is the real part of $\tilde{p} = q\tilde{x}(t)$

$$\tilde{p} = \frac{q^2}{m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0e^{-i\omega t}$$

Permittivity in Nonconductors (II)

N molecules per unit volume; each molecule contains f_j electrons with frequency ω_j and damping γ_j.
The polarization $\mathbf{P}$ is given by the real part of:

$$\tilde{\mathbf{P}} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\tilde{\mathbf{E}} = \varepsilon_0 \tilde{\chi}_e \tilde{\mathbf{E}}$$

$$\tilde{\chi}_e = \frac{Nq^2}{\varepsilon_0 m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right) \leftarrow \text{the complex susceptibility}$$

the complex permittivity $\varepsilon = \varepsilon_0(1 + \tilde{\chi}_e)$

the complex dielectric constant

$$\varepsilon_r = (1 + \tilde{\chi}_e) = 1 + \frac{Nq^2}{\varepsilon_0 m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)$$

Waves in a Dispersive medium

The wave equation for a given frequency reads

$$\nabla^2 \tilde{\mathbf{E}} = \mu\tilde{\varepsilon}\frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2} \qquad \tilde{\varepsilon} = \varepsilon_0(1+\tilde{\chi}_e) = \varepsilon_0 + \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)$$

$$\tilde{k} \equiv \sqrt{\tilde{\varepsilon}\mu_0}\,\omega = k + i\kappa \qquad \tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$

$$I \equiv \langle S \rangle = I_0 e^{-2\kappa z}, \quad \alpha \equiv 2\kappa \text{ (absorption coefficient)}$$

For gases, the second term of $\tilde{\varepsilon}$ is small, i.e. $\tilde{\chi}_e \ll 1$.

$$\tilde{k} \equiv \frac{\omega}{c}\sqrt{\tilde{\varepsilon}_r} \cong \frac{\omega}{c}(1+\frac{1}{2}\tilde{\chi}_e) = \frac{\omega}{c}\left[1+\frac{Nq^2}{2m\varepsilon_0}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\right]$$

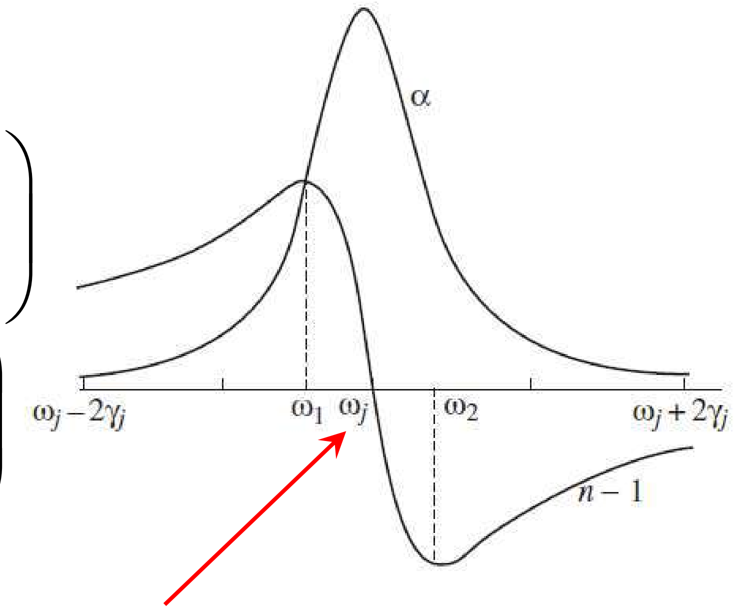
The binomial expansion

Anomalous Dispersion

The index of refraction:

$$n = \frac{ck}{\omega} \cong 1 + \frac{Nq^2}{2m\varepsilon_0}\left(\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}\right)$$

$$\alpha = 2\kappa \cong \frac{Nq^2\omega^2}{m\varepsilon_0 c}\left(\sum_j \frac{f_j\gamma_j^2}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}\right)$$

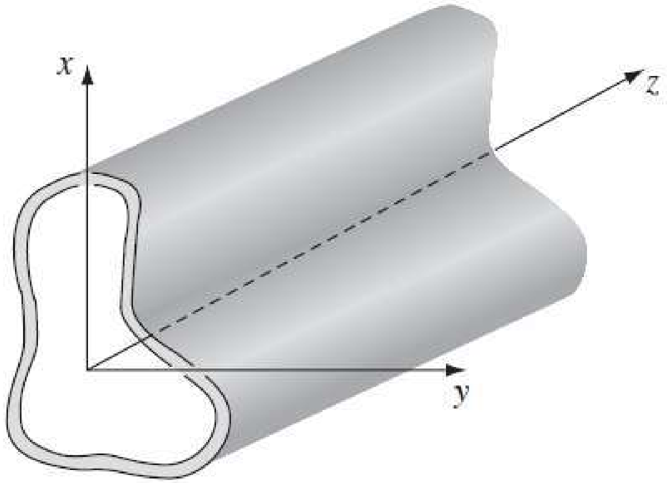


In the immediate neighborhood of a resonance, the index of refraction drops sharply ($n<1$). 🡐 called anomalous dispersion.

Faster Than Light (**FTL**):
Can we find cases where the waves propagate at a speed faster than the speed of light ($v/c=1/n$)? Superluminal effect.

9.5 Guided Waves
9.5.1 Wave Guides

Can the electromagnetic waves propagate in a hollow metal pipe? Yes, they can.



Waveguides generally made of good conductor, so that $\mathbf{E} = 0$ and $\mathbf{B} = 0$ inside the material.

The boundary conditions at the inner wall are: $\mathbf{E}^{//} = 0$ and $B^{\perp} = 0$ …

The generic form of the monochromatic waves:

$$\begin{cases} \tilde{\mathbf{E}}(x,y,z,t) = \tilde{\mathbf{E}}_0(x,y)e^{i(\tilde{k}z-\omega t)} = (\tilde{E}_x\hat{\mathbf{x}} + \tilde{E}_y\hat{\mathbf{y}} + \tilde{E}_z\hat{\mathbf{z}})e^{i(\tilde{k}z-\omega t)} \\ \tilde{\mathbf{B}}(x,y,z,t) = \tilde{\mathbf{B}}_0(x,y)e^{i(\tilde{k}z-\omega t)} = (\tilde{B}_x\hat{\mathbf{x}} + \tilde{B}_y\hat{\mathbf{y}} + \tilde{B}_z\hat{\mathbf{z}})e^{i(\tilde{k}z-\omega t)} \end{cases}$$

The confined waves are not (in general) transverse.

General Properties of Wave Guides

In the interior of the wave guide, the waves satisfy Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 0 \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

Why $\rho_f = 0$ and $J_f = 0$?

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \frac{1}{v^2}\frac{\partial \mathbf{E}}{\partial t}$$

where $v = \dfrac{1}{\sqrt{\varepsilon\mu}}$

We obtain

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \times \mathbf{B} = \frac{1}{v^2}\frac{\partial \mathbf{E}}{\partial t}$$

$$\text{(i)}\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z \qquad \text{(iv)}\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z$$

$$\text{(ii)}\ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\omega B_x \qquad \text{(v)}\ \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = -\frac{i\omega}{c^2}E_x$$

$$\text{(iii)}\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\omega B_y \qquad \text{(vi)}\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y$$

TE, TM, and TEM Waves

Determining the longitudinal components E_z and B_z, we could quickly calculate all the others.

$$E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$

$$B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

We obtain

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{v^2} - k^2\right]E_z = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{v^2} - k^2\right]B_z = 0$$

If $E_z = 0 \Rightarrow$ TE (transverse electric) waves;

If $B_z = 0 \Rightarrow$ TM (transverse magnetic) waves;

If $E_z = 0$ and $B_z = 0 \Rightarrow$ TEM waves.

No TEM Waves in a Hollow Wave Guide

Proof: A hollow wave guide cannot support the TEM wave.

$E_z = 0$, Ampere's law says $\nabla_t \times \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = -\nabla_t \phi_B$

$B_z = 0$, Faraday's law says $\nabla_t \times \mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{E} = -\nabla_t \phi_E$

ϕ_E and ϕ_B are potentials. $\begin{cases} \nabla \cdot \mathbf{E} = 0 \quad \Rightarrow \quad \nabla_t^2 \phi_E = 0 \\ \nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \nabla_t^2 \phi_B = 0 \end{cases}$

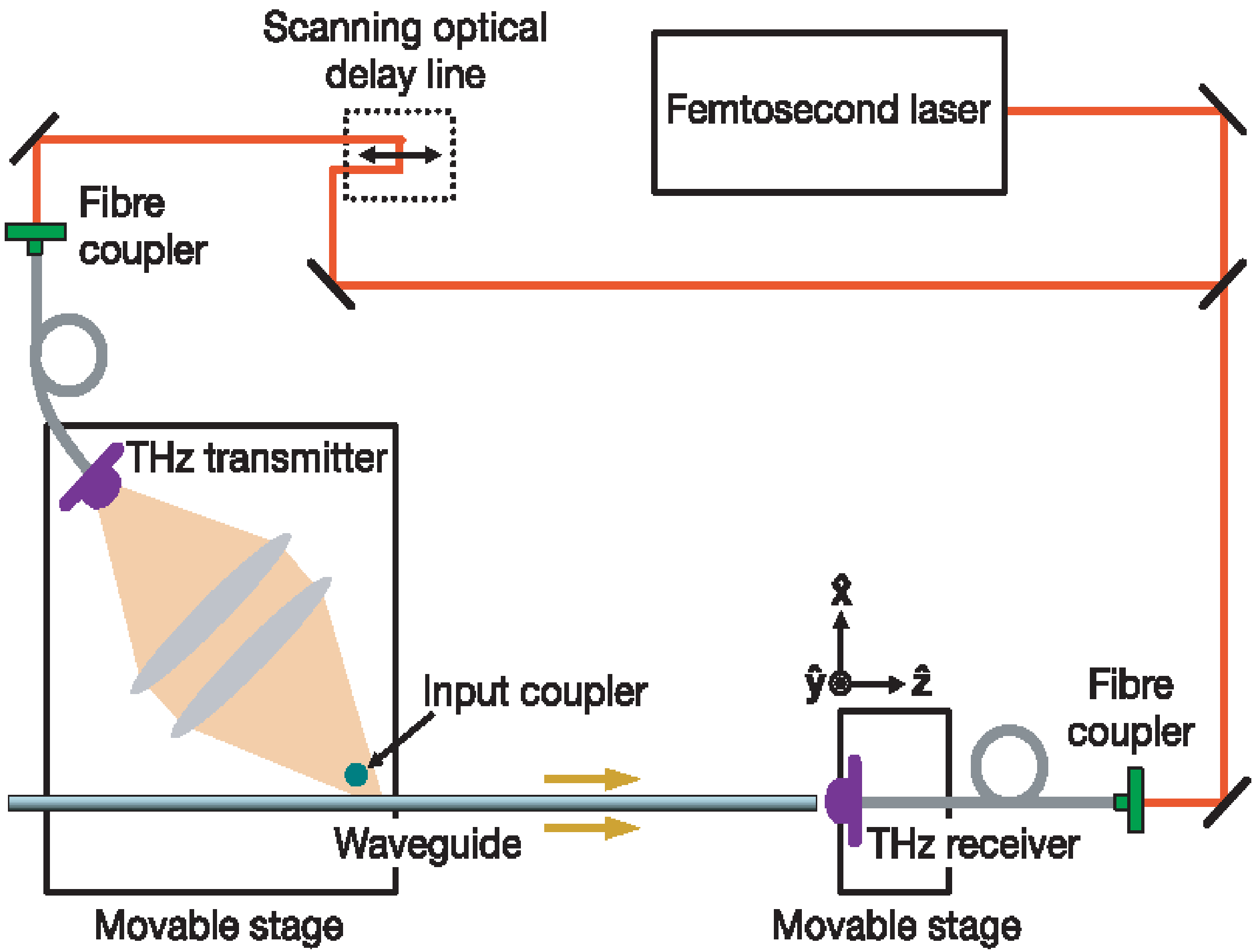
The boundary condition on $\mathbf{E}$ requires that the surface be an equal-potential. $\mathbf{E}^{//} = 0$

Laplace's equation admits no local maxima or minima.

➔ the potential is constant throughout. $\mathbf{E} = 0$ — no wave at all.

Can a metal wire support a wave? Yes.

A Diagram of the Optical Setup

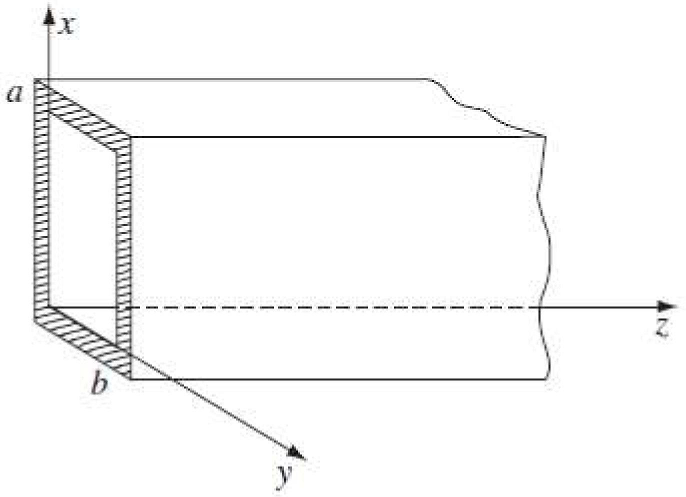


K. Wang and D. M. Mittleman, “Metal wires for terahertz wave guiding”, Nature, vol.432, No. 18, p.376, 2004.

9.5.2 TE Waves in a Rectangular Wave Guide

$E_z = 0,$ and $B_z(x, y) = X(x)Y(y) \leftarrow$ separation of variables

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \left(\frac{\omega^2}{v^2} - k^2\right) = 0$$



$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \quad \text{and} \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2$$

with $\dfrac{\omega^2}{v^2} = k^2 + k_x^2 + k_y^2$

$$X(x) = A\sin k_x x + B\cos k_x x$$
$$Y(y) = C\sin k_y y + D\cos k_y y$$

TE Waves in a Rectangular Wave Guide (II)

$$E_x \propto \frac{\partial B_z}{\partial y} \propto C\cos k_y y - D\sin k_y y$$

$$E_x(@\, y=0)=0 \Rightarrow C=0$$

$$E_x(@\, y=b)=0 \Rightarrow \sin k_y b = 0, k_y = \frac{n\pi}{b} (n=0,1,2,...)$$

$$E_y \propto \frac{\partial B_z}{\partial x} \propto A\cos k_x x - B\sin k_x x$$

$$E_y(@\, x=0)=0 \Rightarrow A=0$$

$$E_y(@\, x=a)=0 \Rightarrow \sin k_x a = 0, k_x = \frac{m\pi}{a} (m=0,1,2,...)$$

$$B_z(x,y) = B_0\cos(m\pi x/a)\cos(n\pi y/b) \leftarrow \text{the } \mathrm{TE_{mn}} \text{ mode}$$

$$k = \sqrt{(\omega/v)^2 - \pi^2[(m/a)^2 + (n/b)^2]}$$

TE Waves in a Rectangular Wave Guide (III)

$$B_z(x,y) = B_0 \cos(m\pi x/a)\cos(n\pi y/b)$$

In vacuum, $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$, $v = c$.

the cutoff frequency

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}, \text{ where } \omega_{mn}^2 = c^2\pi^2[(m/a)^2 + (n/b)^2]$$

If $\omega < \omega_{mn}$, the wave number is imaginary.

The lowest cutoff frequency of TE_{10} mode is: $\omega_{10} = c\pi/a$

The wave velocities are:

$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1-\omega_{mn}^2/\omega^2}} > c \quad \text{phase velocity}$$

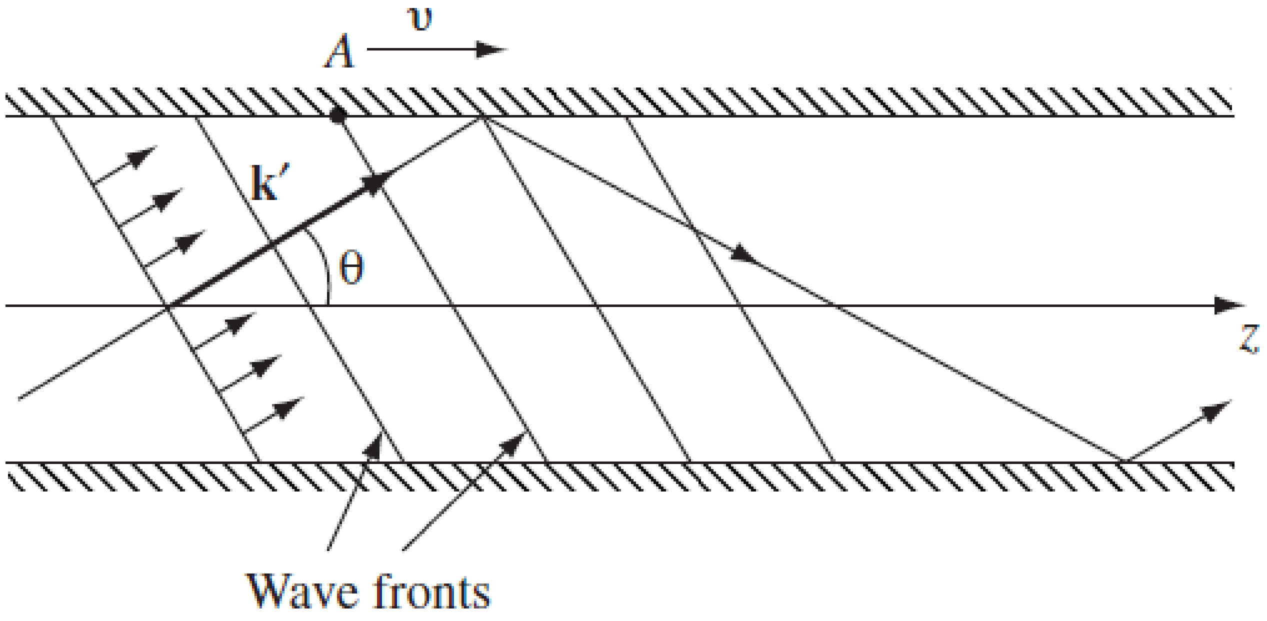
$$v_g = \frac{d\omega}{dk} = c\sqrt{1-\omega_{mn}^2/\omega^2} < c \quad \text{group velocity}$$

Why the Phase Velocity Greater Than The Speed of Light

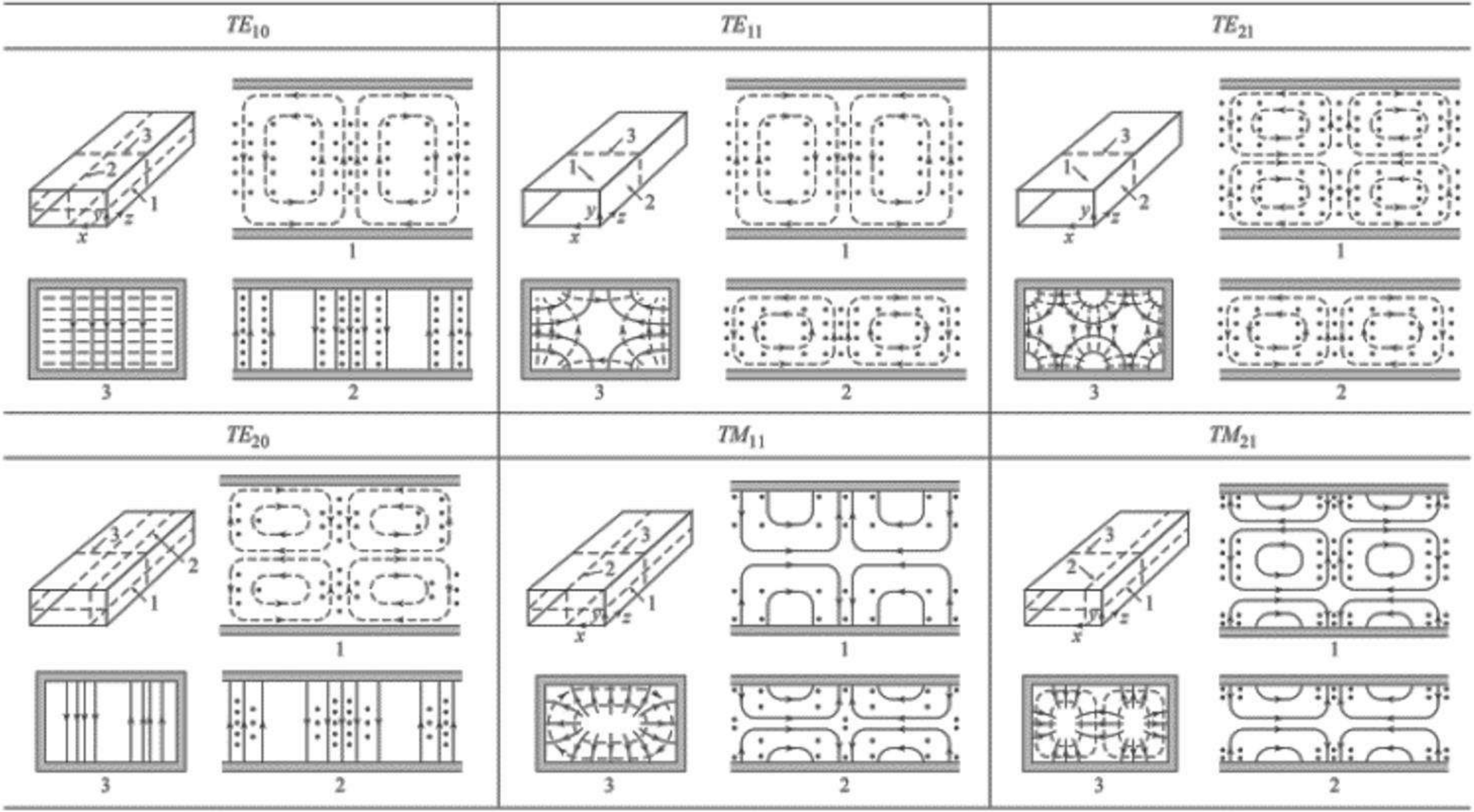
$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1-\omega_{mn}^2/\omega^2}} > c \quad \text{phase velocity}$$

$$v_g = \frac{d\omega}{dk} = c\sqrt{1-\omega_{mn}^2/\omega^2} < c \quad \text{group velocity}$$

$$v_p v_g = c^2$$

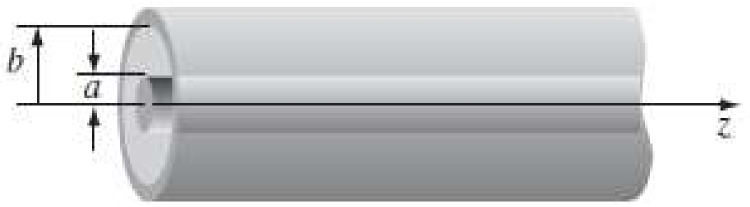


The Field Profiles: Examples



9.5.3 The Coaxial Transmission Line

A hollow wave guide cannot support the TEM wave, but a coaxial transmission line can.



$$\text{(i)}\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega B_z \qquad \text{(iv)}\ \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=-\frac{i\omega}{c^2}E_z$$

$$\text{(ii)}\ \frac{\partial E_z}{\partial y}-ikE_y=i\omega B_x \qquad \text{(v)}\ \frac{\partial B_z}{\partial y}-ikB_y=-\frac{i\omega}{c^2}E_x$$

$$\text{(iii)}\ ikE_x-\frac{\partial E_z}{\partial x}=i\omega B_y \qquad \text{(vi)}\ ikB_x-\frac{\partial B_z}{\partial x}=-\frac{i\omega}{c^2}E_y$$

$$\left.\begin{array}{l}\nabla_t\times\mathbf{E}=0\\ \nabla_t\times\mathbf{B}=0\end{array}\right\}\Rightarrow\left\{\begin{array}{l}\mathbf{E}=-\nabla_t\phi_E,\ \nabla_t{}^2\phi_E=0\ \ \text{electrostatic}\\ \mathbf{B}=-\nabla_t\phi_B,\ \nabla_t{}^2\phi_B=0\ \ \text{magnetostatic}\end{array}\right.$$

ϕ_E and ϕ_B are potentials.

The Coaxial Transmission Line (II)

The problem is reduced to two dimensions.

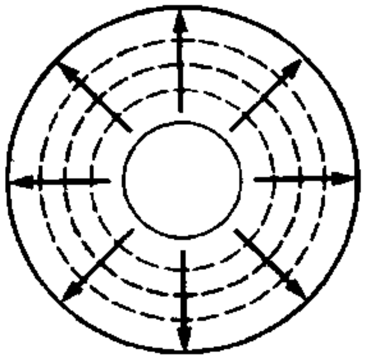
Electrostatic: the infinite line charge;
Magnetostatic: an infinite straight current.

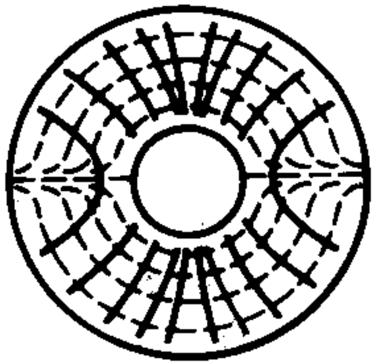
$$\mathbf{E}_0(s,\phi) = \frac{A}{s}\hat{\mathbf{s}}, \quad \mathbf{B}_0(s,\phi) = \frac{A}{cs}\hat{\boldsymbol{\phi}}$$

Taking the real part:

$$\mathbf{E}(s,\phi,z,t) = \frac{A\cos(kz-\omega t)}{s}\hat{\mathbf{s}}$$

$$\mathbf{B}(s,\phi,z,t) = \frac{A\cos(kz-\omega t)}{cs}\hat{\boldsymbol{\phi}}$$


(a) TEM


(b) TE11

Homework of Chap.9 (II)

Problem 9.17 Analyze the case of polarization *perpendicular* to the plane of incidence (i.e. electric fields in the y direction, in Fig. 9.15). Impose the boundary conditions (Eq. 9.101), and obtain the Fresnel equations for $\tilde{E}_{0R}$ and $\tilde{E}_{0r}$. Sketch $(\tilde{E}_{0R}/\tilde{E}_{0I})$ and $(\tilde{E}_{0T}/\tilde{E}_{0I})$ as functions of θ_I, for the case $\beta = n_2/n_1 = 1.5$. (Note that for this β the reflected wave is *always* 180° out of phase.) Show that there is no Brewster's angle for *any* n_1 and n_2: $\tilde{E}_{0R}$ is *never* zero (unless, of course, $n_1 = n_2$ and $\mu_1 = \mu_2$, in which case the two media are optically indistinguishable). Confirm that your Fresnel equations reduce to the proper forms at normal incidence. Compute the reflection and transmission coefficients, and check that they add up to 1.

Problem 9.20

(a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\varepsilon$) is $(2/\sigma)\sqrt{\varepsilon/\mu}$ (independent of frequency). Find the skin depth (in meters) for (pure) water. (Use the static values of ε, μ, and σ; your answers will be valid, then, only at relatively low frequencies.)

(b) Show that the skin depth in a good conductor ($\sigma \gg \omega\varepsilon$) is $\lambda/2\pi$ (where λ is the wavelength *in the conductor*). Find the skin depth (in nanometers) for a typical metal ($\sigma \approx 10^7 (\Omega m)^{-1}$) in the visible range ($\omega \approx 10^{15}/s$). assuming $\varepsilon \approx \varepsilon_0$ and $\mu \approx \mu_0$. Why are metals opaque?

(c) Show that in a good conductor the magnetic field lags the electric field by 45°, and find the ratio of their amplitudes. For a numerical example, use the "typical metal" in part (b).

Homework of Chap.9 (II)

Problem 9.19

(a) Suppose you imbedded some free charge in a piece of glass. About how long would it take for the charge to flow to the surface?

(b) Silver is an excellent conductor, but it's expensive. Suppose you were designing a microwave experiment to operate at a frequency of 10^{10}Hz. How thick would you make the silver coatings?

(c) Find the wavelength and propagation speed in copper for radio waves at MHz. Compare the corresponding values in air (or vacuum).

Problem 9.30 Confirm that the energy in the TE_{mn} mode travels at the group velocity. [*Hint*: Find the time averaged Poynting vector $\langle \mathbf{S} \rangle$ and the energy density $\langle u \rangle$ (use Prob. 9.12 if you wish). Integrate over the cross section of the wave guide to get the energy per unit time and per unit length carried by the wave, and take their ratio.]

Homework of Chap.9 (II)

Problem 9.31 Work out the theory of TM modes for a rectangular wave guide. In particular, find the longitudinal electric field, the cutoff frequencies, and the wave and group velocities. Find the ratio of the lowest TM cutoff frequency to the lowest TE cutoff frequency, for a given wave guide. [*Caution*: What is the lowest TM mode?]

Problem 9.37 A microwave antenna radiating at 10 GHz is to be protected from the environment by a plastic shield of dielectric constant 2.5. What is the minimum thickness of this shielding that will allow perfect transmission (assuming normal incidence)? [*Hint*: Use Eq. 9.199.]

Problem 9.40 Consider the **resonant cavity** produced by closing off the two ends of a rectangular wave guide, at $z = 0$ and at $z = d$, making a perfectly conducting empty box. Show that the resonant frequencies for both TE and TM modes are given by

$$\omega_{lmn} = c\pi\sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2}, \qquad (9.204)$$

for integers l, m, and n. Find the associated electric and magnetic fields.