Chapter 8: Conservation Laws

8.1 Charge and Energy   8.1.1 The Continuity Equation

Conservation laws in electrodynamics
- Charge ← the paradigm KK[`pærəˌdaɪm]
- Energy
- Momentum
- Angular momentum

Global conservation of charge: the total charge in the universe is constant.

Local conservation of charge: If the total charge in some volume changes, then exactly that amount of charge must have passed in or out through the surface.

$$\frac{dQ_{total}}{dt} = 0 = \frac{\partial Q_{enc}}{\partial t} + \oint_S \mathbf{J} \cdot d\mathbf{a}$$

The Continuity Equation

$$\int_V \frac{\partial \rho(\mathbf{r},t)}{\partial t} d\tau = -\int_V (\nabla \cdot \mathbf{J}) d\tau$$ (invoking the divergence theorem)

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} \longleftarrow$$ the continuity equation (in differential form)

This equation is a precise mathematical statement of the local conservation of charge.

It can be derived from Maxwell's equations. since $\nabla \cdot (\nabla \times \mathbf{B}) = 0$

$$\frac{\partial \rho}{\partial t} = \varepsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) = \varepsilon_0 (\nabla \cdot \frac{\partial \mathbf{E}}{\partial t}) = \varepsilon_0 \nabla \cdot (-\frac{1}{\varepsilon_0}\mathbf{J} + \cancel{\frac{1}{\varepsilon_0 \mu_0} \nabla \times \mathbf{B}})$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$ (a consequence of the law of electrodynamics)

Q1: The energy and momentum density → analogous to ρ.

Q2: The energy and momentum "current" → analogous to $\mathbf{J}$.

8.1.2 Poynting's Theorem (I) in vacuum

The work necessary to assemble a static charge distribution

$$W_e = \frac{\varepsilon_0}{2}\int E^2 d\tau$$ (against the Coulomb repulsion of like charges)

The work required to get current going

$$W_m = \frac{1}{2\mu_0}\int B^2 d\tau$$ (against the back emf)

The total energy stored in the electromagnetic fields *might be*

$$U_{em} = W_e + W_m = \frac{1}{2}\int (\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2) d\tau = \int u_{em} d\tau$$

$$\text{where } u_{em} = \frac{1}{2}(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2)$$

Q: Can we derive this equation in a more general and more persuasive way?

Poynting's Theorem (II)

Starting point: How much work, dW, is done by the electromagnetic forces acting on these charges in the interval dt?   (using the Lorentz force law)

$$dW = \sum_j \mathbf{F}_j \cdot d\mathbf{l}_j = \sum_j q_j(\mathbf{E}_j + \mathbf{v}_j \times \mathbf{B}_j) \cdot \mathbf{v}_j dt = \sum_j q_j \mathbf{E}_j \cdot \mathbf{v}_j dt$$

$$\frac{dW}{dt} = \sum_j q_j \mathbf{E}_j \cdot \mathbf{v}_j = \int_V (\mathbf{E} \cdot \mathbf{v})\rho d\tau = \int_V \underline{(\mathbf{E} \cdot \mathbf{J})} d\tau$$

(the work done per unit time, per unit volume, i.e., the power delivered per unit volume)

Q: Can we express this quantity in terms of the fields alone?

Yes, use the Ampere-Maxwell law to eliminate $\mathbf{J}$, analogous to the proof of the continuity equation.

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Poynting's Theorem (III)

$$\mathbf{E}\cdot\mathbf{J}=\mathbf{E}\cdot\left(\frac{1}{\mu_0}\nabla\times\mathbf{B}-\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)=\frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B})-\varepsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}$$

$$\mathbf{E}\cdot(\nabla\times\mathbf{B})=\mathbf{B}\cdot\underline{(\nabla\times\mathbf{E})}-\nabla\cdot(\mathbf{E}\times\mathbf{B})$$ (product rule 6)

(Faraday's law)

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\mathbf{E}\cdot\mathbf{J}=-\left(\varepsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}+\frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)-\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})$$

$$=-\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0E^2+\frac{1}{\mu_0}B^2\right)-\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})$$

$$\frac{dW}{dt}=\int_V(\mathbf{E}\cdot\mathbf{J})d\tau=-\frac{d}{dt}\int_V\frac{1}{2}\left(\varepsilon_0E^2+\frac{1}{\mu_0}B^2\right)d\tau-\oint_S\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$

(invoking the divergence theorem)

Poynting's Theorem and Poynting Vector

$$\frac{dW}{dt} = -\frac{d}{dt}\underbrace{\int_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau}_{\text{(total energy stored in the field, } U_{\text{em}})} - \underbrace{\oint_S \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}}$$

(total energy stored in the field, U_{em})

(the rate at which energy is carried out of V, across its boundary surface S, by the electromagnetic fields.)

Poynting's theorem: "work-energy theorem" of electrodynamics.

Poynting vector: the energy per unit time, per unit area, transported by the fields.

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

($\mathbf{S}$: the energy flux density)

Differential Form of Poynting's Theorem

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \oint_S \mathbf{S}\cdot d\mathbf{a}$$

$$\frac{dW}{dt} = \frac{d}{dt}\int_V u_{\mathrm{mech}}\, d\tau$$ (u_{mech}: the mechanical energy density)

$$\frac{d}{dt}\int_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau = \frac{d}{dt}\int_V u_{\mathrm{em}}\, d\tau$$ (u_{em}: the energy density of the fields)

So $$\frac{d}{dt}\int_V u_{\mathrm{mech}}\, d\tau = -\frac{d}{dt}\int_V u_{\mathrm{em}}\, d\tau - \oint_S \mathbf{S}\cdot d\mathbf{a}$$

(divergence theorem)

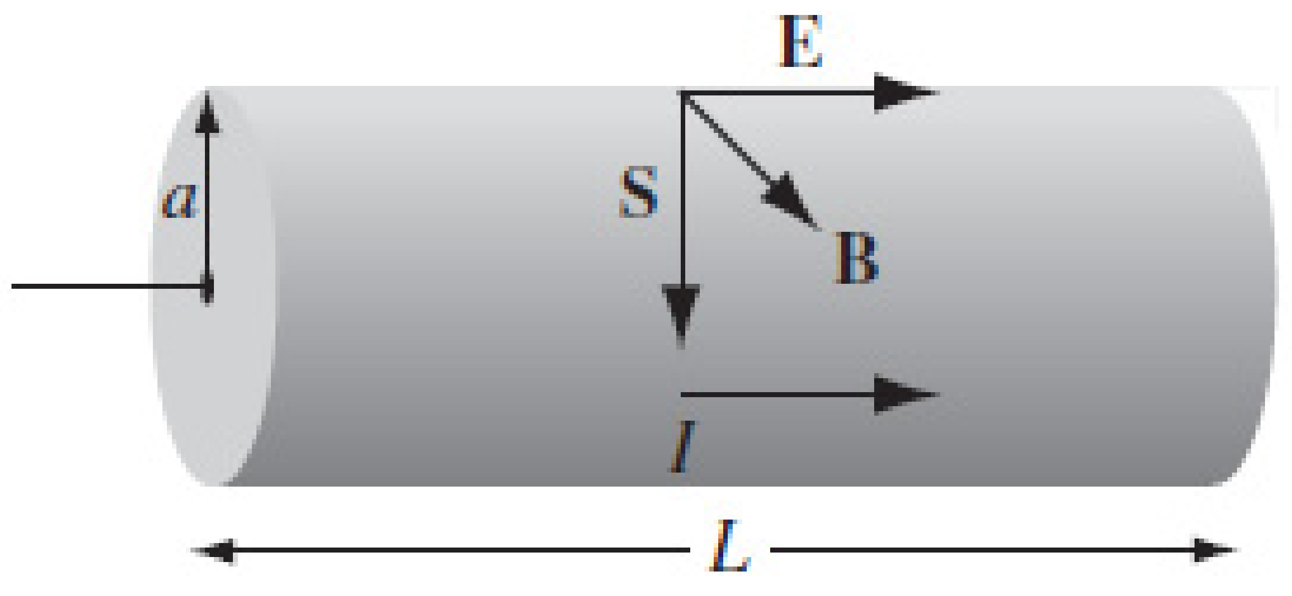
$$\frac{d}{dt}\int_V u_{\mathrm{mech}}\, d\tau = -\frac{d}{dt}\int_V u_{\mathrm{em}}\, d\tau - \int_V (\nabla\cdot\mathbf{S})\, d\tau$$

$$\frac{\partial}{\partial t}(u_{\mathrm{mech}} + u_{\mathrm{em}}) = -\nabla\cdot\mathbf{S}$$ (the differential form of Poynting's theorem)

Q: What's the difference between $\frac{d}{dt}$ and $\frac{\partial}{\partial t}$?

Example 8.1

When current flows down a wire, work is done, which shows up as Joule heating of the wire. Find the energy per unit time delivered to the wire using Poynting vector?



Sol:

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) \qquad \begin{cases} \mathbf{E} = \dfrac{V}{L}\hat{\mathbf{z}} \\ \mathbf{B}(r=a) = \dfrac{\mu_0 I}{2\pi a}\hat{\boldsymbol{\phi}} \end{cases}$$

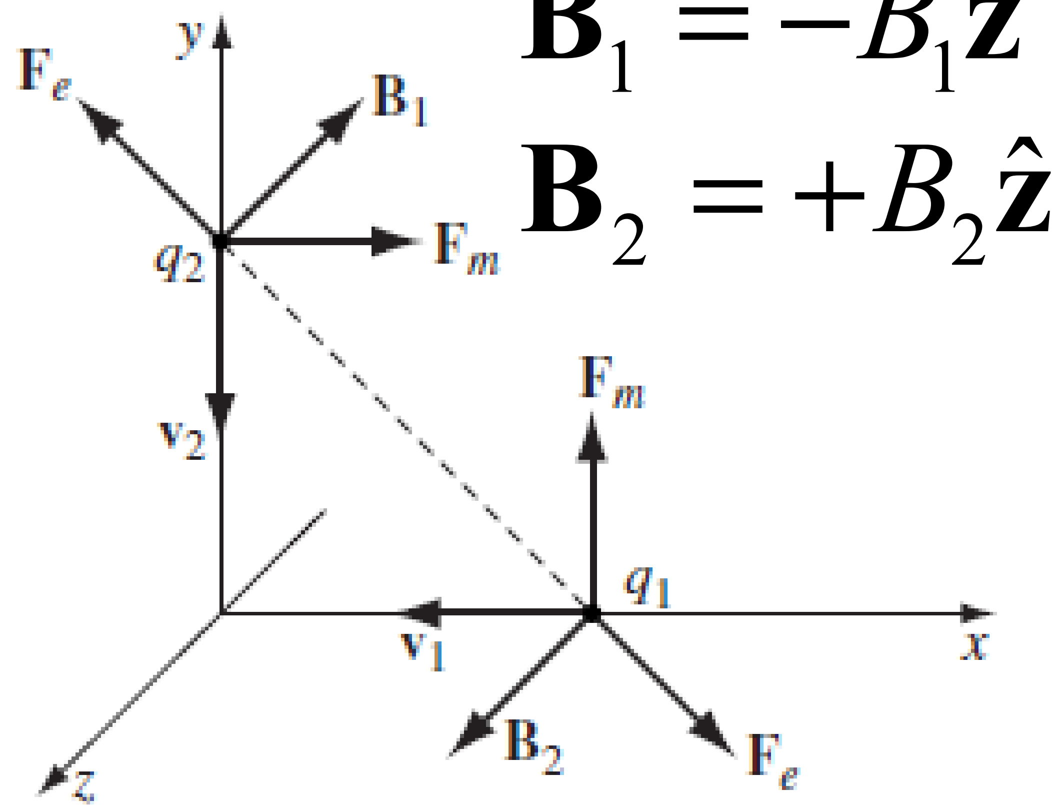
So $\mathbf{S} = \dfrac{1}{\mu_0}(\dfrac{V}{L}\hat{\mathbf{z}}\times\dfrac{\mu_0 I}{2\pi a}\hat{\boldsymbol{\phi}}) = -\dfrac{VI}{2\pi a L}\hat{\mathbf{r}}$ (point radially inward)

The energy per unit time passing through the surface of the wire is:

$$-\oint_S \mathbf{S}\cdot d\mathbf{a} = S(2\pi a L) = VI = \frac{dW}{dt} \qquad \frac{dU_{\text{em}}}{dt} = 0 \text{ (static fields)}$$

8.2 Momentum

8.2.1 Newton's Third Law in Electrodynamics



$$\mathbf{B}_1 = -B_1\hat{\mathbf{z}}$$
$$\mathbf{B}_2 = +B_2\hat{\mathbf{z}}$$

Suppose two charges, q_1 and q_2, travel in along x axis and y axis, respectively. They can only slide on the axes with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ as shown in the figure.

Q: Is Newton's third law valid?

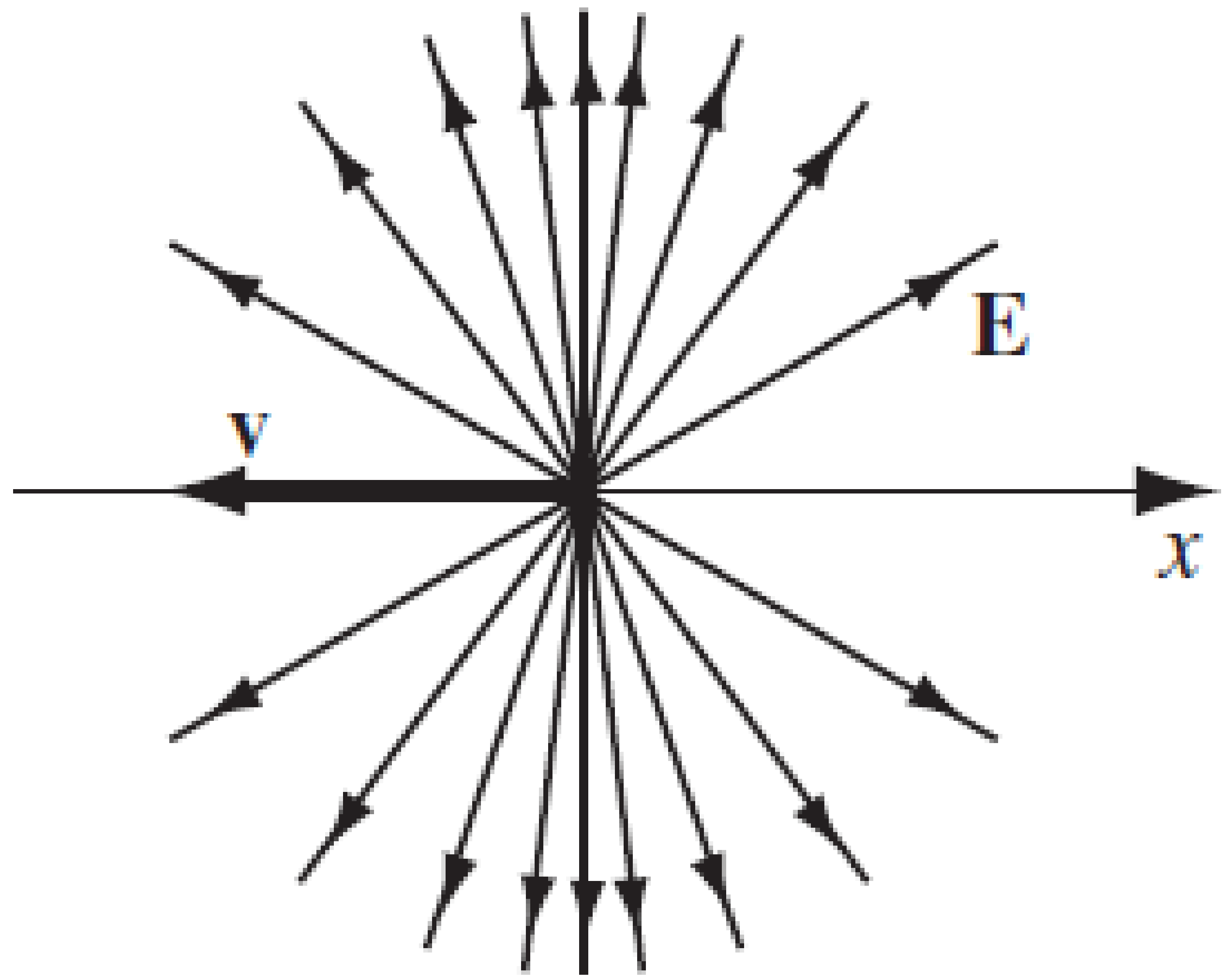
The electric force between them satisfies the third law, but the magnetic force does not hold (same magnitudes, but their directions are not opposite).

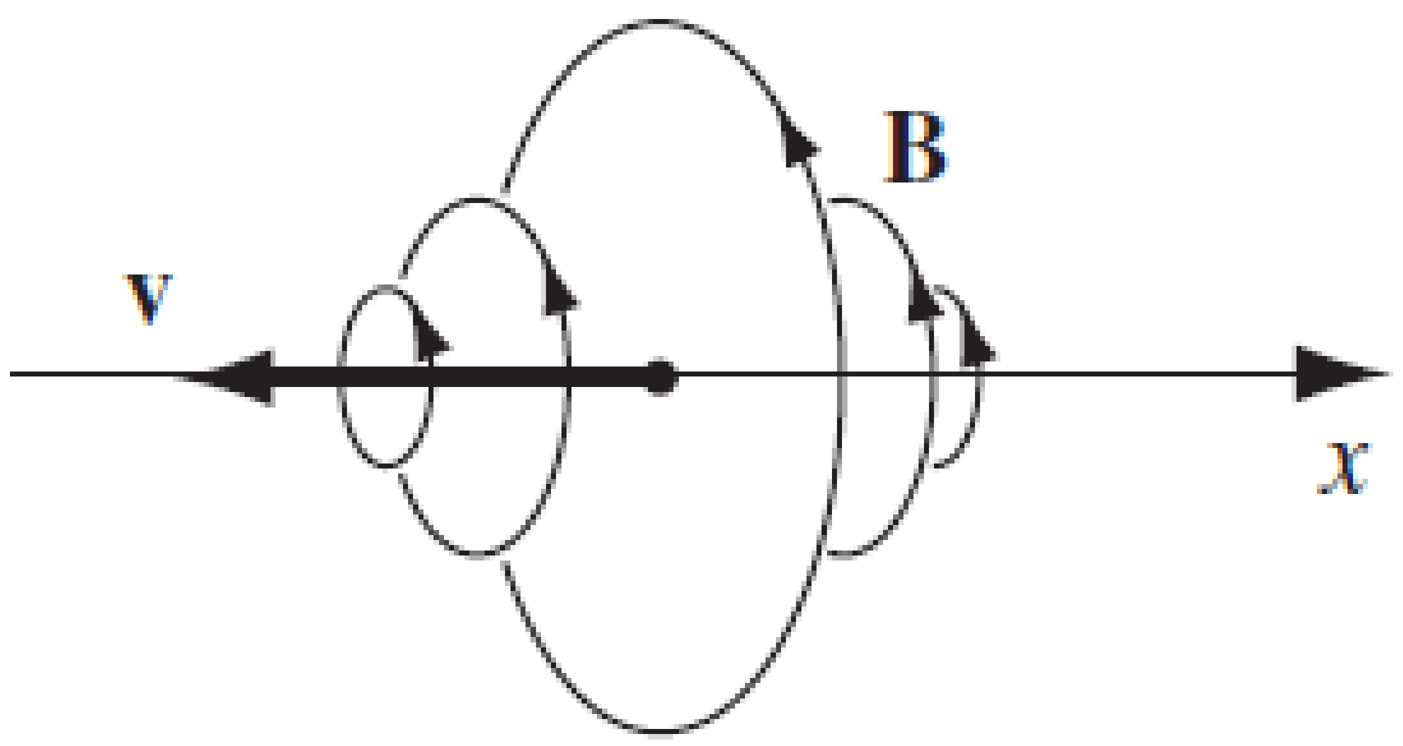
The proof of conservation of the momentum, however, rests on the cancellation of the internal forces, which follows from the third law. In electrodynamics the third law does not hold.

Q: How to rescue the momentum conservation?

The fields themselves carry momentum. (Surprise!)

The Fields of a Moving Charge





The electric field of a moving charge is not given by Coulomb’s law.

The magnetic field of a moving charge does not constitute a steady current. Thus it is not given by Biot-Sarvart law.

8.2.2 Maxwell's Stress Tensor

The total electromagnetic force on the charges in volume *V*:

$$\mathbf{F} = \int_V \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}) d\tau = \int_V (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) d\tau = \int_V \mathbf{f} d\tau$$

Where $\mathbf{f}$ denotes the force per unit volume.

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$$

Eliminate ρ and $\mathbf{J}$ by using Maxwell's equations.

$$\begin{cases} \rho = \varepsilon_0 (\nabla \cdot \mathbf{E}) \\ \mathbf{J} = \dfrac{1}{\mu_0} \nabla \times \mathbf{B} - \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

$$\mathbf{f} = \varepsilon_0 (\nabla \cdot \mathbf{E})\mathbf{E} + \left(\frac{1}{\mu_0} \nabla \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B}$$

$$= \varepsilon_0 (\nabla \cdot \mathbf{E})\mathbf{E} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \varepsilon_0 \left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}\right)$$

Maxwell's Stress Tensor (II)

$$\mathbf{f} = \varepsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} - \varepsilon_0\left(\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}\right)$$

$$\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) = \frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} + \mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t}$$

(Faraday's law) $$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$$

$$\mathbf{f} = \varepsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} - \varepsilon_0\mathbf{E}\times(\nabla\times\mathbf{E}) - \varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{E}\times(\nabla\times\mathbf{E}) = \frac{1}{2}\nabla E^2 - (\mathbf{E}\cdot\nabla)\mathbf{E}$$

$$(\nabla\times\mathbf{B})\times\mathbf{B} = -\mathbf{B}\times(\nabla\times\mathbf{B}) = -\frac{1}{2}\nabla B^2 + (\mathbf{B}\cdot\nabla)\mathbf{B}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

$$\nabla(\mathbf{B}\cdot\mathbf{B}) = \mathbf{B}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{B}) + (\mathbf{B}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{B}$$

$$\Rightarrow \nabla B^2 = 2\mathbf{B}\times(\nabla\times\mathbf{B}) + 2(\mathbf{B}\cdot\nabla)\mathbf{B}$$

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Maxwell's Stress Tensor (III)

$$\mathbf{f} = \varepsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} - \frac{1}{2\mu_0}\nabla B^2 + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} - \varepsilon_0 \frac{1}{2}\nabla E^2 + \varepsilon_0(\mathbf{E}\cdot\nabla)\mathbf{E} - \varepsilon_0 \frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})$$

$= 0$

$$= \varepsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{E}] + \frac{1}{\mu_0}[(\mathbf{B}\cdot\nabla)\mathbf{B} + (\nabla \cdot \mathbf{B})\mathbf{B}]$$

$$-\frac{1}{2\mu_0}\nabla B^2 - \frac{\varepsilon_0}{2}\nabla E^2 - \varepsilon_0 \frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})$$

It can be simplified by introducing the **Maxwell stress tensor**.

$$T_{ij} \equiv \varepsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)$$

(the Kronecker delta, another example see Prob. 3.45)

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Maxwell’s Stress Tensor (IV)

$$T_{ij} \equiv \varepsilon_0 (E_i E_j - \frac{1}{2}\delta_{ij} E^2) + \frac{1}{\mu_0}(B_i B_j - \frac{1}{2}\delta_{ij} B^2) \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\begin{cases} T_{xx} = \dfrac{\varepsilon_0}{2}(E_x^2 - E_y^2 - E_z^2) + \dfrac{1}{2\mu_0}(B_x^2 - B_y^2 - B_z^2) \\ T_{yy} = \dfrac{\varepsilon_0}{2}(E_y^2 - E_z^2 - E_x^2) + \dfrac{1}{2\mu_0}(B_y^2 - B_z^2 - B_x^2) \\ T_{zz} = \dfrac{\varepsilon_0}{2}(E_z^2 - E_y^2 - E_x^2) + \dfrac{1}{2\mu_0}(B_z^2 - B_y^2 - B_x^2) \\ T_{xy} = T_{yx} = \varepsilon_0 (E_x E_y) + \dfrac{1}{\mu_0}(B_x B_y) \\ T_{yz} = T_{zy} = \varepsilon_0 (E_y E_z) + \dfrac{1}{\mu_0}(B_y B_z) \\ T_{zx} = T_{xz} = \varepsilon_0 (E_z E_x) + \dfrac{1}{\mu_0}(B_z B_x) \end{cases}$$

Because T_{ij} carries two indices, it is sometimes written with a double arrow $\overleftrightarrow{\mathbf{T}}$.

Q: How does the tensor operate?

See, “Vector Analysis”, Chap.8, M. E. Spiegel, McGRAW-HILL.

Maxwell's Stress Tensor (V)

On can form the dot product of tensor $\overleftrightarrow{\mathbf{T}}$ with a vector $\mathbf{a}$:

(row vector)

$$(\mathbf{a}\cdot\overleftrightarrow{\mathbf{T}})_j = \sum_{i=x,y,z} a_i T_{ij}$$

$$(\overline{A_1}\ \overline{A_2}\ \overline{A_3}) = (A_1\ A_2\ A_3)\begin{pmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^1}{\partial \bar{x}^2} & \frac{\partial x^1}{\partial \bar{x}^3} \\ \frac{\partial x^2}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^2} & \frac{\partial x^2}{\partial \bar{x}^3} \\ \frac{\partial x^3}{\partial \bar{x}^1} & \frac{\partial x^3}{\partial \bar{x}^2} & \frac{\partial x^3}{\partial \bar{x}^3} \end{pmatrix}$$

$$\begin{pmatrix} \overline{A_1} \\ \overline{A_2} \\ \overline{A_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^1} & \frac{\partial x^3}{\partial \bar{x}^1} \\ \frac{\partial x^1}{\partial \bar{x}^2} & \frac{\partial x^2}{\partial \bar{x}^2} & \frac{\partial x^3}{\partial \bar{x}^2} \\ \frac{\partial x^1}{\partial \bar{x}^3} & \frac{\partial x^2}{\partial \bar{x}^3} & \frac{\partial x^3}{\partial \bar{x}^3} \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

(column vector)

$$(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{a})_i = \sum_{j=x,y,z} T_{ij} a_j$$

The divergence of the Maxwell stress tensor is:

$$(\nabla\cdot\overleftrightarrow{\mathbf{T}})_j = \varepsilon_0[(\nabla\cdot\mathbf{E})E_j + (\mathbf{E}\cdot\nabla)E_j - \frac{1}{2}\nabla_j E^2]$$
$$+\frac{1}{\mu_0}[(\mathbf{B}\cdot\nabla)B_j + (\nabla\cdot\mathbf{B})B_j - \frac{1}{2}\nabla_j B^2]$$

Maxwell's Stress Tensor (VI)

The force per unit volume:

$$\mathbf{f} = \nabla \cdot \overleftrightarrow{\mathbf{T}} - \varepsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t}$$

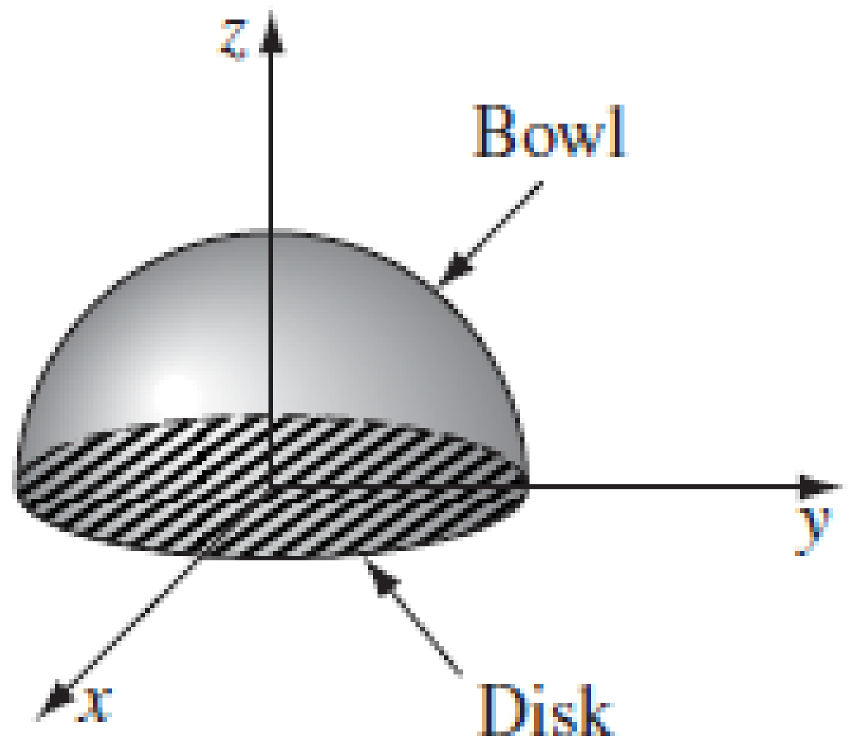
The total force on the charges in V is:

$$\mathbf{F} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \varepsilon_0 \mu_0 \frac{d}{dt} \int_V \mathbf{S}\, d\tau$$

Physically, the Maxwell stress tensor is the force per unit area acting on the surface.

Example 8.2

A uniformly charged solid sphere of radius R and charge Q is cut into two hemispheres. Find the force required to prevent the hemispheres from separating.



Sol: This is an electrostatics, no magnetic field involved.

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Qr}{R^3}\hat{\mathbf{r}} \qquad \mathbf{F} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$$

The boundary surface consists of two parts---bowl and disk.

The net force is obviously in the z direction.

$$dF_z = (\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = T_{zx}da_x + T_{zy}da_y + T_{zz}da_z$$

Express the electric component in Cartesian coordinates.

Example 8.2 (ii)

On the bowl, $r = R$.

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{\mathbf{r}} \qquad \hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$\begin{cases} T_{zx} = \varepsilon_0 E_z E_x = \varepsilon_0 (\frac{Q}{4\pi\varepsilon_0 R^2})^2 \sin\theta\cos\theta\cos\phi \\ T_{zy} = \varepsilon_0 E_z E_y = \varepsilon_0 (\frac{Q}{4\pi\varepsilon_0 R^2})^2 \sin\theta\cos\theta\sin\phi \\ T_{zz} = \frac{\varepsilon_0}{2}(E_z^2 - E_y^2 - E_x^2) = \frac{\varepsilon_0}{2}(\frac{Q}{4\pi\varepsilon_0 R^2})^2 (\cos^2\theta - \sin^2\theta) \end{cases}$$

$$d\mathbf{a} = R^2 \sin\theta d\theta d\phi\hat{\mathbf{r}} \quad \begin{cases} da_x = d\mathbf{a}\cdot\hat{\mathbf{x}} = R^2 \sin\theta\sin\theta\cos\phi d\theta d\phi \\ da_y = d\mathbf{a}\cdot\hat{\mathbf{y}} = R^2 \sin\theta\sin\theta\sin\phi d\theta d\phi \\ da_z = d\mathbf{a}\cdot\hat{\mathbf{z}} = R^2 \sin\theta\cos\theta d\theta d\phi \end{cases}$$

Example 8.2 (iii)

The force on the bowl is:

$$dF_z = T_{zx}da_x + T_{zy}da_y + T_{zz}da_z$$

$$= \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R^2}\right)^2 R^2 \begin{bmatrix} 2\sin^2\theta\sin\theta\cos\theta\cos^2\phi \\ +2\sin^2\theta\sin\theta\cos\theta\sin^2\phi \\ +\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) \end{bmatrix} d\theta d\phi$$

$$= \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R}\right)^2 \sin\theta\cos\theta d\theta d\phi$$

$$F_{\text{bowl}} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R}\right)^2 \sin\theta\cos\theta d\theta d\phi$$

$$= \int_{\theta=0}^{\pi/2} \frac{\varepsilon_0 2\pi}{2}\left(\frac{Q}{4\pi\varepsilon_0 R}\right)^2 \frac{1}{2}\sin 2\theta d\theta = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{8R^2}$$

Example 8.2 (iv)

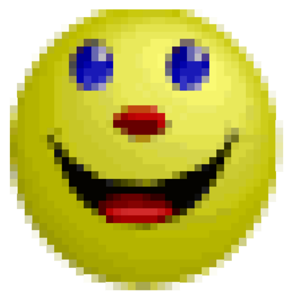
Contd.:

The force on the disk is:

$$F_{\text{disk}} = \int_{r=0}^{R} \int_{\phi=0}^{2\pi} \frac{\varepsilon_0}{2} \left(\frac{Q}{4\pi\varepsilon_0 R^3} \right)^2 r^3 dr d\phi = \frac{1}{4\pi\varepsilon_0} \frac{Q^2}{16R^2}$$

The net force on the northern hemisphere is:

$$F = \frac{1}{4\pi\varepsilon_0} \frac{3Q^2}{16R^2}$$



Q: Can we solve this problem using a simpler approach?

Yes, we can use the potential energy to find the net force.

8.2.3 Conservation of Momentum

Newton's second law ➔ the force on an object is equal to the rate of change of its momentum.

$$\mathbf{F} = \frac{d\mathbf{p}_{\text{mech}}}{dt} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \varepsilon_0 \mu_0 \frac{d}{dt} \int_V \mathbf{S}\, d\tau$$

where $\mathbf{p}_{\text{mech}}$ is the total (mechanical) momentum of the particles contained in the volume V.

$$\begin{cases} \dfrac{d\mathbf{p}_{\text{mech}}}{dt} = \dfrac{d}{dt} \displaystyle\int_V \mathbf{g}_{\text{mech}}\, d\tau \\ \dfrac{d\mathbf{p}_{\text{em}}}{dt} = \dfrac{d}{dt} \varepsilon_0 \mu_0 \displaystyle\int_V \mathbf{S}\, d\tau, \quad \text{where } \mathbf{p}_{\text{em}} = \displaystyle\int_V (\varepsilon_0 \mu_0 \mathbf{S})\, d\tau = \int_V \mathbf{g}_{\text{em}}\, d\tau \end{cases}$$

(an analogous interpretation, not a rigorous proof)

(momentum stored in the electromagnetic fields themselves)

(momentum density)

Conservation of Momentum (II)

$\oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$ (the momentum per unit time flowing in through the surface.

Conservation of momentum in electrodynamics:
Any increase in the total momentum (mechanical plus electromagnetic) is equal to the momentum brought in by the fields.

$$\frac{\partial}{\partial t}(\mathbf{g}_{\text{mech}} + \mathbf{g}_{\text{em}}) = -\nabla \cdot (\underline{-\overleftrightarrow{\mathbf{T}}})$$ (in differential form)

(**momentum flux density**, playing the role of $\mathbf{J}$ in continuity equation, or $\mathbf{S}$ in Poynting's theorem)

Conservation of Momentum (III)

The roles of Poynting's vector:

$\mathbf{S}$ the energy per unit area, per unit time, transported by electromagnetic fields.

$\mu_0\varepsilon_0\mathbf{S}$ the momentum per unit volume stored in those fields.

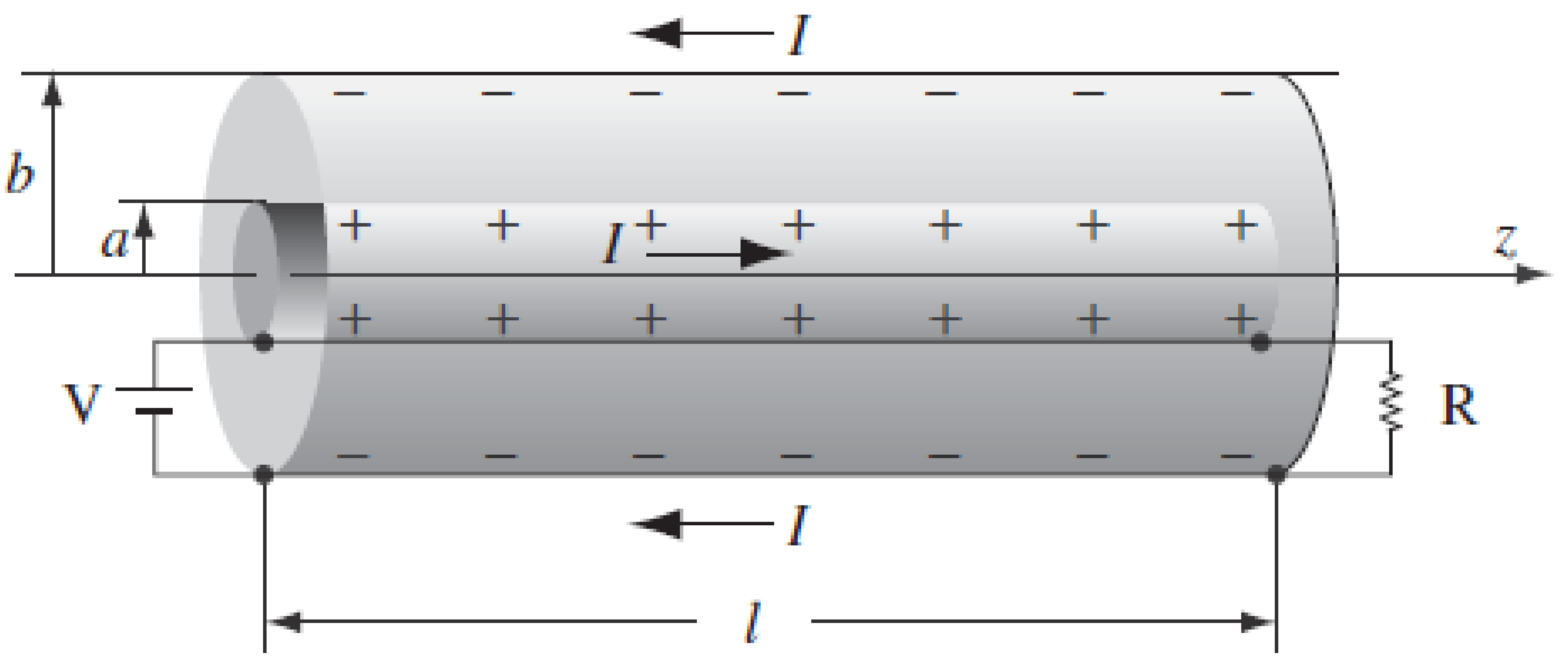
The roles of momentum stress tensor:

$\overleftrightarrow{\mathbf{T}}$ the electromagnetic stress acting on a surface.

$-\overleftrightarrow{\mathbf{T}}$ the flow of momentum transported by the fields.

Example 8.3 (hidden momentum)

A long coaxial cable, of length l, consists of an inner conductor (radius a) and an outer conductor (radius b). It is connected to a battery at one end and a resistor at the other. The inner conductor carries a uniform charge per unit length λ, and a steady current to the right; the outer conductor has the opposite charge and current. What is the electromagnetic momentum stored in the fields.



Sol: The fields are

$$\begin{cases} \mathbf{E} = \dfrac{1}{2\pi\varepsilon_0}\dfrac{\lambda}{s}\hat{\mathbf{s}} \\ \mathbf{B} = \dfrac{\mu_0}{2\pi}\dfrac{I}{s}\hat{\boldsymbol{\phi}} \end{cases} \qquad \mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\lambda I}{4\pi^2\varepsilon_0 s^2}\hat{\mathbf{z}} \triangleright$$

The momentum in the fields is: (an astonishing result!)

$$\mathbf{p}_{\text{em}} = \int \mu_0\varepsilon_0 \mathbf{S}\, d\tau = \frac{\mu_0\lambda I}{4\pi^2}\int_a^b \frac{1}{s^2} l 2\pi s\, ds\hat{\mathbf{z}} = \frac{\mu_0\lambda I l}{2\pi}\ln\left(\frac{b}{a}\right)\hat{\mathbf{z}}$$

In fact, if the center of mass of a localized system is at rest, its total momentum must be zero.

There is "hidden" mechanical momentum associated with the flow of current, and this exactly cancels the momentum in the fields. (This is a relativistic effect: See Example 12.12)

8.2.4 Angular Momentum

The electromagnetic fields carry energy and momentum, not merely mediators of forces between charges.

$$\begin{cases} u_{\text{em}} = \frac{1}{2}(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2) \\ \mathbf{g}_{\text{em}} = \varepsilon_0 \mu_0 \mathbf{S} = \varepsilon_0 (\mathbf{E} \times \mathbf{B}) \end{cases}$$

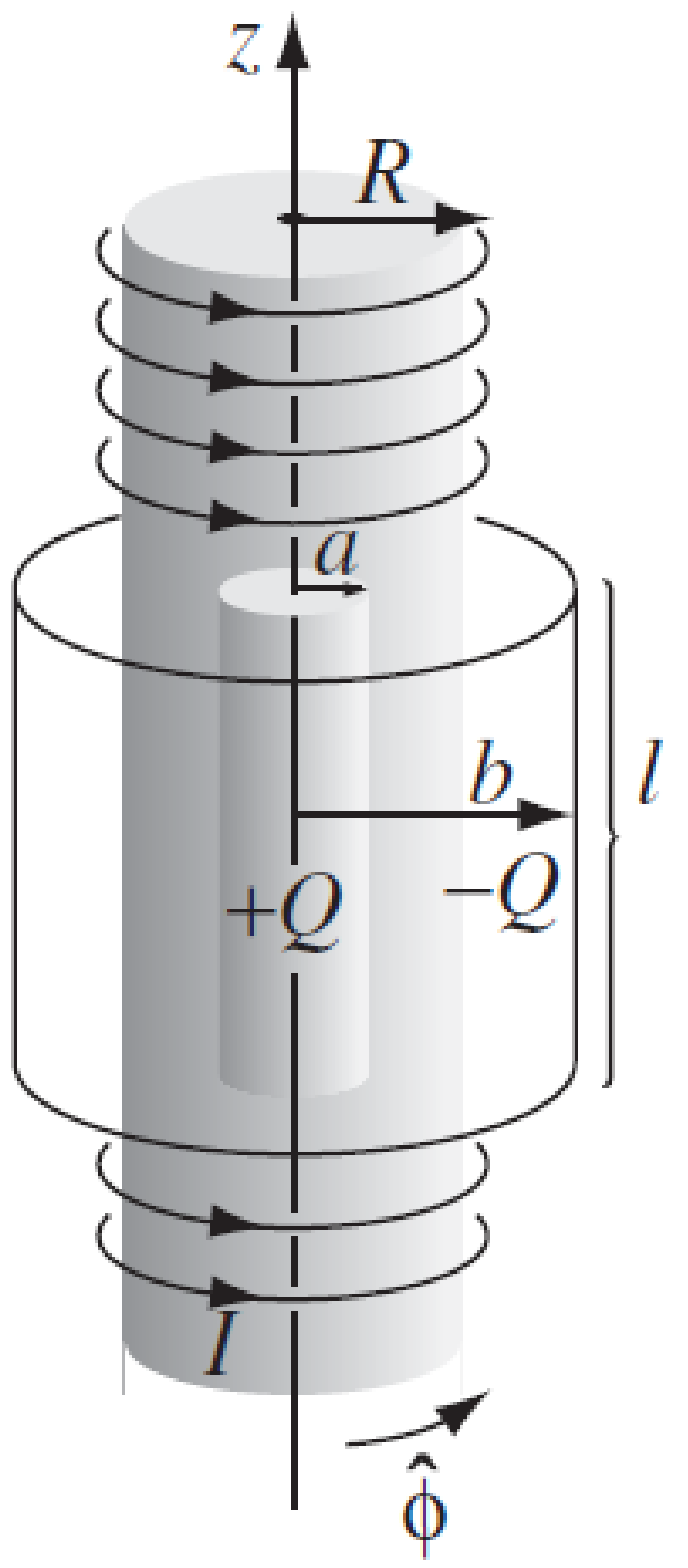
How about the angular momentum?

$$\ell_{\text{em}} = \mathbf{r} \times \mathbf{g}_{em} = \varepsilon_0 [\mathbf{r} \times (\mathbf{E} \times \mathbf{B})]$$ (again, not a rigorous proof)

Even perfectly static fields can harbor momentum and angular momentum. See the following example.

Example 8.4

Imagine a very long solenoid with radius R, n turns per unit length, and current I. Coaxial with the solenoid are two long cylindrical shells of length l---one, inside the solenoid at radius a, carries a charge $+Q$, uniformly distributed over the surface; the other, outside the solenoid at radius b, carries charge $-Q$. When the current in the solenoid is gradually reduced, the cylinders begin to rotate, as we found in Ex. 7.8. Where does the angular momentum come from?



Sol: The fields are

$$\begin{cases} \mathbf{E} = \dfrac{1}{2\pi\varepsilon_0}\dfrac{Q}{ls}\hat{\mathbf{s}} \quad (a < s < b) \\ \mathbf{B} = \mu_0 nI\hat{\mathbf{z}} \quad (s < R) \end{cases}$$

The momentum density is:

$$\mathbf{g}_{\text{em}} = \mu_0\varepsilon_0\mathbf{S} = \varepsilon_0\mathbf{E}\times\mathbf{B} = -\frac{\mu_0 nIQ}{2\pi ls}\hat{\boldsymbol{\phi}} \quad (a < s < R)$$

The angular momentum density is:

$$\ell_{\text{em}} = \mathbf{r}\times\mathbf{g}_{\text{em}} = \varepsilon_0[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})] = -\frac{\mu_0 nIQ}{2\pi l}\hat{\mathbf{z}} \quad (a < s < R)$$

The total angular momentum in the fields is:

$$L_{\text{em}} = \int \ell_{\text{em}} d\tau = -\frac{\mu_0 nIQ}{2}(R^2 - a^2)\hat{\mathbf{z}}$$

(an astonishing result!)

Conservation Laws

Conservation laws in electrodynamics $\begin{cases} \text{Charge} \\ \text{Energy} \\ \text{Momentum} \\ \text{Angular momentum} \end{cases}$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

$$\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla \cdot \mathbf{S}$$

$$\frac{\partial}{\partial t}(\mathbf{g}_{\text{mech}} + \mathbf{g}_{\text{em}}) = -\nabla \cdot (-\overleftrightarrow{\mathbf{T}})$$

What is the conservation law for angular momentum ?
Extra bonus (send your answer to me).

$$\ell_{\text{em}} = \mathbf{r} \times \mathbf{g}_{\text{em}} = \varepsilon_0 [\mathbf{r} \times (\mathbf{E} \times \mathbf{B})]$$

Homework of Chap.8

Problem 8.1 Calculate the power (energy per unit time) transported down the cables of Ex. 7.13 and Prob. 7.62, assuming the two conductors are held at potential difference V, and carry current I (down one and back up the other).

Problem 8.4

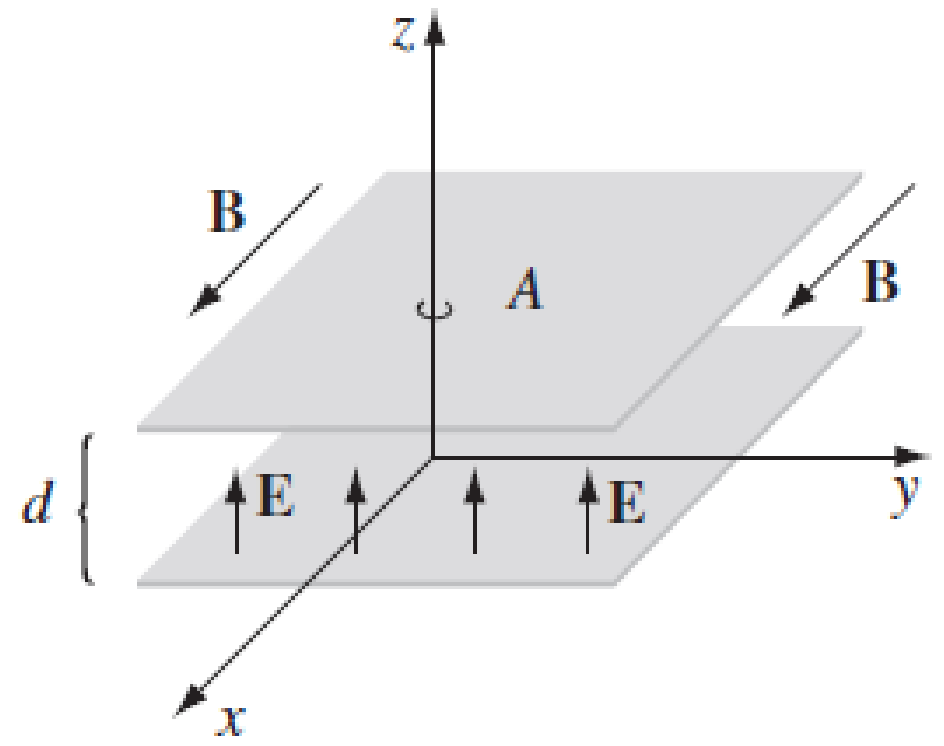
(a) Consider two equal point charges q, separated by a distance $2a$. Construct the plane equidistant from the wo charges. By integrating Maxwell's stress tensor over this plane, determine the force of one charge on the other.

(b) Do the same for charges that are opposite in sign.

Problem 8.6 A charged parallel-plate capacitor (with uniform electric field $\mathbf{E} = E\hat{\mathbf{z}}$) is placed in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{x}}$, as shown in Fig.8.6.

(a) Find the electromagnetic momentum in the space between the plates.

(b) Now a resistive wire is connected between the plates, along the z axis, so that the capacitor slowly discharges. The current through the wire will experience a magnetic force; what is the total impulse delivered to the system, during the discharge?[7]



Problem 8.16[17] A sphere of radius R carries a uniform polarization $\mathbf{P}$ and a uniform magnetization $\mathbf{M}$ (not necessarily in the same direction). Find the electromagnetic momentum of this configuration. [*Answer*: $(4/9)\pi\mu_0 R^3(\mathbf{M}\times\mathbf{P})$]

Homework of Chap.8

Problem 8.19[19] Suppose you had an electric charge q_e and a magnetic monopole q_m. The field of the electric charge is $\mathbf{E}=\dfrac{1}{4\pi\varepsilon_0}\dfrac{q_e}{\mathscr{r}^2}\hat{\mathscr{r}}$.

(of course), and the field of the magnetic monopole is $\mathbf{B}=\dfrac{\mu_0}{4\pi}\dfrac{q_m}{\mathscr{r}^2}\hat{\mathscr{r}}$.

Find the total angular momentum stored in the fields, if the two charges are separated by a distance d. [*Answer*: $(\mu_0/4\pi)q_e q_m$.][20]

Problem 8.23

(a) Carry through the argument in Sect. 8.1.2, starting with Eq. 8.6, but using J_f in place of **J**. Show that the Poynting vector becomes

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}, \tag{8.46}$$

and the rate of change of the energy density in the fields is

$$\frac{\partial u}{\partial t}=\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}\cdot$$

For *linear* media, show that[24]

$$u=\frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H}). \tag{8.47}$$

(b) In the same spirit, reproduce the argument in Sect. 8.2.2, starting with Eq. 8.15, with ρ_f and J_f in place of ρ and J. Don't bother to construct the Maxwell stress tensor, but do show that the momentum density is[25]

$$\mathbf{g}=\mathbf{D}\times\mathbf{B}. \tag{8.48}$$