Chap. 7 (Contd)

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For the EM Course Lectured by Prof. Tsun-Hsu Chang

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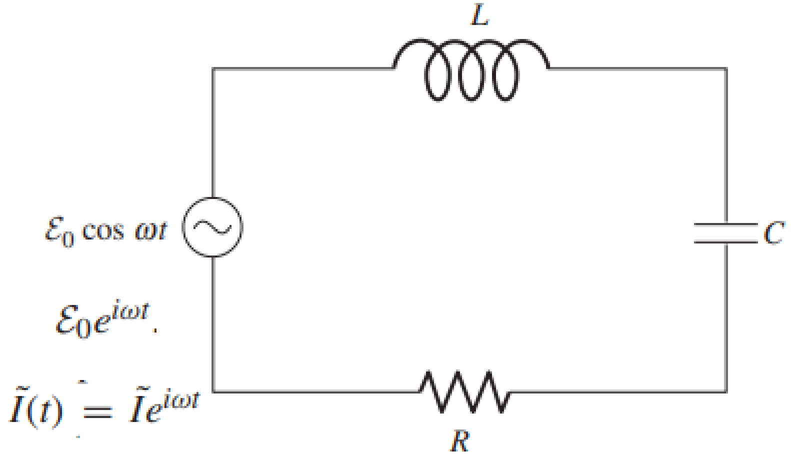
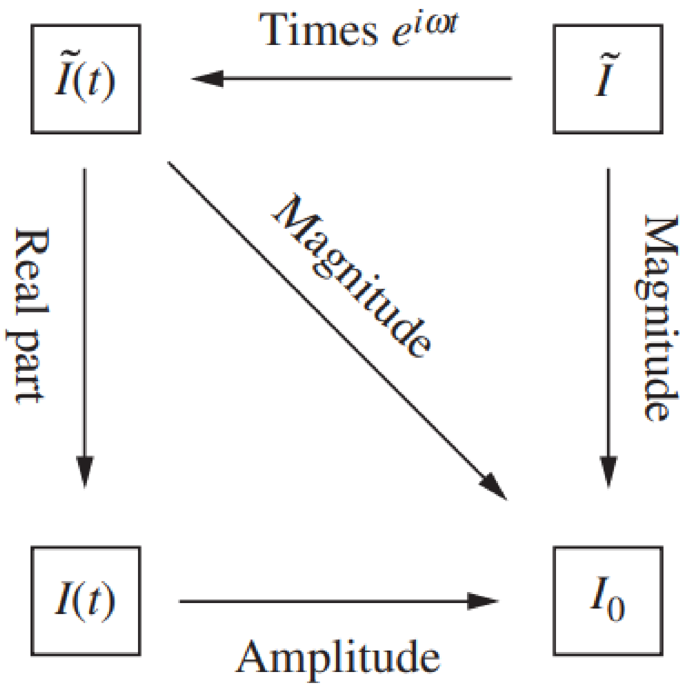
More on AC analysis



$$L\frac{d\tilde{I}(t)}{dt} + R\tilde{I}(t) + \frac{\tilde{Q}(t)}{C} = \mathcal{E}_0 e^{i\omega t}.$$

$$L\frac{d}{dt}\mathrm{Re}[\tilde{I}(t)] + R\,\mathrm{Re}[\tilde{I}(t)] + \frac{1}{C}\int \mathrm{Re}[\tilde{I}(t)]\,dt = \mathcal{E}_0 \cos \omega t.$$

$$Li\omega\tilde{I}e^{i\omega t} + R\tilde{I}e^{i\omega t} + \frac{\tilde{I}e^{i\omega t}}{i\omega C} = \mathcal{E}_0 e^{i\omega t}.$$



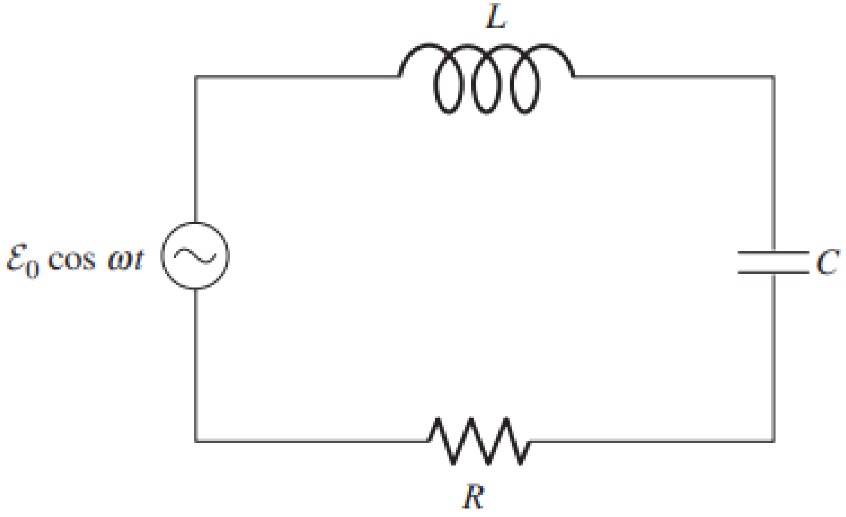
$$\tilde{I} = \frac{\mathcal{E}_0}{R^2 + (\omega L - 1/\omega C)^2} \cdot \sqrt{R^2 + (\omega L - 1/\omega C)^2}\, e^{i\phi}$$

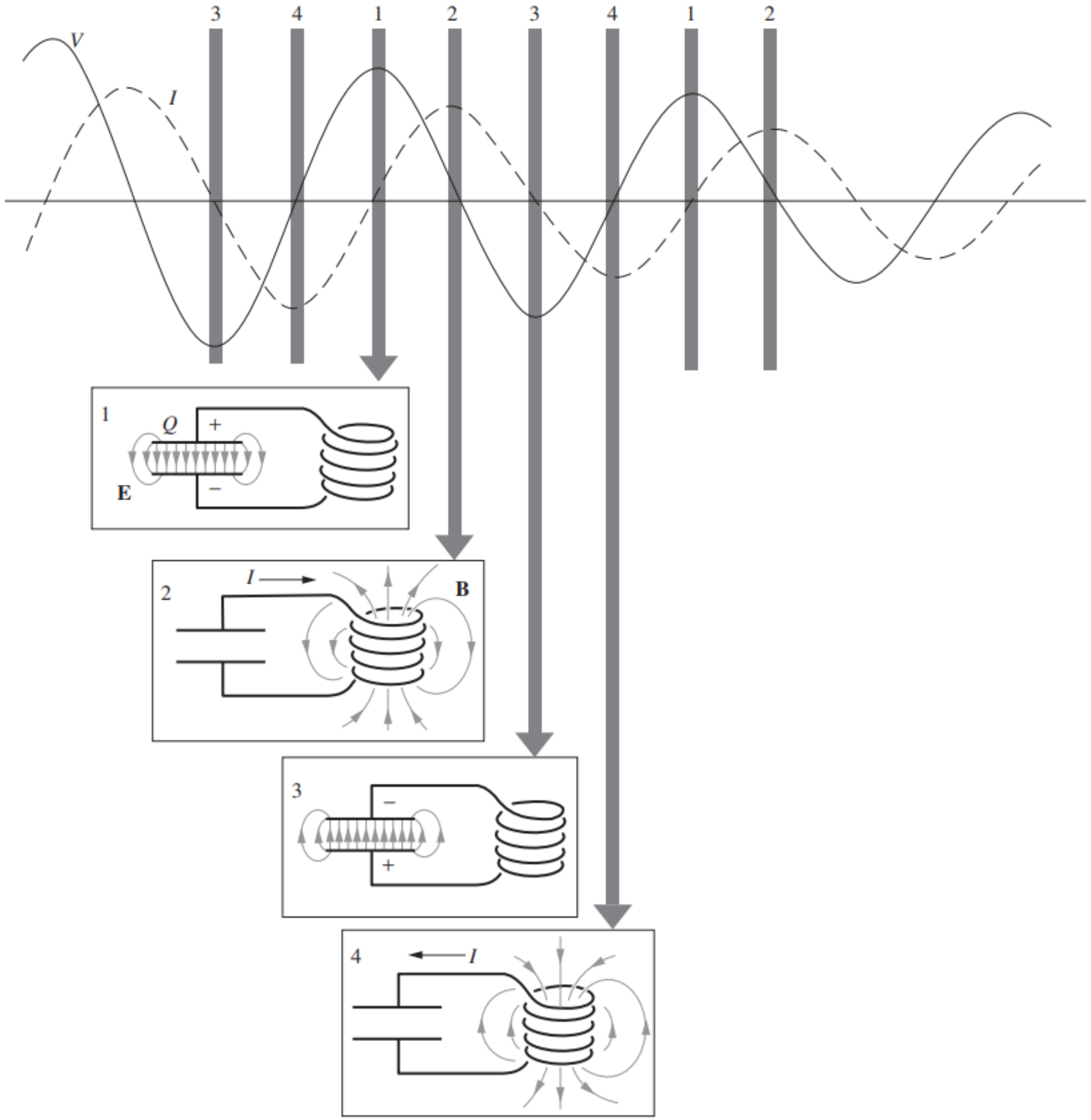
where

$$I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \quad \text{and} \quad \tan\phi = \frac{1}{R\omega C} - \frac{\omega L}{R}.$$

$$I(t) = \mathrm{Re}\left[\tilde{I}e^{i\omega t}\right] = \mathrm{Re}\left[I_0 e^{i\phi} e^{i\omega t}\right] = I_0 \cos(\omega t + \phi)$$
$$= \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \cos(\omega t + \phi),$$

More on AC analysis





More on AC analysis

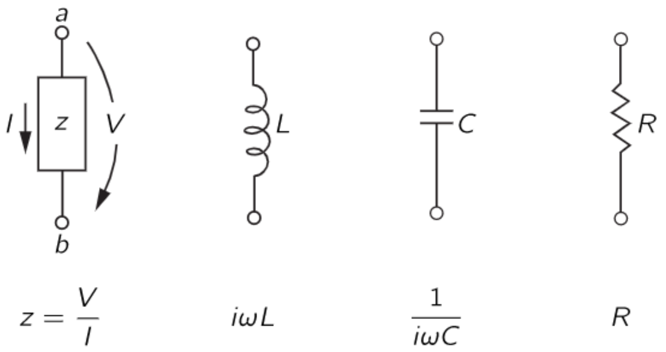
$$V(t) = \hat{V} e^{i\omega t}$$

$$I = \hat{I}\ e^{i\omega t} \quad \text{(current)}$$

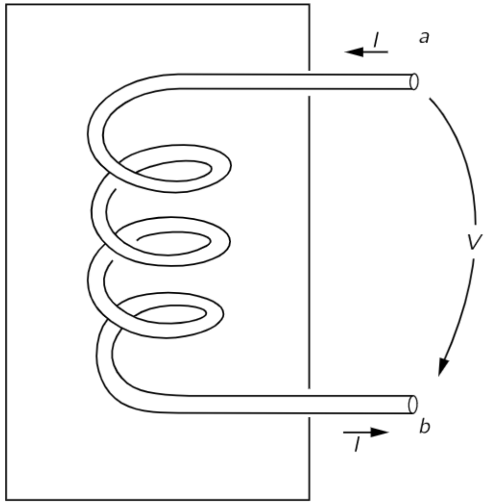
$$\mathcal{E} = \hat{\mathcal{E}}\ e^{i\omega t} \quad \text{(emf)}$$

$$\boldsymbol{E} = \hat{\boldsymbol{E}}\, e^{i\omega t} \quad \text{(electric field)}$$

The actual time-varying voltage *V(t)* is given by the **real part of the complex function** on the right-hand side of the equation.



$$z\,\text{(resistance)} = z_R = R$$



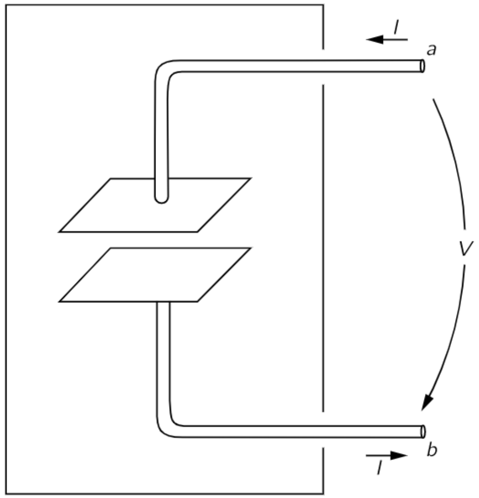
$$V = -\mathcal{E} = L\frac{dI}{dt}$$

$$dI/dt = i\omega I$$

$$V = i\omega L I$$

$$\frac{V}{I} = \frac{\hat{V}}{\hat{I}} = z$$

$$z\,\text{(inductance)} = z_L = i\omega L$$



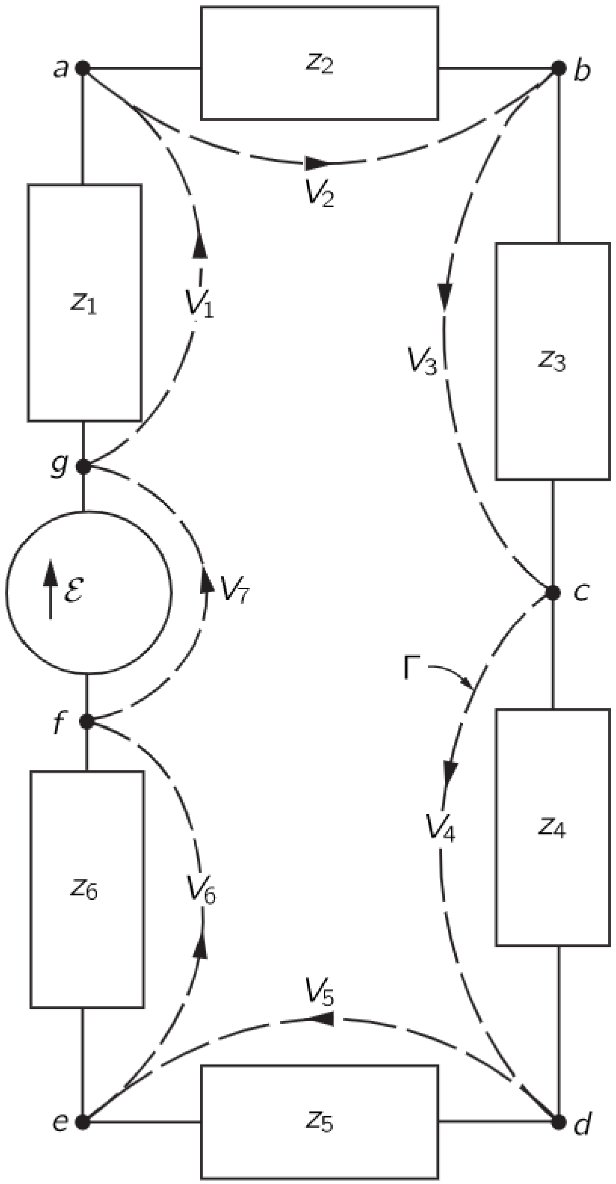
$$V = \frac{Q}{C}$$

dV/dt as $i\omega V$

$$i\omega V = \frac{I}{C}$$

$$z\,\text{(capacitor)} = z_C = \frac{1}{i\omega C}$$

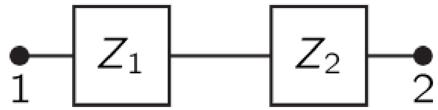
More on AC analysis



$$\oint \boldsymbol{E} \cdot d\boldsymbol{s} = \sum V_n$$

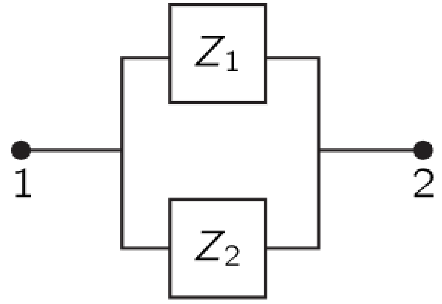
$$\sum_{\substack{\text{around} \\ \text{any loop}}} V_n = 0$$

From one of Maxwell's equations—that in a region where there are no magnetic fields the line integral of *E* around any complete loop is zero



$$\hat{V} = \hat{V}_1 + \hat{V}_2 = (\hat{Z}_1 + \hat{Z}_2)\hat{I}$$

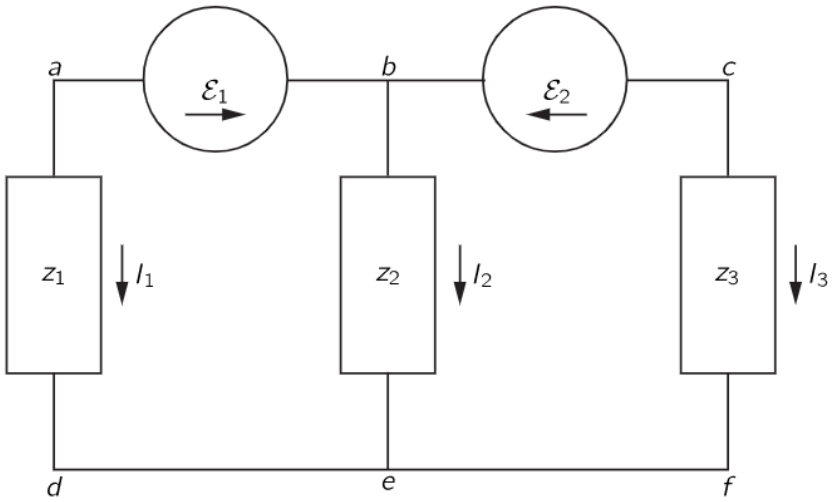
$$\hat{Z}_s = \hat{Z}_1 + \hat{Z}_2$$



$$\hat{V} = \frac{\hat{I}}{(1/\hat{Z}_1) + (1/\hat{Z}_2)} = \hat{I}\,\hat{Z}_p$$

$$1/\hat{Z}_p = 1/\hat{Z}_1 + 1/\hat{Z}_2$$

More on AC analysis



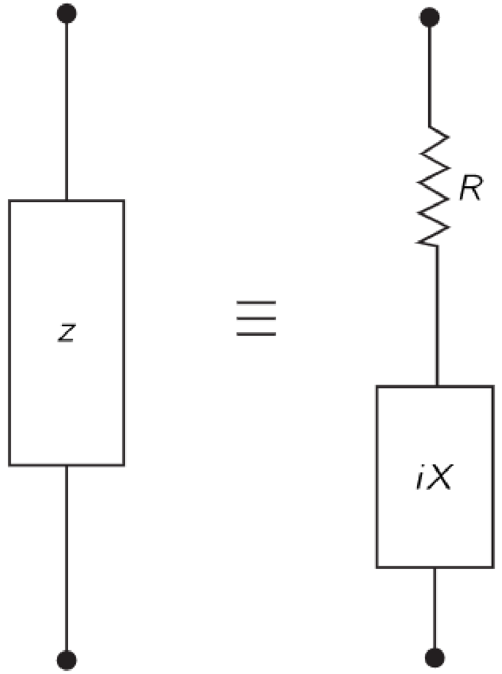
$$I_1 + I_2 + I_3 = 0.$$

$$-\mathcal{E}_1 + I_2 z_2 - I_1 z_1 = 0$$

$$\mathcal{E}_2 - (I_1 + I_2) z_3 - I_2 z_2 = 0.$$

$$I_1 = \frac{z_2 \mathcal{E}_2 - (z_2 + z_3)\mathcal{E}_1}{z_1(z_2 + z_3) + z_2 z_3}$$

$$I_2 = \frac{z_1 \mathcal{E}_2 + z_3 \mathcal{E}_1}{z_1(z_2 + z_3) + z_2 z_3}$$

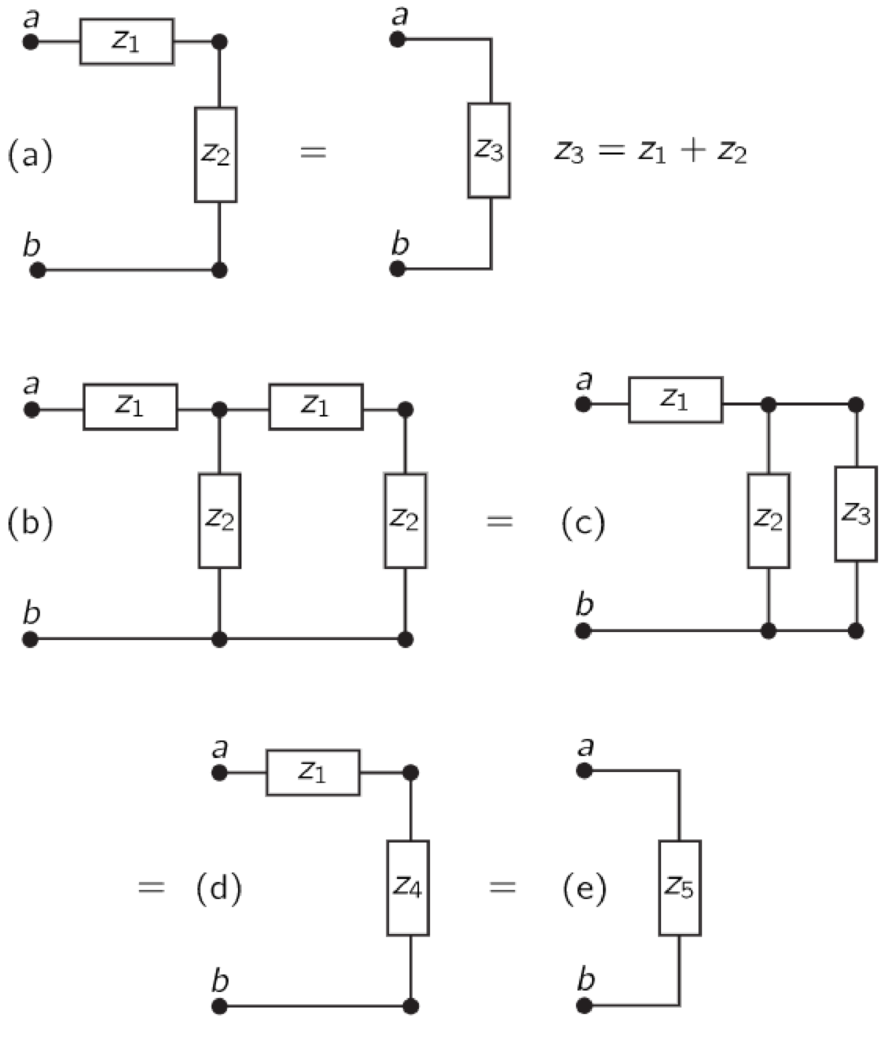


Any impedance is equivalent to a series combination of a pure resistance and a pure reactance.

$$z = R + iX$$

Dissipative *Nondissipative*

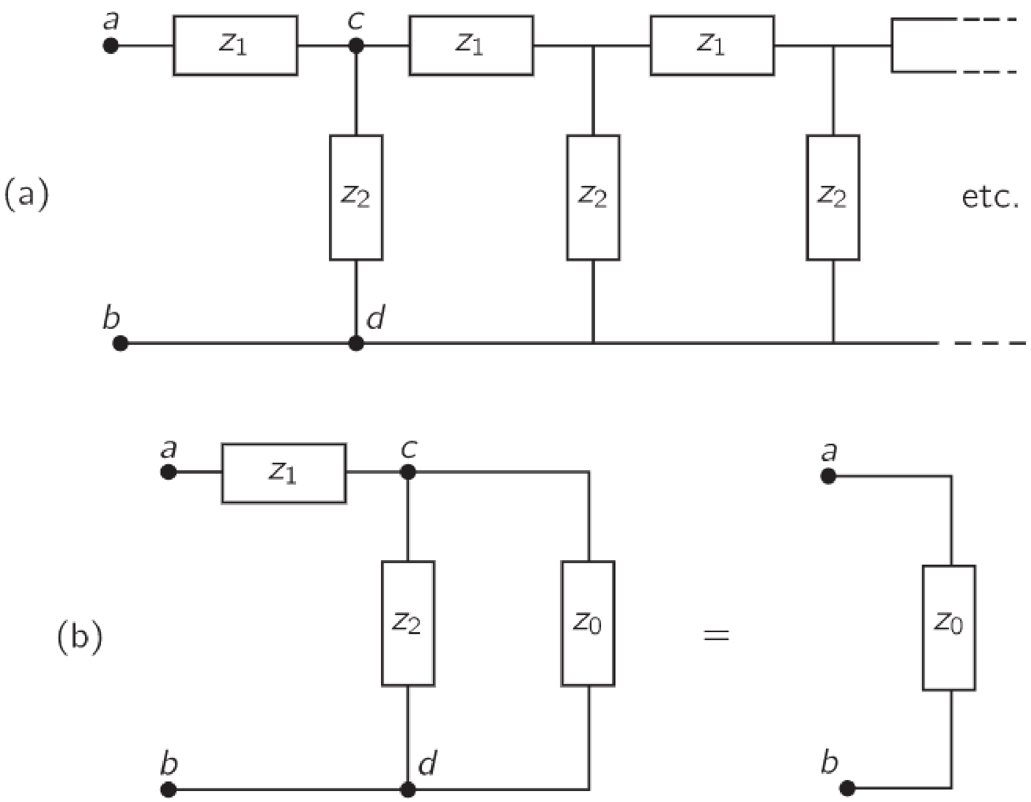
More on AC analysis



$$\frac{1}{z_4} = \frac{1}{z_2} + \frac{1}{z_3}$$

$$z_5 = z_1 + z_4$$

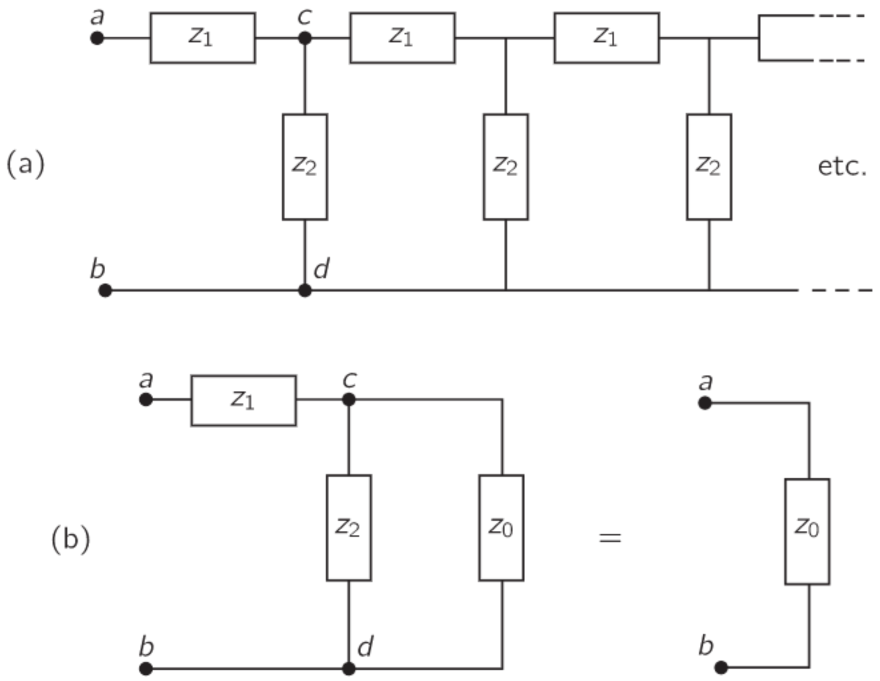
A ladder network



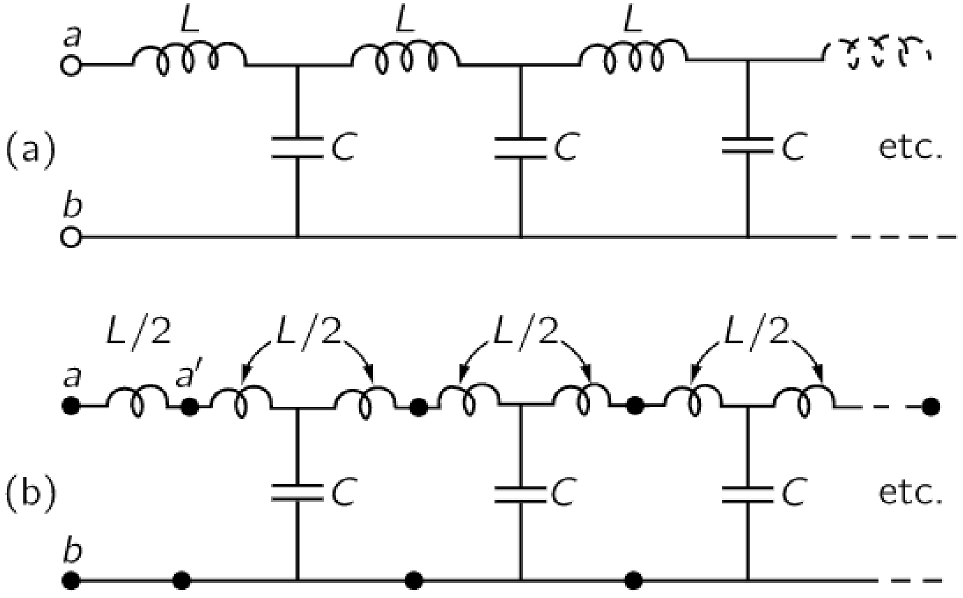
$$z_0 = z_1 + \frac{z_2 z_0}{z_2 + z_0}$$

$$z_0 = \frac{z_1}{2} + \sqrt{(z_1^2/4) + z_1 z_2}$$

More on AC analysis



$$z_0 = \frac{z_1}{2} + \sqrt{(z_1^2/4) + z_1 z_2}$$



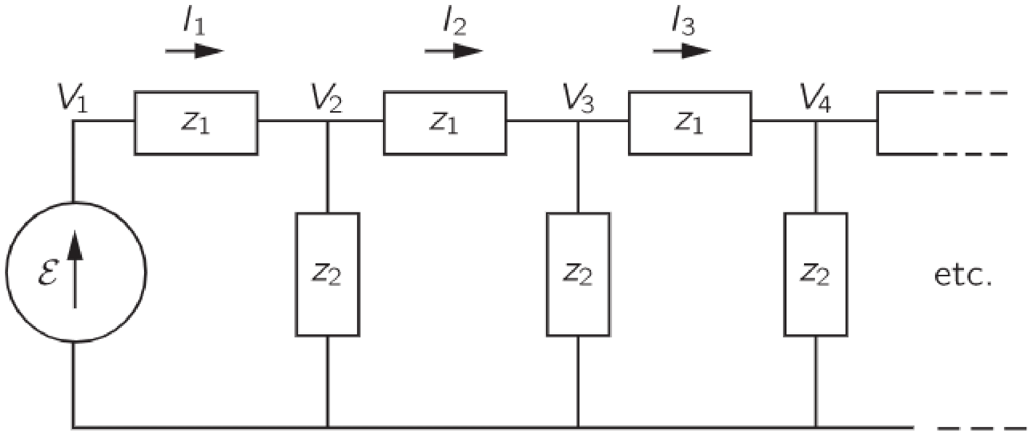
$$z_1 = i\omega L \text{ and } z_2 = 1/i\omega C$$

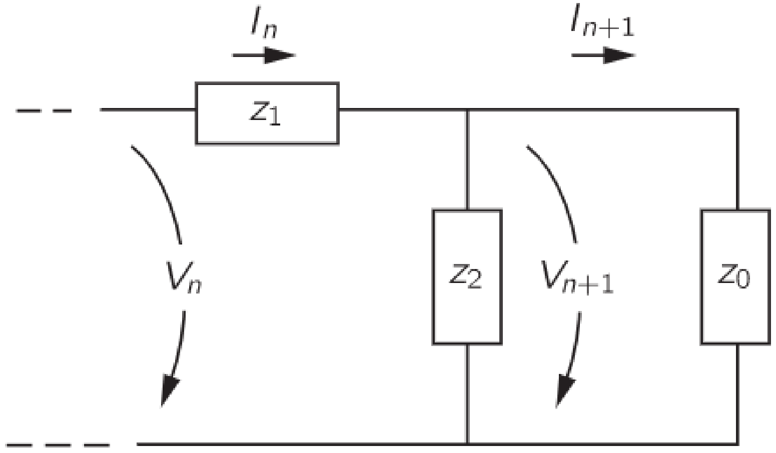
$$z_0 = \sqrt{(L/C) - (\omega^2 L^2/4)}$$

ω greater than $\sqrt{4/LC}$ $\qquad z_0 = i\sqrt{(\omega^2 L^2/4) - (L/C)}$

For lower frequencies the impedance is a pure resistance and will therefore absorb energy. But how can the circuit continuously absorb energy, as a resistance does, if it is made only of inductances and capacitances?

More on AC analysis





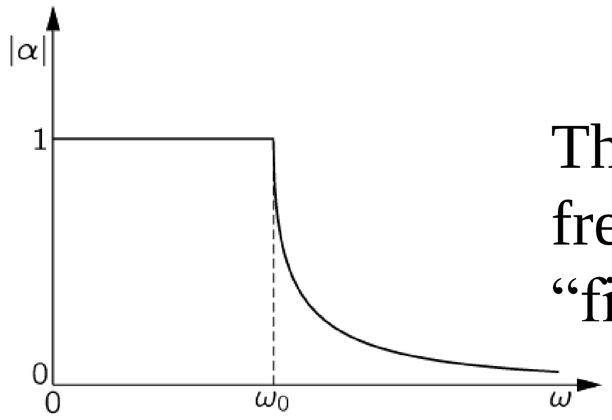
$$V_n - V_{n+1} = I_n z_1 = V_n \frac{z_1}{z_0}$$

$$\frac{V_{n+1}}{V_n} = 1 - \frac{z_1}{z_0} = \frac{z_0 - z_1}{z_0}$$

$$V_n = \alpha^n \mathcal{E}$$

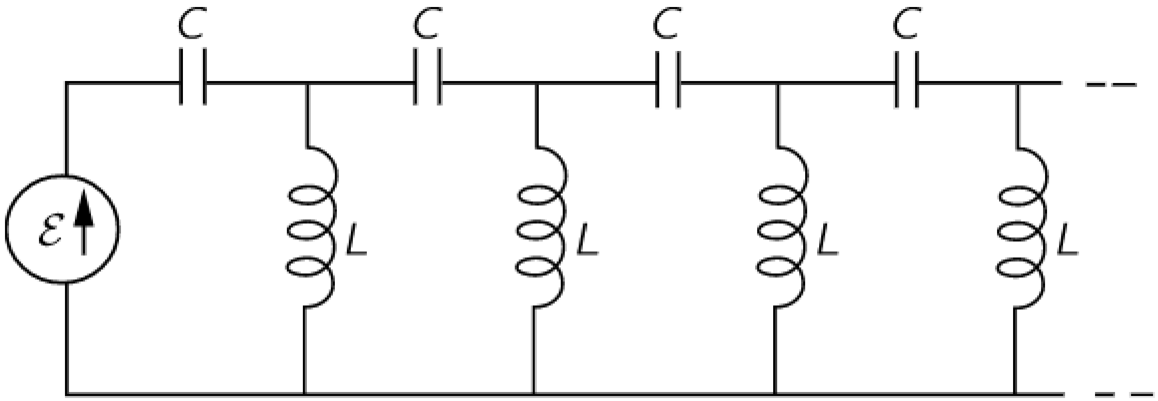
$$\alpha = \frac{\sqrt{(L/C) - (\omega^2 L^2/4)} - i(\omega L/2)}{\sqrt{(L/C) - (\omega^2 L^2/4)} + i(\omega L/2)} = e^{i\delta}$$

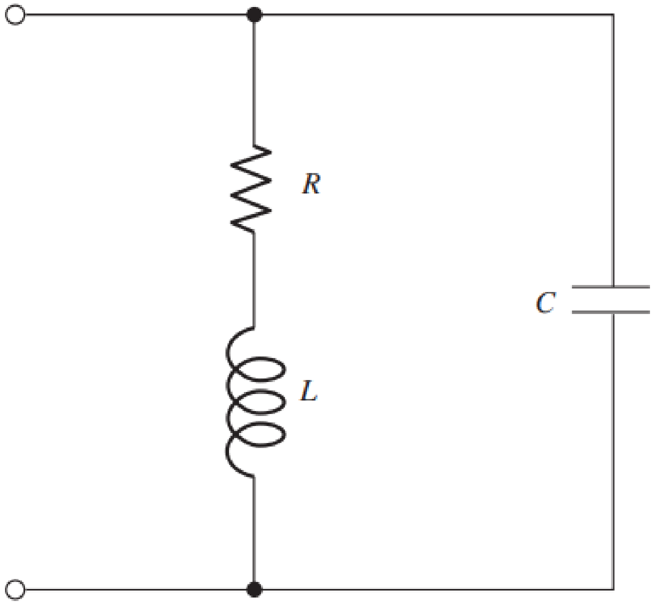
$$\alpha = \frac{\sqrt{(\omega^2 L^2/4) - (L/C)} - (\omega L/2)}{\sqrt{(\omega^2 L^2/4) - (L/C)} + (\omega L/2)}$$

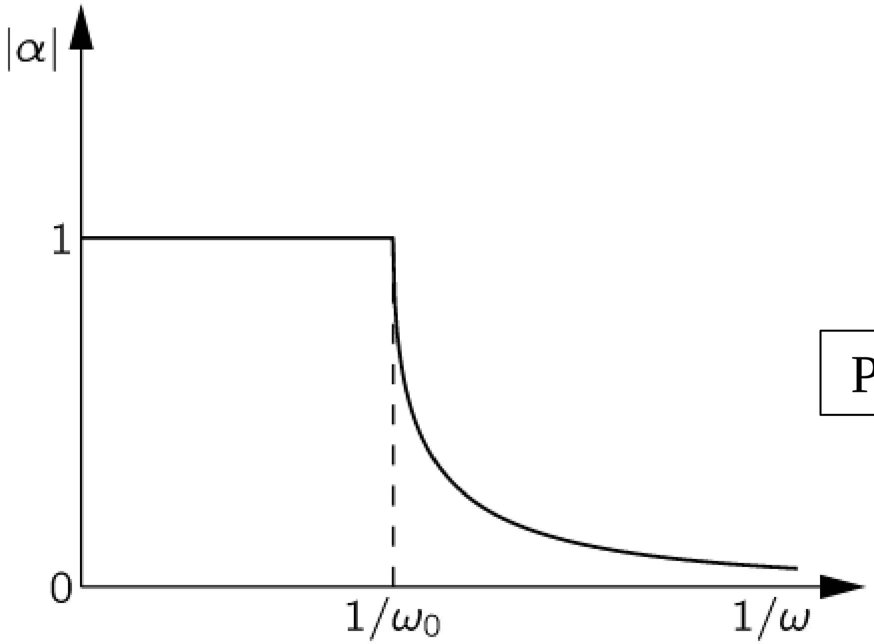


The network "passes" low frequencies and "rejects" or "filters out" the high frequencies.

More on AC analysis





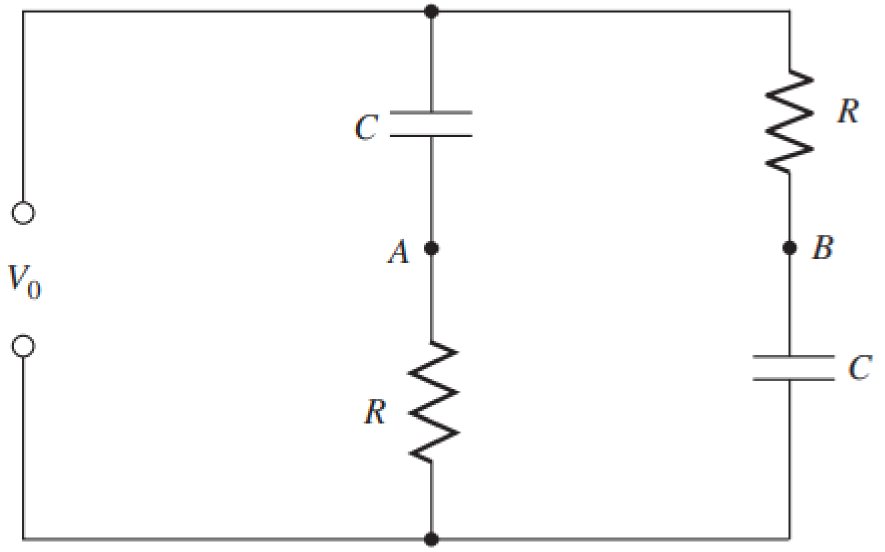


$$\frac{1}{Z} = \frac{1}{R + i\omega L} + i\omega C = \frac{R - i\omega L}{R^2 + \omega^2 L^2} + i\omega C$$

Purely real $\quad \dfrac{\omega L}{R^2 + \omega^2 L^2} = \omega C \implies \omega^2 = \dfrac{1}{LC} - \dfrac{R^2}{L^2}$

$$R^2 < L/C$$

More on AC analysis

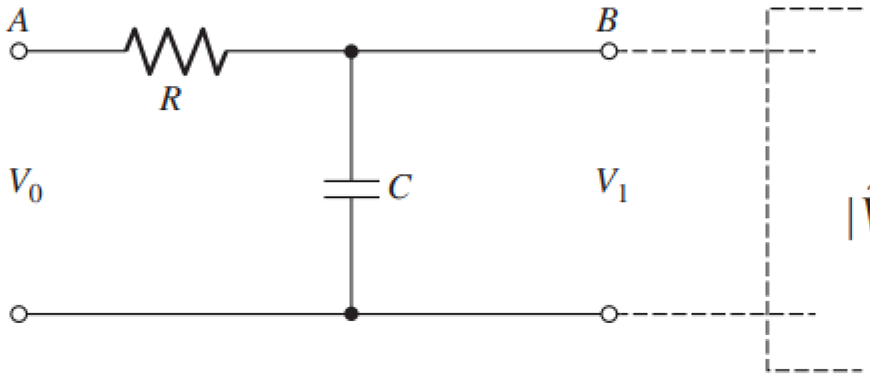


$$V_0 = \tilde{I}(R + 1/i\omega C) \implies \tilde{I} = \frac{V_0}{R + 1/i\omega C}$$

$$\tilde{V}_{AB} \equiv \tilde{V}_B - \tilde{V}_A = \tilde{I}(-R + 1/i\omega C)$$

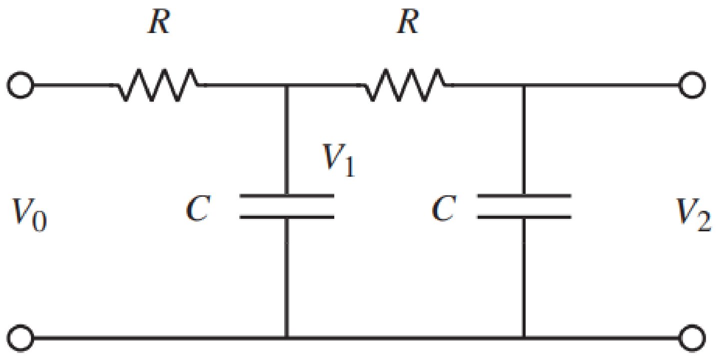
$$= V_0 \frac{-R + 1/i\omega C}{R + 1/i\omega C} = V_0 \frac{1 - i\omega RC}{1 + i\omega RC}$$

$$\boxed{|V_{AB}|^2 = V_0^2}$$



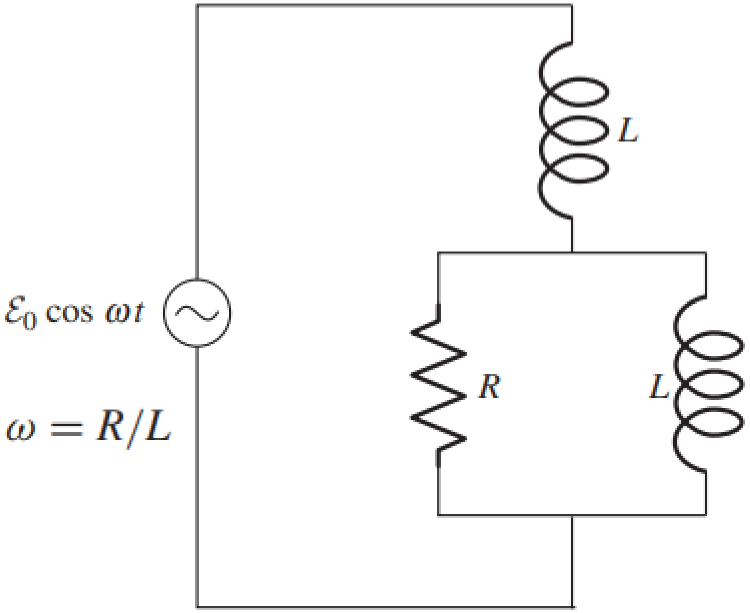
$$|\tilde{V}_1/V_0|^2 \propto 1/\omega^2$$

$$\frac{\tilde{V}_1}{V_0} = \frac{1}{1 + i\omega RC} \implies \left|\frac{\tilde{V}_1}{V_0}\right|^2 = \frac{1}{1 + \omega^2 R^2 C^2}$$

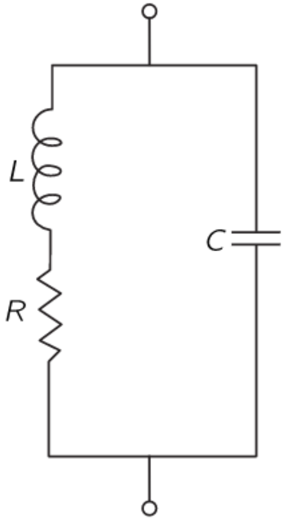


$$|V_2/V_0|^2 \propto 1/\omega^4$$

More on AC analysis

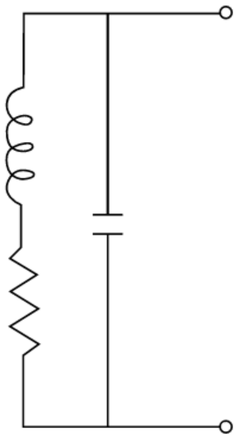


Real R

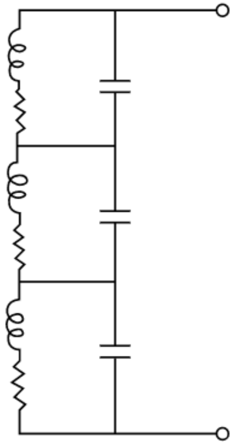


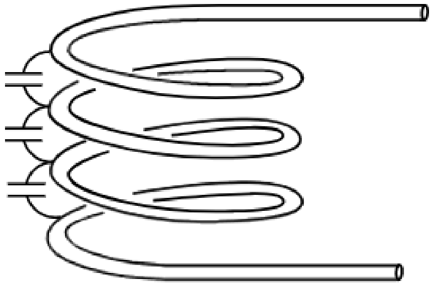
Real L

Low-f



High-f



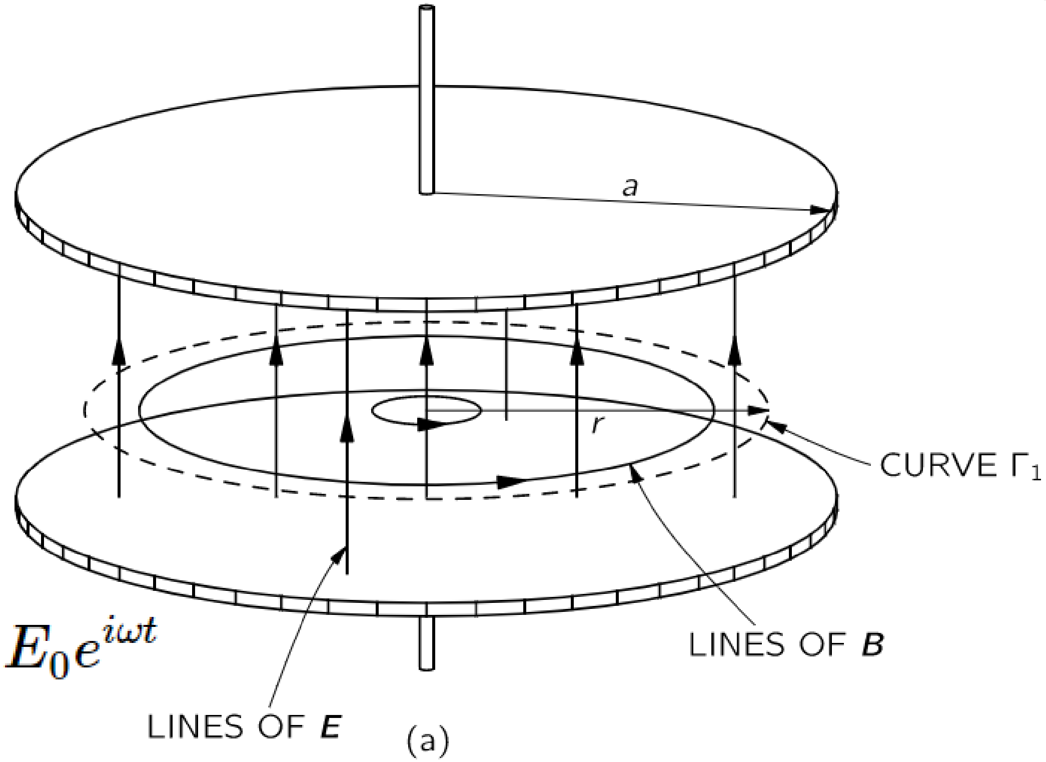


$$Z = Z_L + \frac{Z_R Z_L}{Z_R + Z_L} = iR + \frac{R(iR)}{R + iR} = R\frac{-1+2i}{1+i}$$

$$\tilde{I} = \frac{\mathcal{E}_0}{Z} = \frac{\mathcal{E}_0}{R}\frac{1+i}{-1+2i} = \frac{\mathcal{E}_0}{R}\frac{1-3i}{5} = \frac{\mathcal{E}_0}{R}\frac{\sqrt{10}}{5}e^{i\phi}$$

$$I(t) = \mathrm{Re}\left[\tilde{I}e^{i\omega t}\right] = \mathrm{Re}\left[\frac{\sqrt{10}}{5}\frac{\mathcal{E}_0}{R}e^{i\phi}e^{i\omega t}\right] = \frac{\sqrt{10}}{5}\frac{\mathcal{E}_0}{R}\cos(\omega t + \phi)$$

More on AC analysis



$$E = E_0 e^{i\omega t}$$

$$c^2 \oint_\Gamma \boldsymbol{B}\cdot d\boldsymbol{s} = \frac{d}{dt}\int_{\text{inside }\Gamma} \boldsymbol{E}\cdot\boldsymbol{n}\,da$$

$$c^2 B\cdot 2\pi r = \frac{\partial}{\partial t}E\cdot \pi r^2$$

$$\frac{\partial}{\partial t}E \rightarrow i\omega E_0 e^{i\omega t}$$

$$B = \frac{i\omega r}{2c^2}E_0 e^{i\omega t}$$

When there is a varying magnetic field, there will be induced electric fields and the capacitor will begin to act a little bit like an inductance. As the frequency goes up, the magnetic field gets stronger; it is proportional to the rate of change of E, and so to ω. The impedance of the capacitor will no longer be simply $1/i\omega C$.

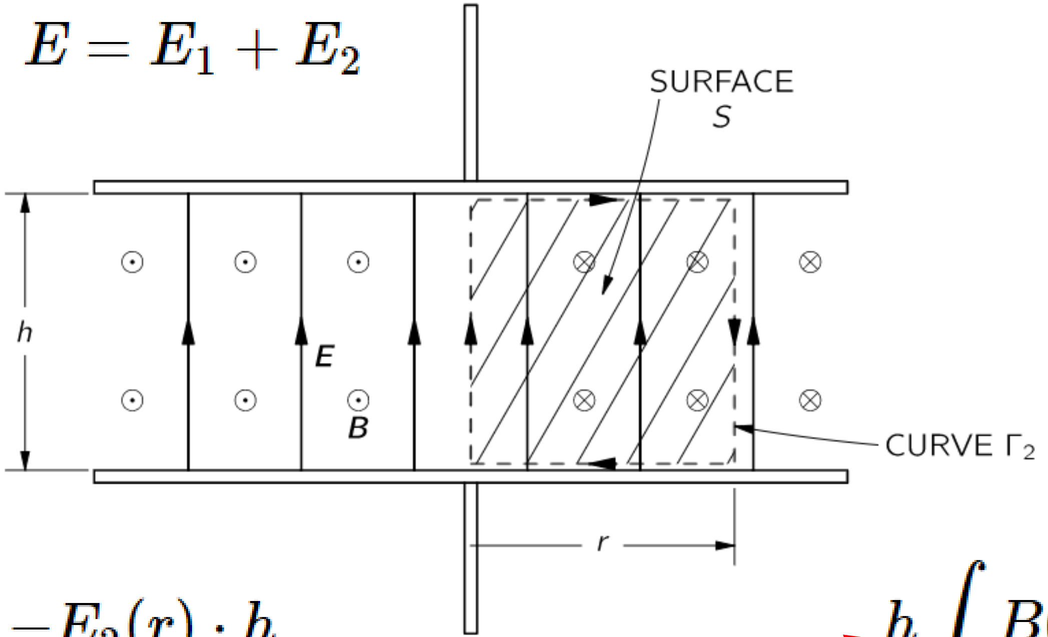
$$E = E_1 + E_2$$

Correction

$$\oint_\Gamma \boldsymbol{E}\cdot d\boldsymbol{s} = -\frac{d}{dt}(\text{flux of } \boldsymbol{B})$$

More on AC analysis

$$E = E_1 + E_2$$



$$-E_2(r) \cdot h \qquad h \int B(r)\, dr$$

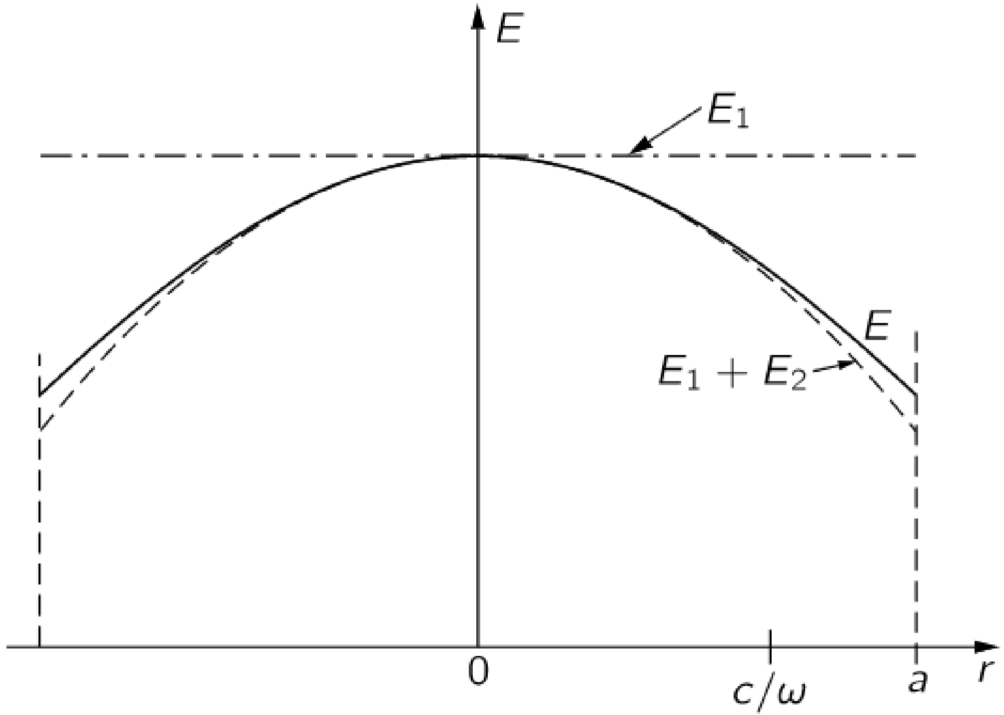
$$\oint_\Gamma \boldsymbol{E} \cdot d\boldsymbol{s} = -\frac{d}{dt}(\text{flux of } \boldsymbol{B})$$

$$E_2(r) = \frac{\partial}{\partial t} \int B(r)\, dr$$

$$B = \frac{i\omega r}{2c^2} E_0 e^{i\omega t}$$

$$E_2(r) = -\frac{\omega^2 r^2}{4c^2} E_0 e^{i\omega t}$$

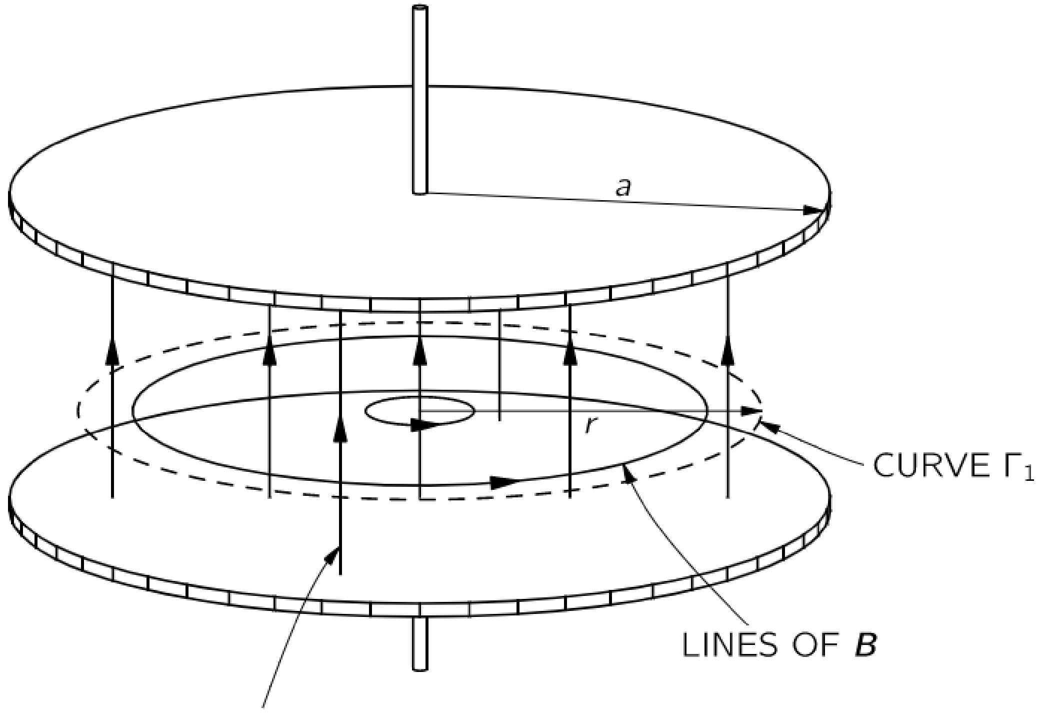
$$E = E_1 + E_2 = \left(1 - \frac{1}{4}\frac{\omega^2 r^2}{c^2}\right) E_0 e^{i\omega t}$$



More on AC analysis

$$E = E_1 + E_2 = \left(1 - \frac{1}{4}\frac{\omega^2 r^2}{c^2}\right) E_0 e^{i\omega t}$$

$$B = B_1 + B_2$$



(a)

$$c^2 B_2 \cdot 2\pi r = \frac{d}{dt}(\text{flux of } E_2 \text{ through } \Gamma_1)$$

$$E_2(r) = -\frac{\omega^2 r^2}{4c^2} E_0 e^{i\omega t} \quad B_2(r) = \frac{1}{rc^2}\frac{\partial}{\partial t}\int E_2(r) r\, dr$$

$$B_2(r) = -\frac{i\omega^3 r^3}{16c^4} E_0 e^{i\omega t}$$

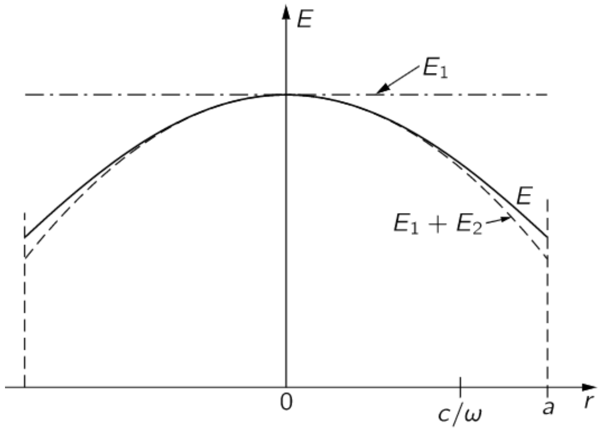
$$E_3(r) = \frac{\partial}{\partial t}\int B_2(r)\, dr \qquad E_3(r) = +\frac{\omega^4 r^4}{64c^4} E_0 e^{i\omega t}$$

$$E = E_0 e^{i\omega t}\left[1 - \frac{1}{2^2}\left(\frac{\omega r}{c}\right)^2 + \frac{1}{2^2 \cdot 4^2}\left(\frac{\omega r}{c}\right)^4\right]$$

$$E_4 = -\frac{1}{2^2 \cdot 4^2 \cdot 6^2}\left(\frac{\omega r}{c}\right)^6 E_0 e^{i\omega t}$$

More on AC analysis

$$E = E_0 e^{i\omega t}\left[1 - \frac{1}{(1!)^2}\left(\frac{\omega r}{2c}\right)^2 + \frac{1}{(2!)^2}\left(\frac{\omega r}{2c}\right)^4 - \frac{1}{(3!)^2}\left(\frac{\omega r}{2c}\right)^6 \pm \cdots\right]$$



$$J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 \pm \cdots$$

$$E = E_0 e^{i\omega t} J_0\left(\frac{\omega r}{c}\right)$$

The function J_0 is to cylindrical waves like the cosine function is to waves on a straight line. A man named Bessel got his name attached to it. The subscript zero means that Bessel invented many different functions, and this is just the first of them. The other functions of Bessel—J_1, J_2, and so on—have to do with cylindrical waves, which vary their strength with the angle around the cylinder's axis.

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