Chap.7 Supplement

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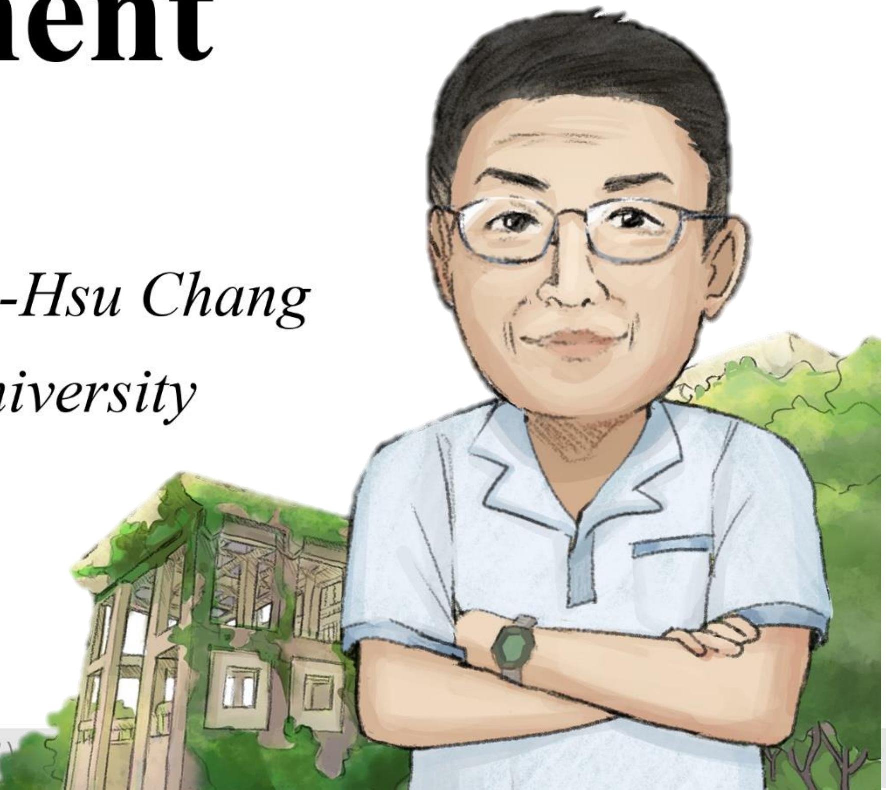
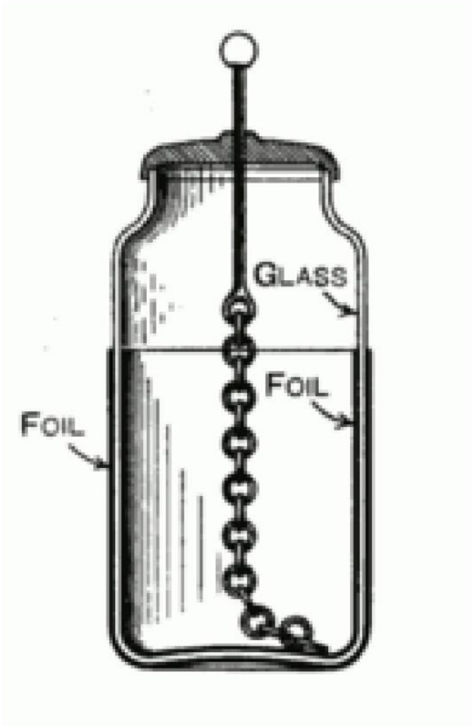
For the EM Course Lectured by Prof. Tsun-Hsu Chang

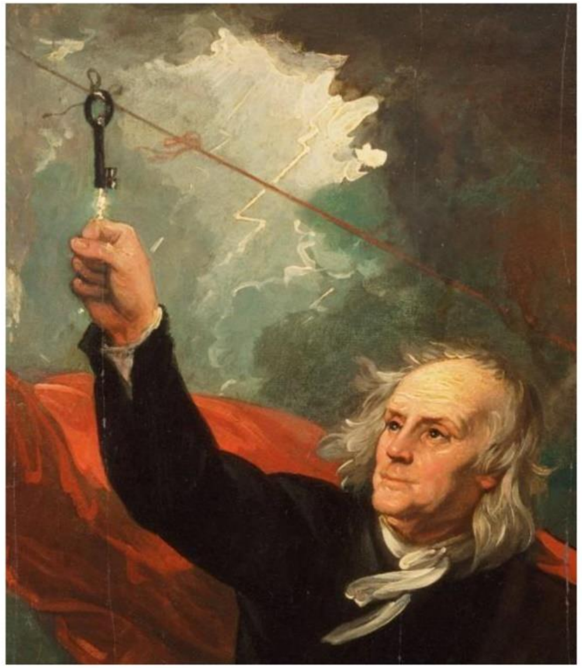
2023 Spring at National Tsing Hua University



Brief History of Discovering RLC #1



Leyden jar
(1745)

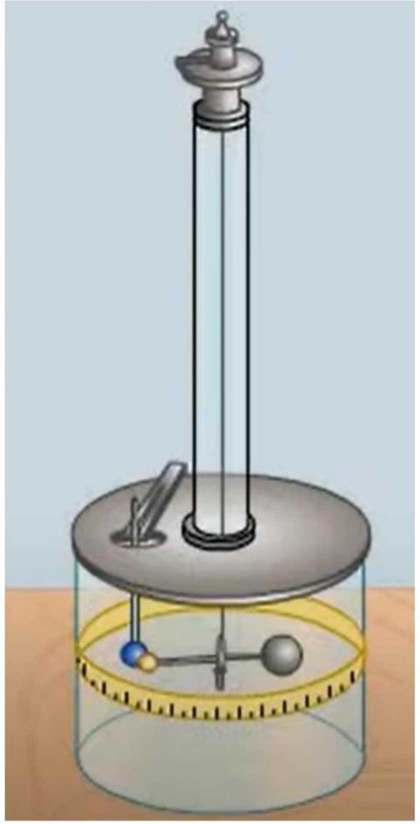


Kite experiment
(1752)

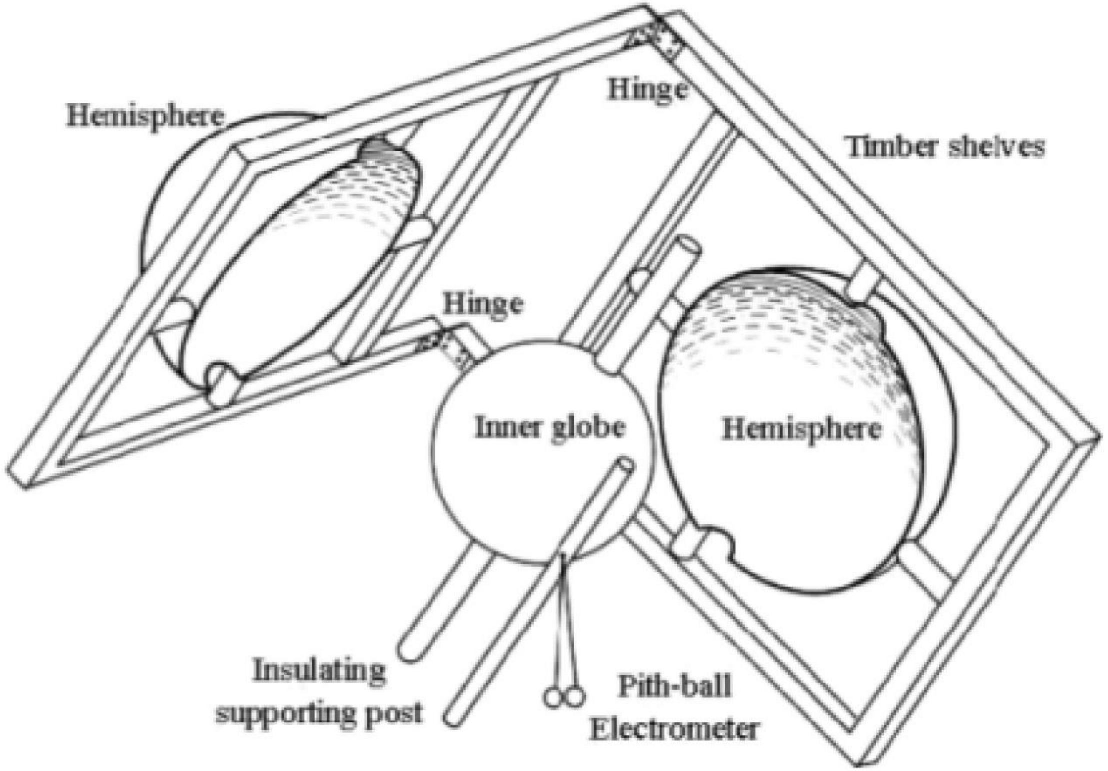
Lightning Rod
(Franklin Rod)

Laser-guided lightning,
Nat. Photon. (2023).

Brief History of Discovering RLC #1



Coulomb's Torsion Balance (1785)



Cavendish's Shielding Test (1773)

For more discussions, see AJP **60**, 988 (1992)

Justification of Inverse-Square Law

Cavendish and Maxwell conducted experiments to test the inverse-square nature of electrical force. This problem gives the theory behind their experiments.

(a) Assume that Coulomb's law takes the form of $kq_1q_2/r^{2+\delta}$. Given a hollow spherical shell with radius R and uniformly distributed charge Q, show that the potential at radius r is (with $f(x) = x^{1-\delta}$ and $k \equiv 1/4\pi\epsilon_0$)

$$\phi(r) = \frac{kQ}{2(1-\delta^2)rR}\big[f(R+r) - f(R-r)\big] \qquad (\text{for } r < R),$$

$$\phi(r) = \frac{kQ}{2(1-\delta^2)rR}\big[f(R+r) - f(r-R)\big] \qquad (\text{for } r > R).$$

(b) Consider two concentric shells with radii a and b (with $a > b$) and uniformly distributed charges Q_a and Q_b. Show that the potentials on the shells are given by

$$\phi_a = \frac{kQ_a}{2a^2}f(2a) + \frac{kQ_b}{2ab}\big[f(a+b) - f(a-b)\big],$$

$$\phi_b = \frac{kQ_b}{2b^2}f(2b) + \frac{kQ_a}{2ab}\big[f(a+b) - f(a-b)\big].$$

(c) Show that if the shells are connected, so that they are at the same potential ϕ, then the charge on the inner shell is

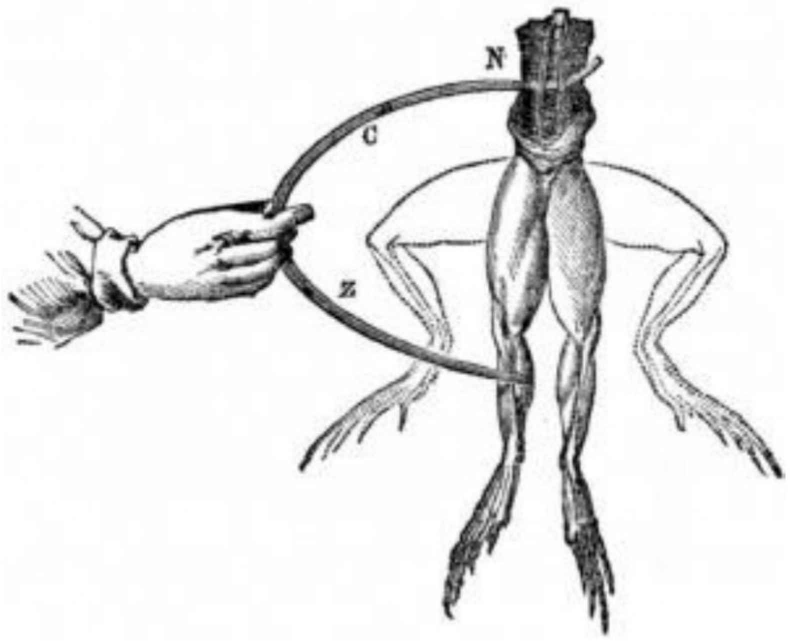
$$Q_b = \frac{2b\phi}{k} \cdot \frac{bf(2a) - a\big[f(a+b) - f(a-b)\big]}{f(2a)f(2b) - \big[f(a+b) - f(a-b)\big]^2}.$$

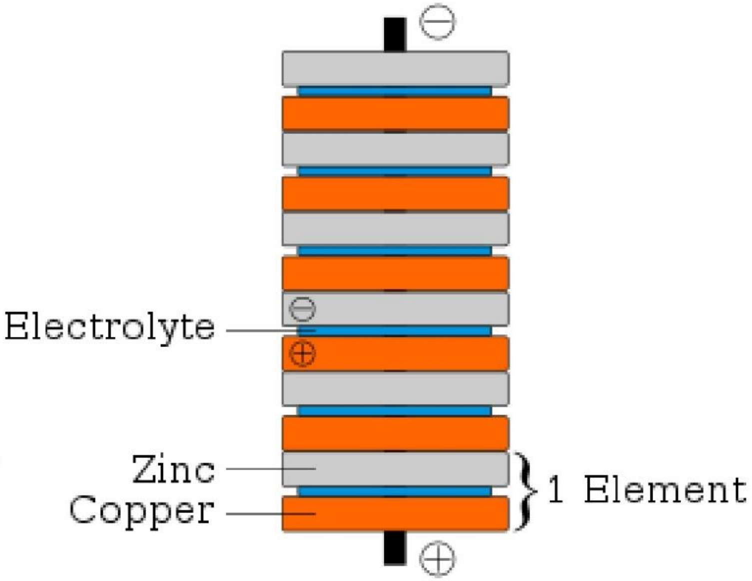
If $\delta = 0$ so that $f(x) = x$, then Q_b equals zero, as it should. So if Q_b is measured to be nonzero, then δ must be nonzero.

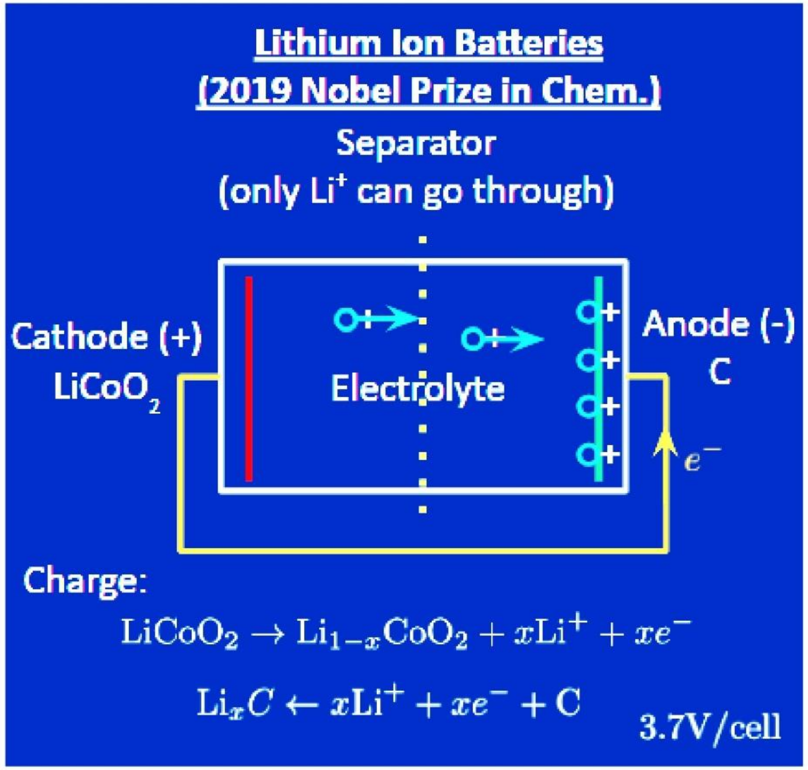
For small δ it is possible to expand Q_b to first order in δ by using the approximation $f(x) = xe^{-\delta \ln x} \approx x(1 - \delta \ln x)$, but this gets very messy. You are encouraged instead to use a computer to calculate and plot Q_b for various values of a, b, and δ. You can also trivially expand Q_b to first order in δ by using the `Series` operation in *Mathematica*.

Photo Credits: Google

Brief History of Discovering RLC #1





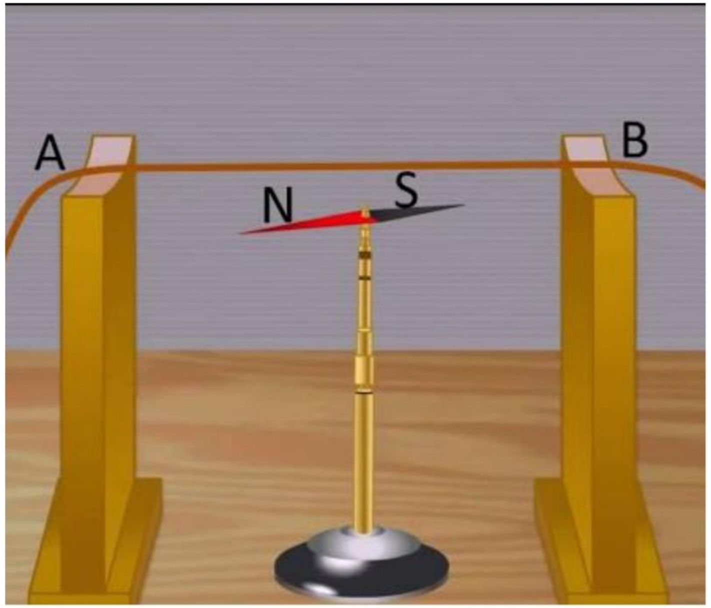


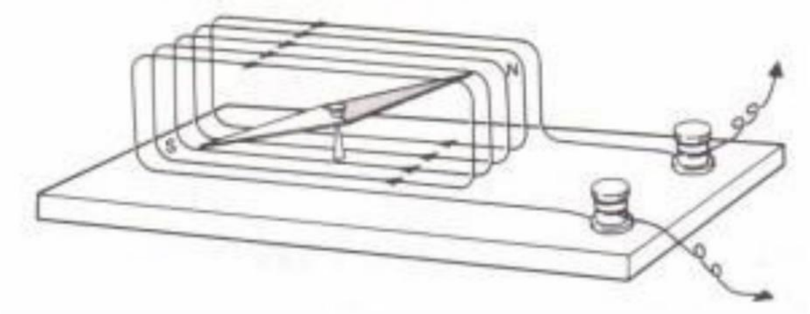
Galvani's animal electricity vs. Volta's metallic electricity
(1780~1800)

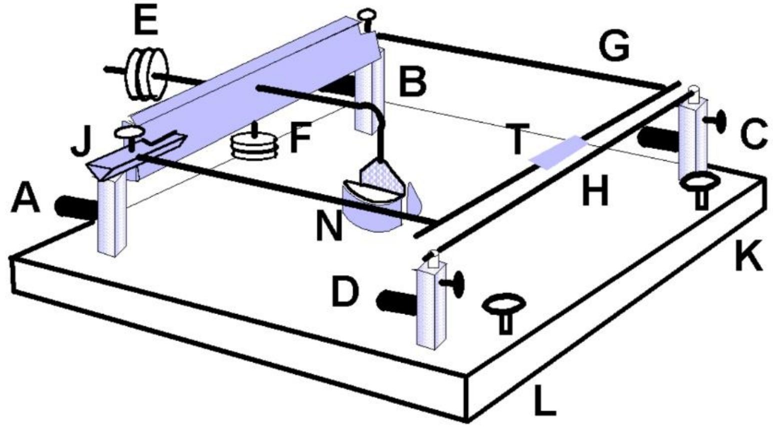
Note: Electron is found by Thomson in 1897.

Brief History of Discovering RLC #2

$$\frac{F_m}{L} = 2k_{\mathrm{A}}\frac{I_1 I_2}{r}$$







Oested's Experiment(1820)

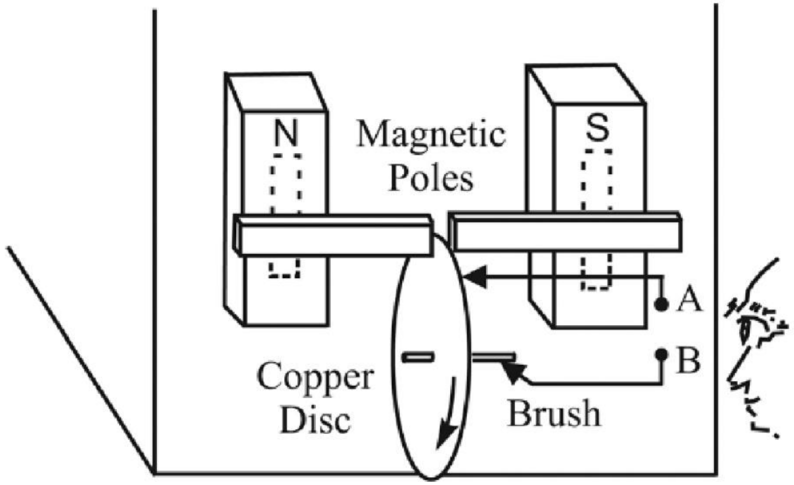
Schweigger's galvanometer (1820)

Ampère's current balance (1820)
→ Ampère's Force Law

[See *AJP* **85**, 369 (2017)]

[See *AJP* **77**, 721 (2009)]

Brief History of Discovering RLC #2



Faraday's dynamo(1831)
(why not Ampere's?)

[See *AJP* **81**, 907 (2013)]

a historical misnomer

$$\oint_C \mathbf{B} \cdot d\vec{\ell} = \mu_0 \sum_i I_i$$

Ampere's circuital Law & Right-hand Rule(1826)

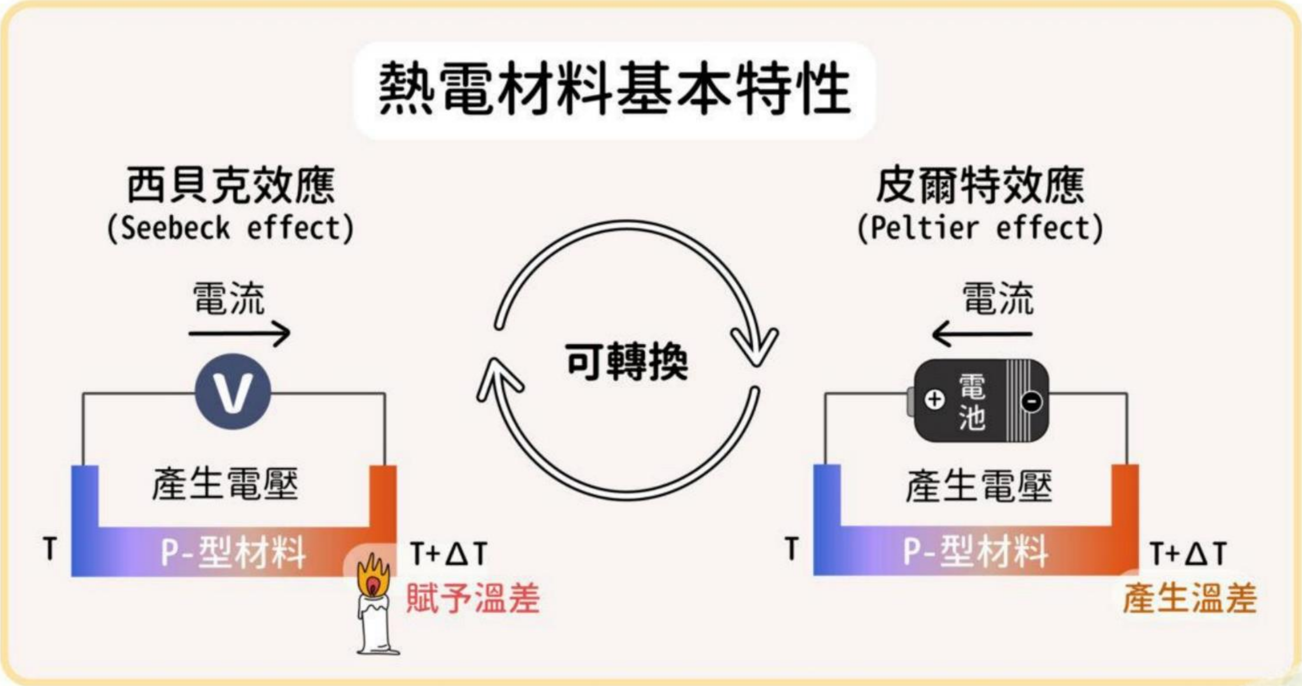
"The total intensity of magnetizing force in a closed curve passing through and embracing the closed current is constant, and may therefore be made a measure of the quantity of the current."

— *Maxwell*

[See *AJP* **67**, 448 (1999)]

Brief History of Discovering RLC #2

Georg S. Ohm (1789 ~1854)



Credit: Academia Sinica.

Brief History of Discovering RLC #2

Georg S. Ohm (1789 ~1854)

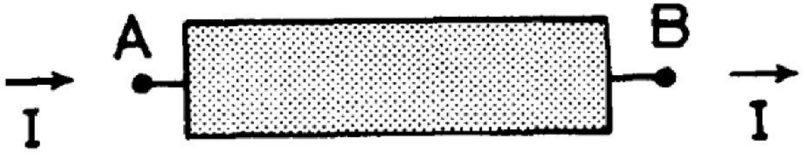
Ohm's work (published in 1827) was not appreciated or even understood in his time for several *possible* reasons:

1. Too much mathematics
2. Different conclusion from Ampère in 1820
3. Introduction of electromotive force

He received Copley Medal in 1841, "for his research into the laws of electric currents."

[See *AJP* **31**, 536 (1963)]

What do Voltmeters measure?



Potential Drop: $\tilde{V} = V_A - V_B$

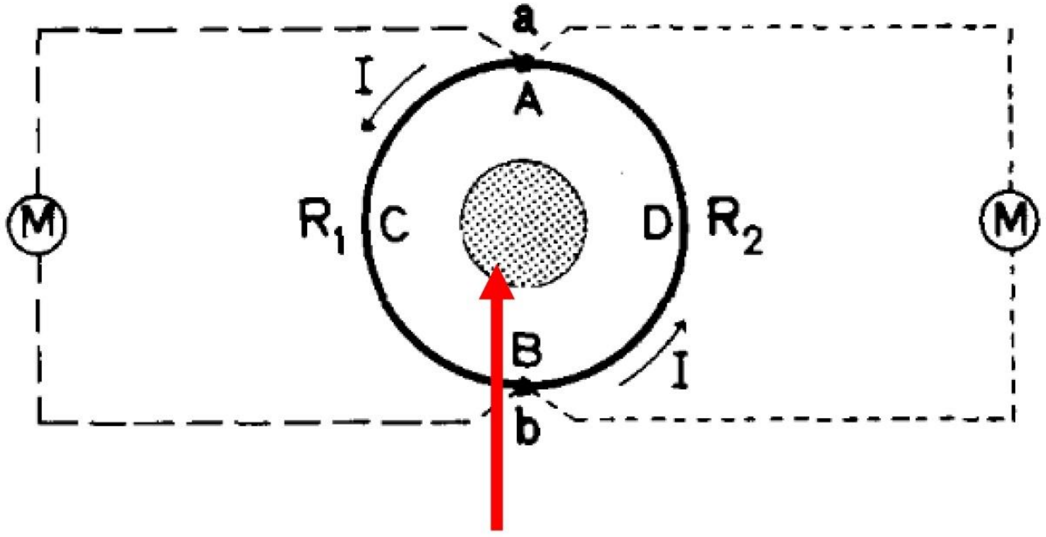
Ohm's Law: $IR = \tilde{V} + \varepsilon$

$\tilde{V}$: Potential drop, by "conservative Coulomb forces"

ε : EMF, by "non-Coulomb forces"

Special-case check: $IR = \tilde{V} + \varepsilon = 0$

$\Rightarrow \varepsilon = V_B - V_A$



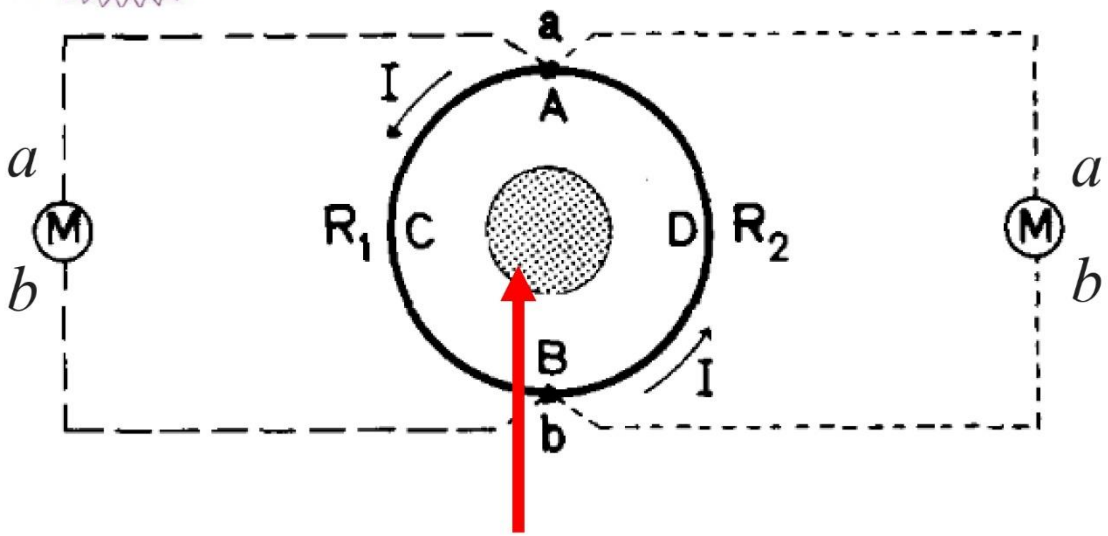
Uniformly-Changing B-field
→Constant EMF

Left loop: $R_1 I = \left(V_A - V_B\right) + \dfrac{\varepsilon}{2}$

Right loop: $R_2 I = \left(V_B - V_A\right) + \dfrac{\varepsilon}{2}$

See *AJP* **50**, 1048 (1982); *AJP* **50**, 1089 (1982)

What do Voltmeters measure?



Uniformly-Changing B-field
→Constant EMF

Left loop: $R_1 I = \left(V_A - V_B\right) + \dfrac{\varepsilon}{2}$

Right loop: $R_2 I = \left(V_B - V_A\right) + \dfrac{\varepsilon}{2}$

$$R_1\left(\frac{\varepsilon}{R_1+R_2}\right) = \left(V_A - V_B\right) + \frac{\varepsilon}{2}$$

$$\Rightarrow V_A - V_B = \frac{1}{2}\frac{R_1 - R_2}{R_1 + R_2}\varepsilon$$

Generally, the indicated voltage is equal to $\tilde{V} + \varepsilon$

Therefore, the voltage indicated by the meters are

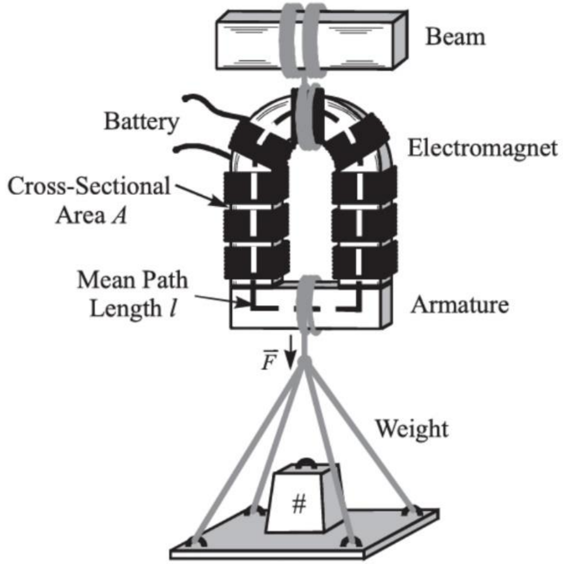
Left: $\left(V_A - V_B\right) + \dfrac{\varepsilon}{2} = \dfrac{\varepsilon R_1}{R_1 + R_2}$ (V_a is higher)

Right: $\left(V_A - V_B\right) - \dfrac{\varepsilon}{2} = \dfrac{-\varepsilon R_2}{R_1 + R_2}$ (V_a is lower)

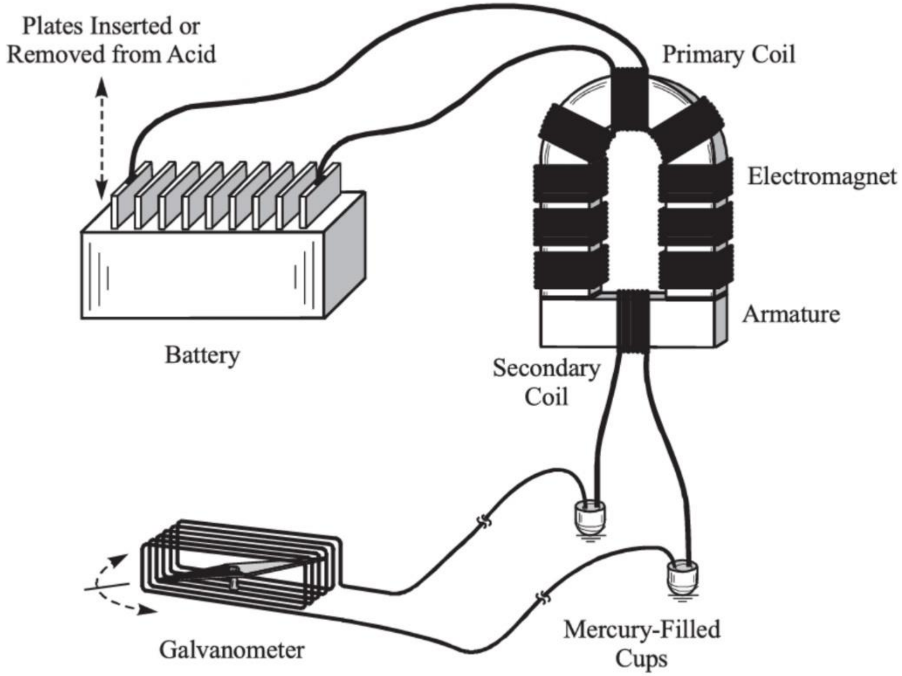
See *EJP* **38**, 015207 (2017)

Brief History of Discovering RLC #3

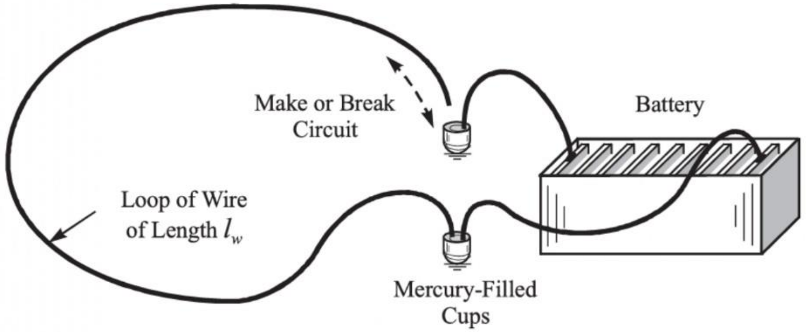
Joseph Henry (1797~1878)



Henry's electromagnet (1831)



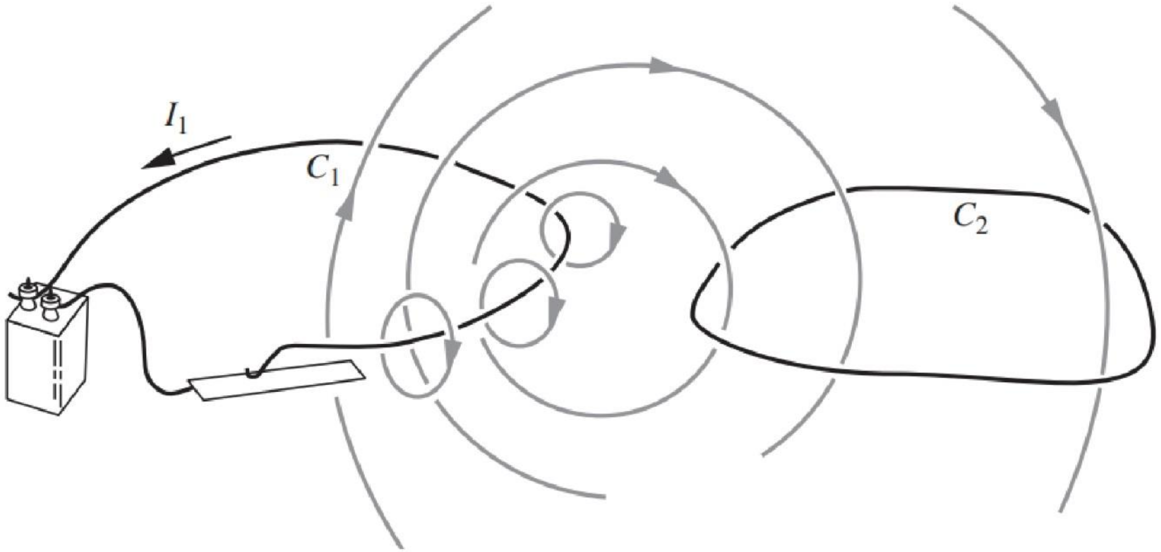
Henry's mutual inductance (1832)

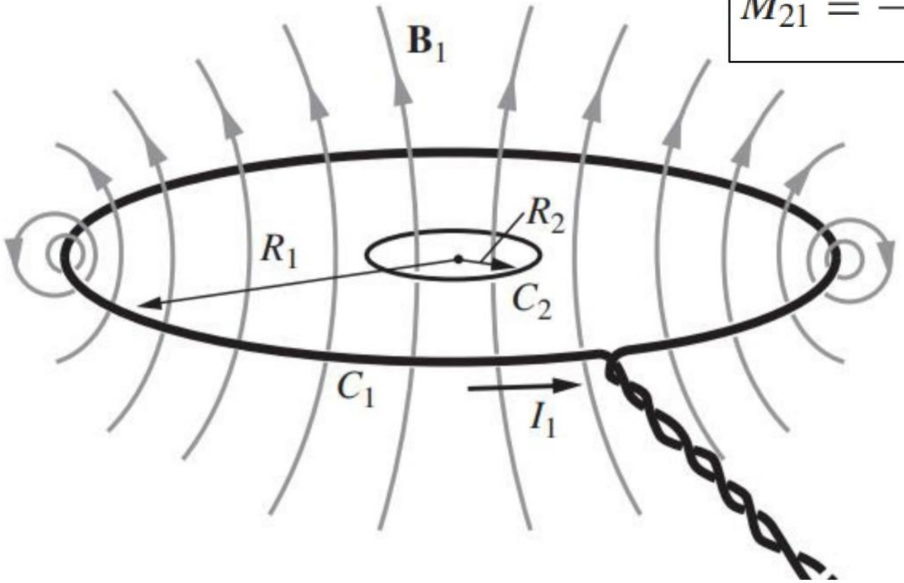


Henry's self-inductance (1832)

Brief History of Discovering RLC #3

$$M_{21} = \frac{\mu_0 \pi N_1 N_2 R_2^2}{2R_1}$$





$$\Phi_{21} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{a}_2 \qquad \frac{\Phi_{21}}{I_1} = \text{constant} \equiv M_{21}$$

$$\mathcal{E}_{21} = -\frac{d\Phi_{21}}{dt} \implies \mathcal{E}_{21} = -M_{21}\frac{dI_1}{dt}$$

$$1 \text{ henry} = 1 \frac{\text{volt} \cdot \text{second}}{\text{amp}} = 1 \text{ ohm} \cdot \text{second}$$

$$B_1 = \frac{\mu_0 I_1}{2R_1} \qquad \Phi_{21} = (\pi R_2^2)\frac{\mu_0 I_1}{2R_1} = \frac{\mu_0 \pi I_1 R_2^2}{2R_1}$$

$$M_{21} = \frac{\Phi_{21}}{I_1} = \frac{\mu_0 \pi R_2^2}{2R_1}$$

$$\mathcal{E}_{21} = -M_{21}\frac{dI_1}{dt} = -\frac{\mu_0 \pi R_2^2}{2R_1}\frac{dI_1}{dt}$$

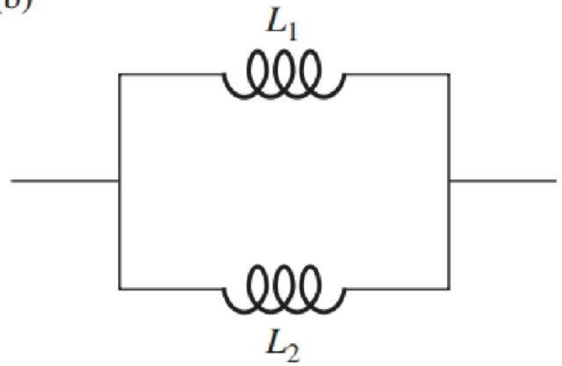
Photo Credits: Google

Brief History of Discovering RLC #3

(a)

$$L\frac{dI}{dt} = L_1\frac{dI}{dt} + L_2\frac{dI}{dt} \implies L = L_1 + L_2.$$

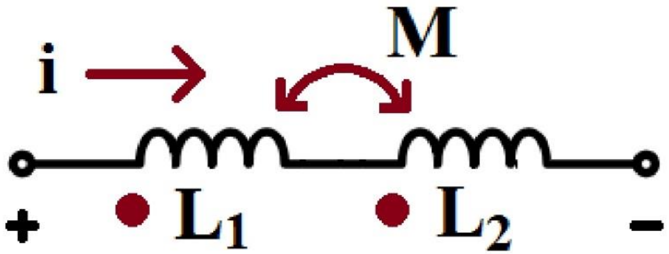
(b)



$$dI/dt = V/L.$$

$$I = I_1 + I_2$$

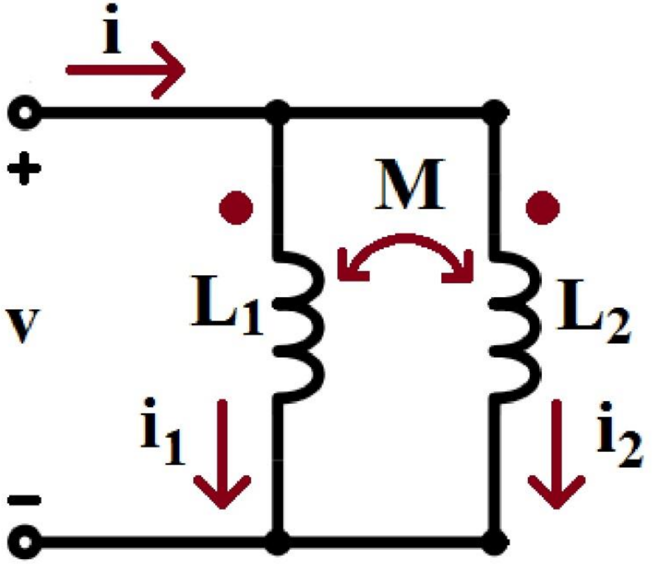
$$\frac{V}{L} = \frac{V}{L_1} + \frac{V}{L_2} \implies \frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}.$$



$$-v + L_1\frac{\mathrm{d}i}{\mathrm{d}t} + M\frac{\mathrm{d}i}{\mathrm{d}t} + L_2\frac{\mathrm{d}i}{\mathrm{d}t} + M\frac{\mathrm{d}i}{\mathrm{d}t} = 0$$

$$v = (L_1 + L_2 + 2M)\frac{\mathrm{d}i}{\mathrm{d}t} \qquad L_{eq} = L_1 + L_2 + 2M$$

Brief History of Discovering RLC #3



$$-v + L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} + M\frac{\mathrm{d}i_2}{\mathrm{d}t} = 0 \qquad \frac{di_1}{dt} = \frac{v(L_2 - M)}{L_1L_2 - M^2}$$

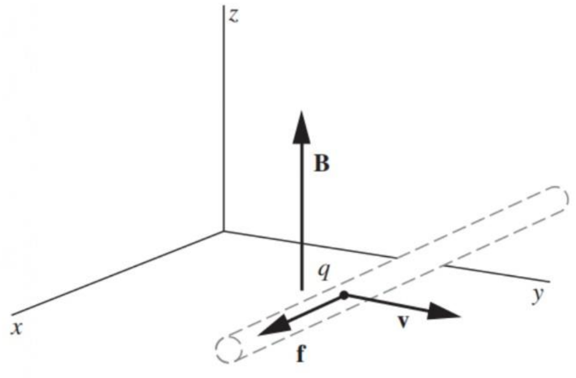
$$-v + L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} + M\frac{\mathrm{d}i_1}{\mathrm{d}t} = 0 \qquad \frac{di_2}{dt} = \frac{v(L_1 - M)}{L_1L_2 - M^2}$$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} = \frac{v(L_1 + L_2 - 2M)}{L_1L_2 - M^2}$$

$$v = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}\,\frac{\mathrm{d}i}{\mathrm{d}t}$$

$$L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$

Electrodynamic Tethers (EDTs)

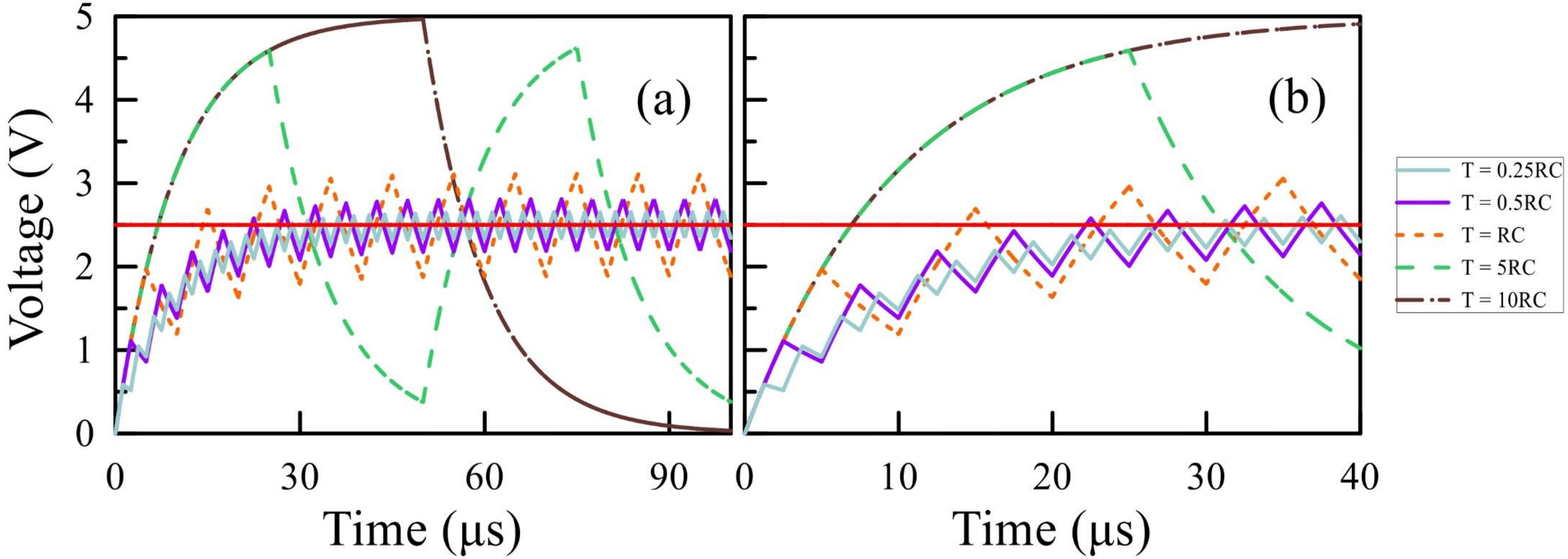


As the satellite and tether orbit the earth, they pass through its magnetic field. As a moving rod, an emf is generated along the wire. If this were the story, the charges would pile up on the ends. But the satellite is moving through the ionosphere, which contains enough ions to yield a return path for the charge. A complete circuit is therefore formed.

Photo Credits: Google

Electric Guitar

RC circuit: a square-wave input



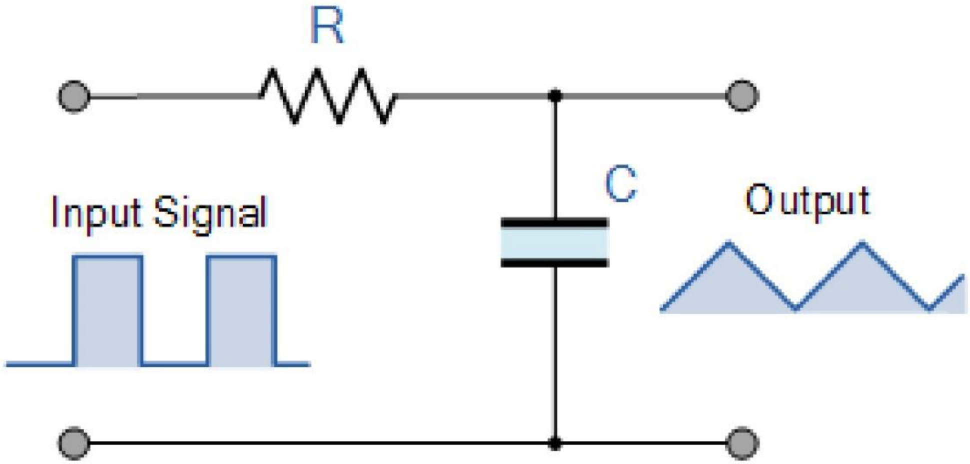
As the input square wave period decreases, the output signal across the capacitor is equipped with a "smaller amplitude." In contrast, all the signals in the steady state have the same "averaged amplitude." Why?

RC circuit: a square-wave input

$$\begin{cases} \varepsilon - \dfrac{Q(t)}{C} - R\dfrac{dQ(t)}{dt} = 0 & \text{(first half; charging)} \\ \dfrac{Q(t)}{C} - R\dfrac{dQ(t)}{dt} = 0 & \text{(second half; discharging)} \end{cases}$$

$$\Rightarrow \begin{cases} Q_{P,1} = C\varepsilon_0\left(1-e^{-T/2\tau}\right) \\ Q_{V,1} = Q_{P,1}e^{-T/2\tau} = C\varepsilon_0\left(1-e^{-T/2\tau}\right)e^{-T/2\tau} \end{cases}$$

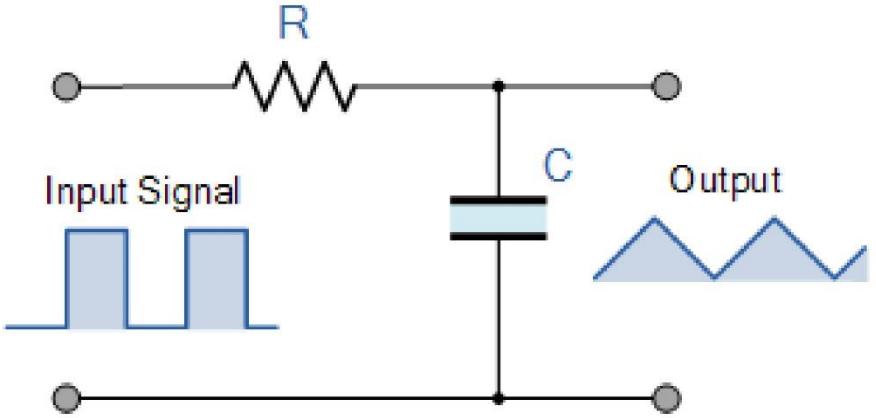
$$\Rightarrow \begin{pmatrix} Q_{P,N} \\ Q_{V,N} \end{pmatrix} \xrightarrow{N\gg 1} \frac{1}{1-e^{-T/\tau}}\begin{pmatrix} Q_{P,1} \\ Q_{V,1} \end{pmatrix} = \begin{pmatrix} \dfrac{C\varepsilon_0}{1+e^{-T/2\tau}} \\ \dfrac{C\varepsilon_0}{1+e^{T/2\tau}} \end{pmatrix}$$

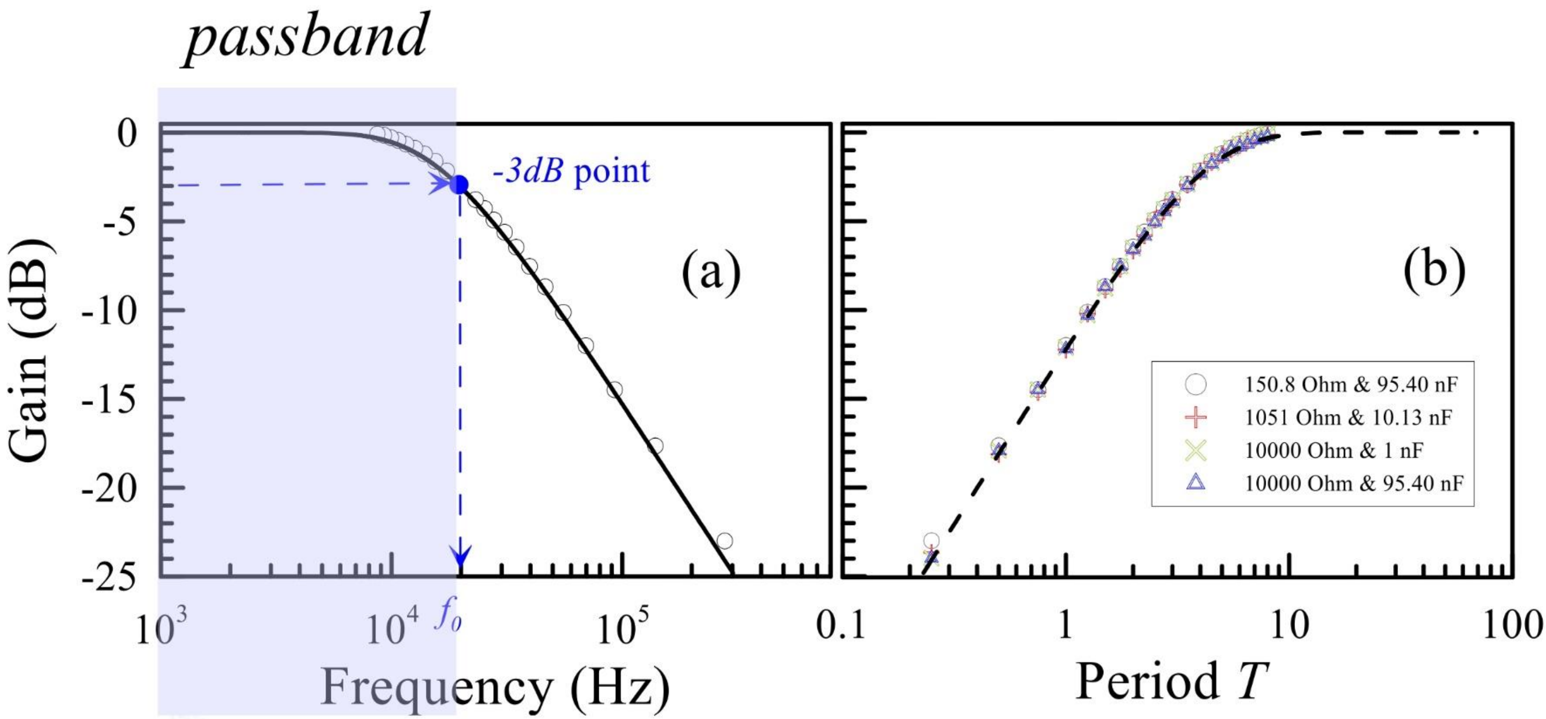


$$Q_{ave,N} = \frac{Q_{P,N}+Q_{V,N}}{2} \xrightarrow{N\gg 1} \frac{Q_{P,1}+Q_{V,1}}{2\left(1-e^{-T/\tau}\right)} = \frac{C\varepsilon_0}{2}\frac{\left(1-e^{-T/2\tau}\right)\left(1+e^{-T/2\tau}\right)}{1-e^{-T/\tau}} = \frac{C\varepsilon_0}{2}$$

$$Q_{diff,N} = Q_{P,N} - Q_{V,N} \xrightarrow{N\gg 1} \frac{Q_{P,1}-Q_{V,1}}{1-e^{-T/\tau}} = C\varepsilon_0\frac{1-e^{-T/2\tau}}{1+e^{-T/2\tau}} = C\varepsilon_0 \tanh\left(\frac{T}{4\tau}\right)$$

RC circuit: a square-wave input

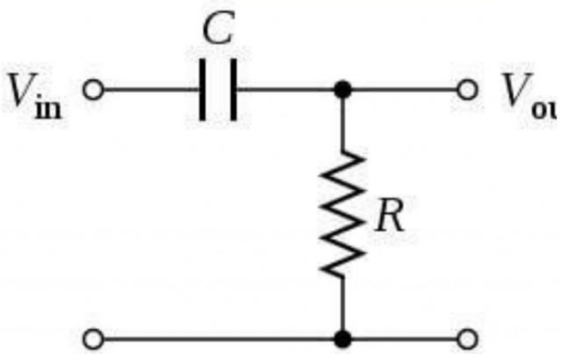


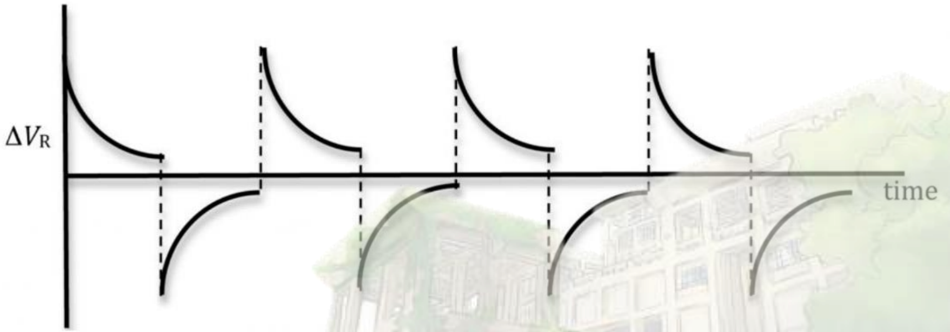


Gain(in dB): $-20\log\left(V_{diff}/\varepsilon\right)$

$$Q_{diff,N} \xrightarrow{N\gg 1} C\varepsilon_0 \tanh\left(\frac{T}{4\tau}\right)$$

Known as a lowpass filter

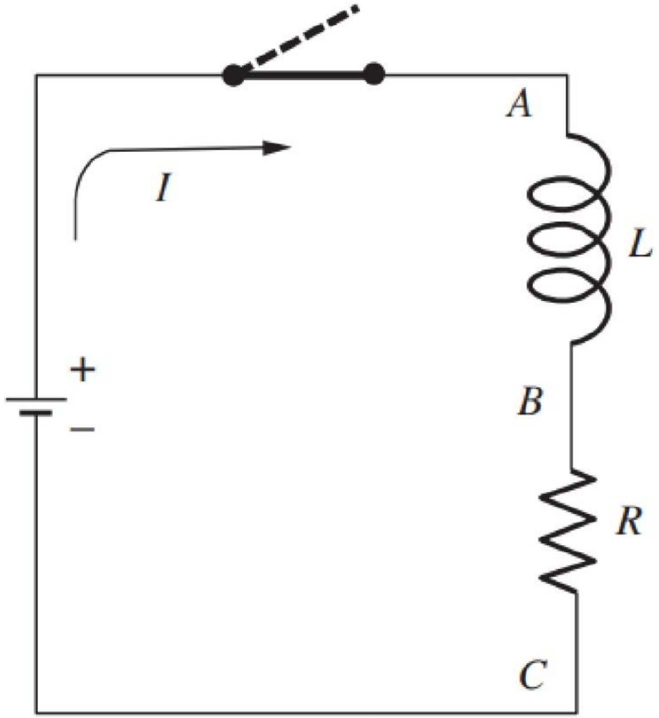




Highpass filter

 Email: yang168@gapp.nthu.edu.tw

RL circuit

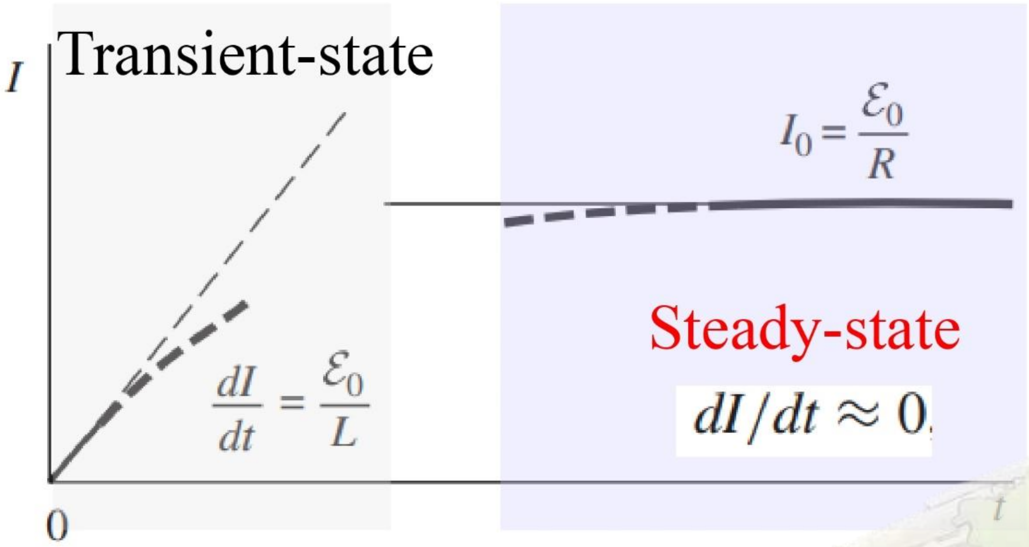


$$\boxed{\mathcal{E}_0 - L\frac{dI}{dt} = RI}$$

Summation of Electromotive force

$$\boxed{\mathcal{E}_0 = L\frac{dI}{dt} + RI}$$

Summation of Voltage difference



$$I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-(R/L)t}\right)$$

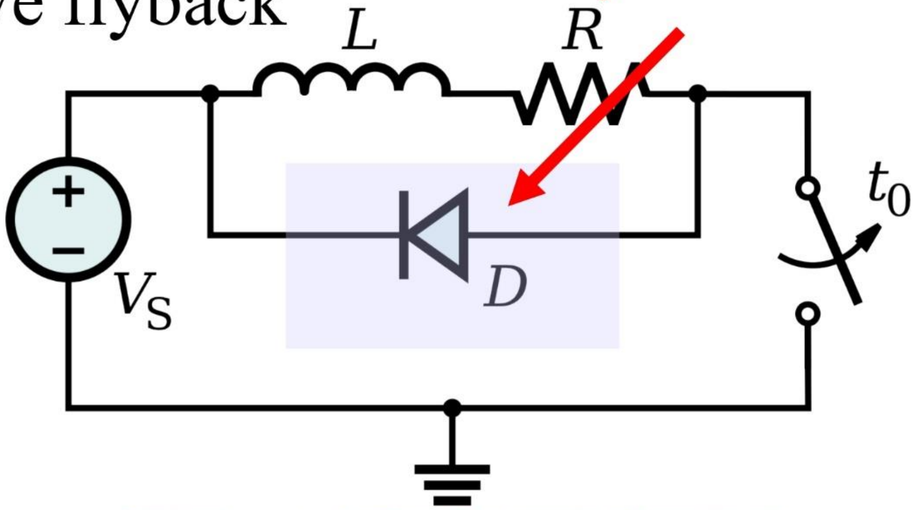
What happens if we open the switch after the current I_0 has been established, thus forcing the current to drop abruptly to zero?

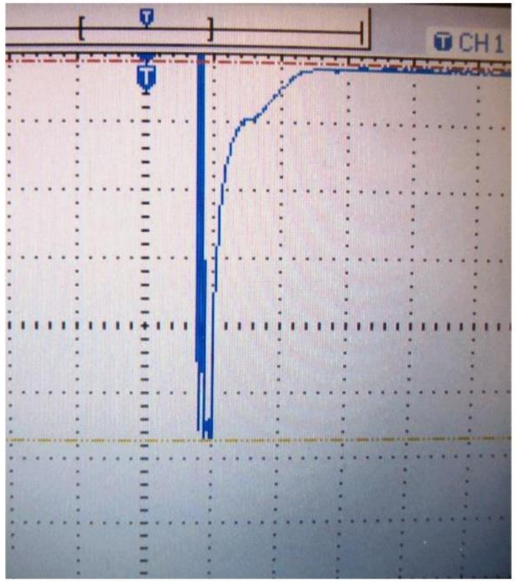
RL circuit

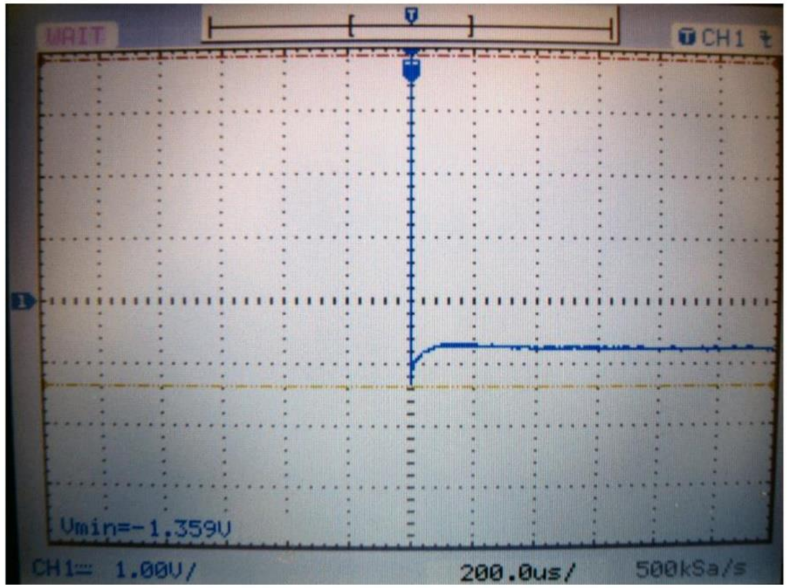
What happens if we open the switch after the current I_0 has been established, thus forcing the current to drop abruptly to zero?

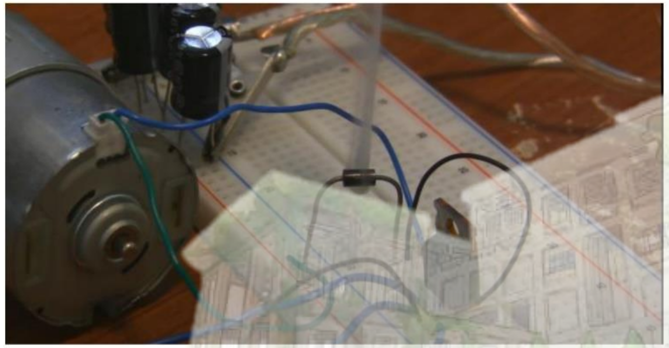
Inductive Spike, aka inductive flyback

Flyback diode

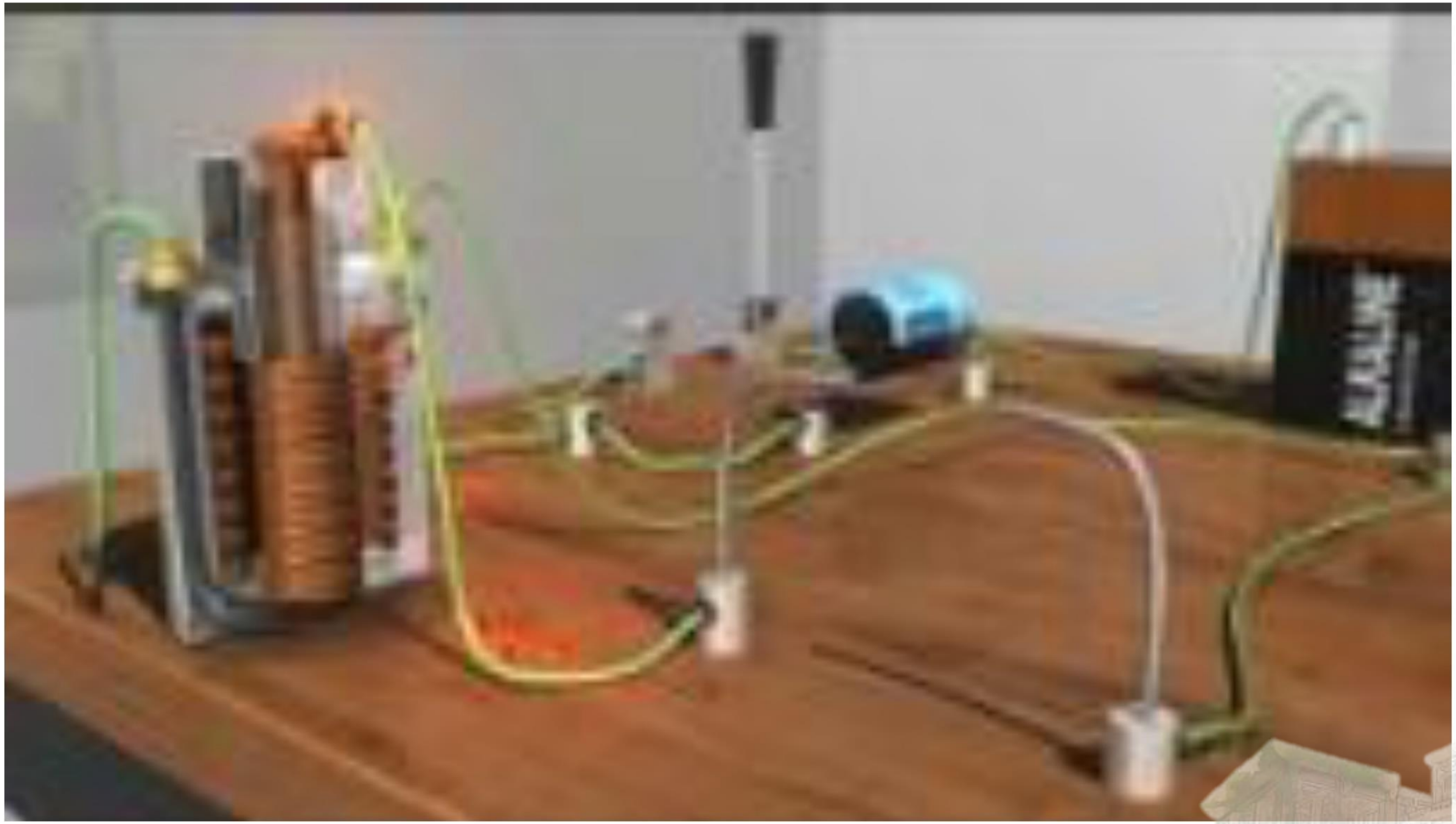




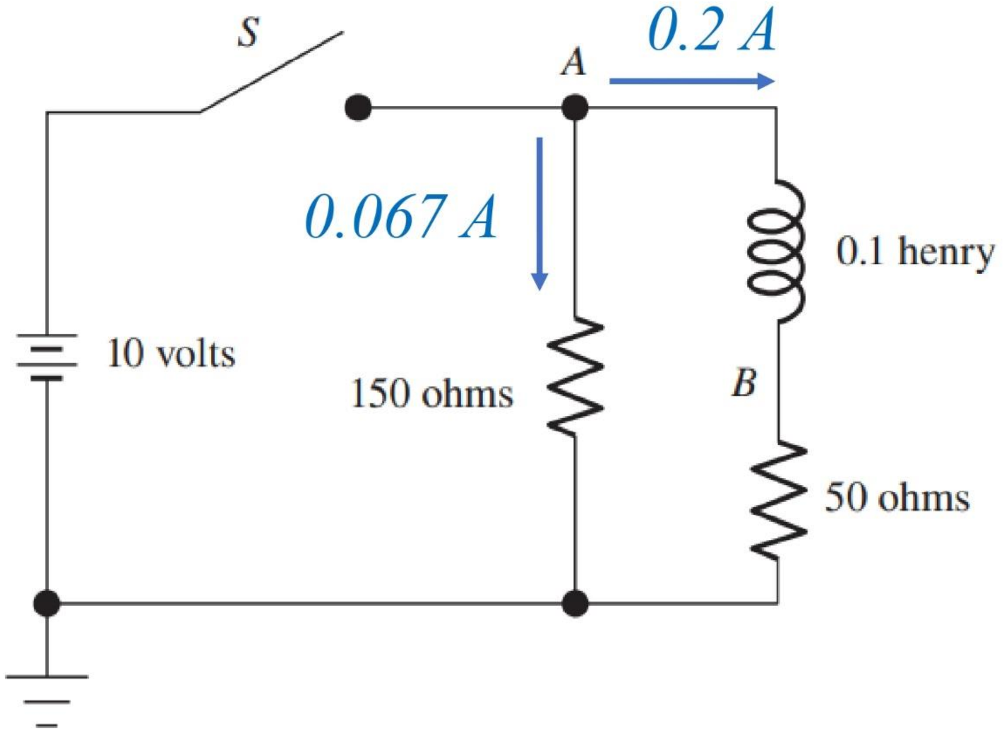


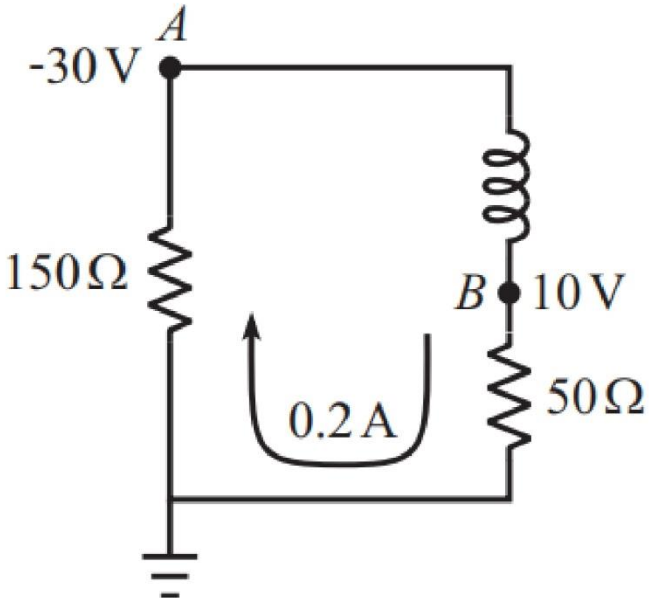


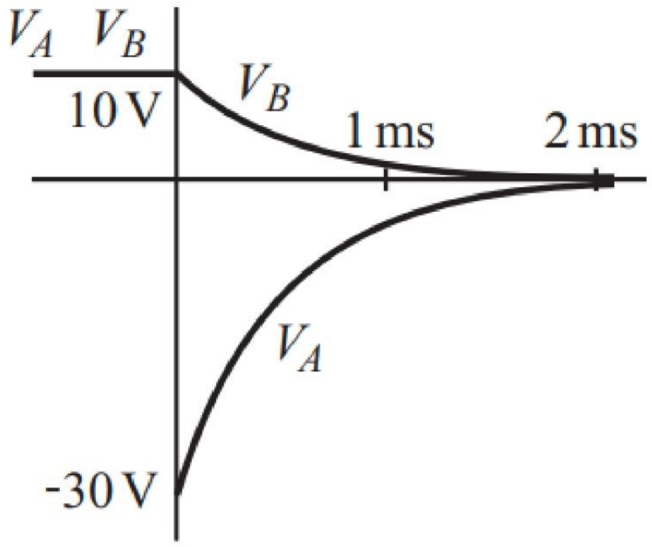
Ignition Coils



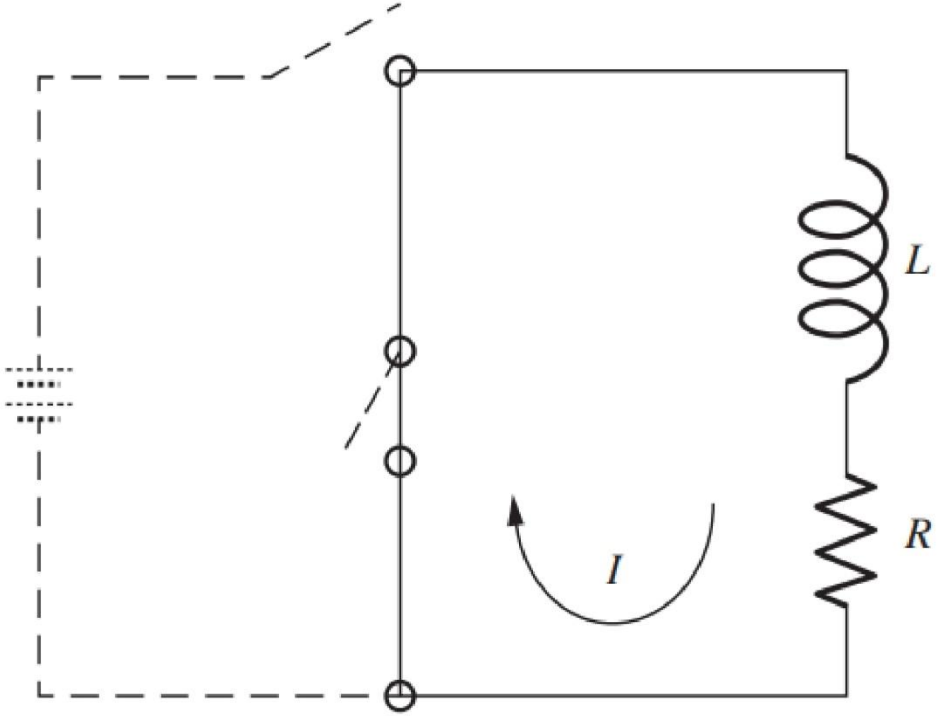
Open RL circuit

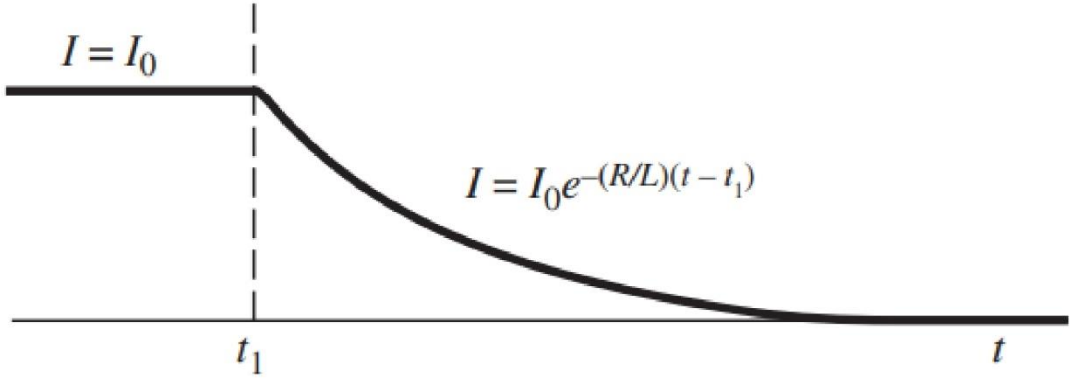






RL circuit





$$U = \int_{t_1}^{\infty} RI^2\, dt = \int_{t_1}^{\infty} RI_0^2 e^{-(2R/L)(t-t_1)}\, dt$$

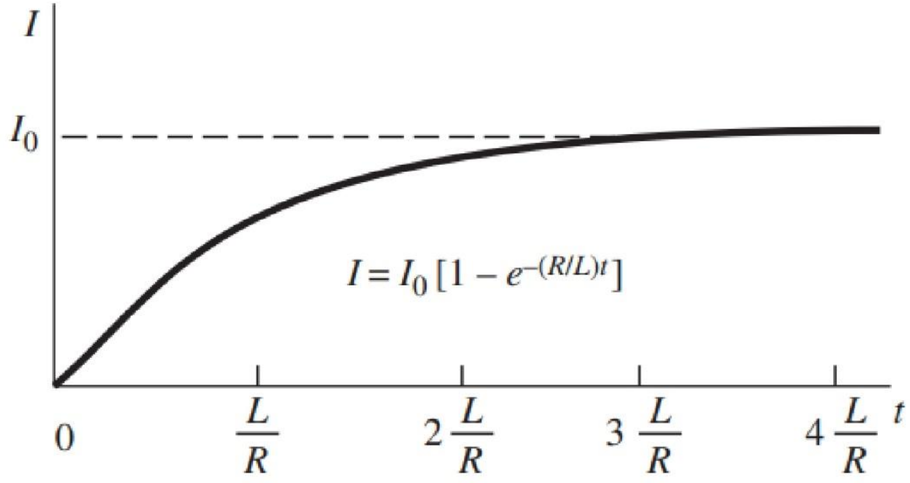
$$= -RI_0^2 \left(\frac{L}{2R}\right) e^{-(2R/L)(t-t_1)} \Big|_{t_1}^{\infty} = \frac{1}{2} L I_0^2.$$

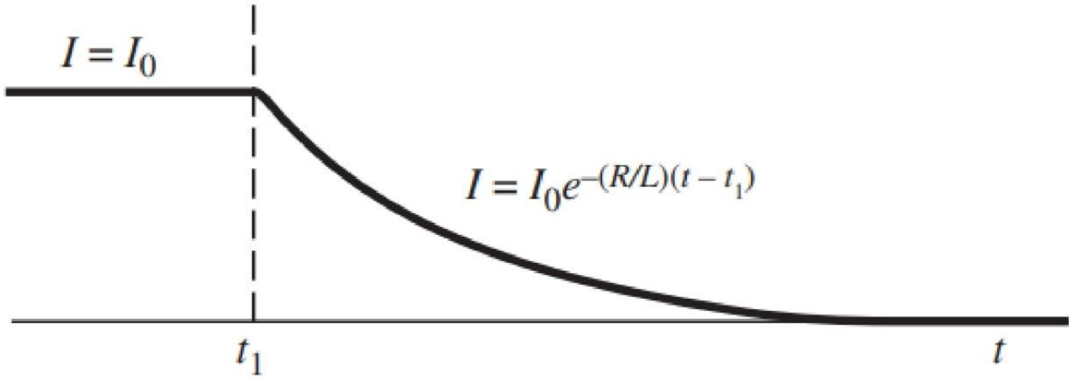
$$0 = L\frac{dI}{dt} + RI$$

Or by definition of emf

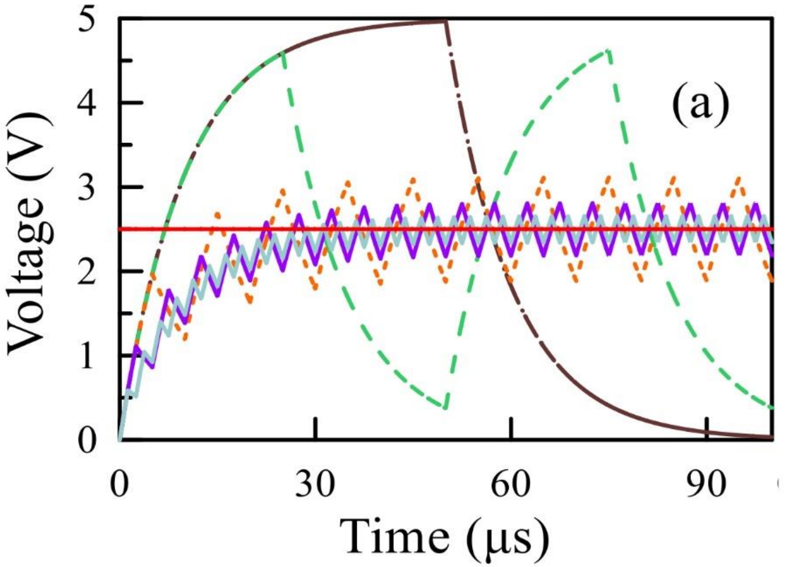
$$dW = L\frac{dI}{dt}(I\, dt) = LI\, dI = \frac{1}{2} L\, d(I^2)$$

RL circuit





Look similar to charging and discharging behavior



Can an RL circuit serve as a "filter" as an RC circuit does?

Yes. One may find it out by mimicking the procedure before. Or simply use *phasor analysis*.

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Total Charge of Induction

A circular coil of wire, with N turns of radius a, is located in the field of an electromagnet. The magnetic field is perpendicular to the coil (that is, parallel to the axis of the coil), and its strength has the constant value B_0 over that area. The coil is connected by a pair of twisted leads to an external resistance. The total resistance of this closed circuit, including that of the coil itself, is R. Suppose the electromagnet is turned off, its field dropping more or less rapidly to zero. The induced electromotive force causes current to flow around the circuit. Derive a formula for the total charge $Q = \int I\,dt$ that passes through the resistor, and explain why it does not depend on the rapidity with which the field drops to zero.

$$\Phi = N\pi a^2 B$$

$$\mathcal{E} = d\Phi/dt = N\pi a^2 (dB/dt)$$

B changes slowly
→ emf (thus I) smaller
But the process takes longer!

$$Q = \int I\,dt = \int \frac{N\pi a^2}{R}\frac{dB}{dt}\,dt = \frac{N\pi a^2}{R}\int_{B_0}^{0} dB = -\frac{N\pi a^2 B_0}{R}$$

Magnetic energy near a neutron star

It has been estimated that the magnetic field strength at the surface of a neutron star, or *pulsar,* may be as high as 10^{10} tesla. What is the energy density in such a field? Express it, using the mass–energy equivalence, in kilograms per m^3.

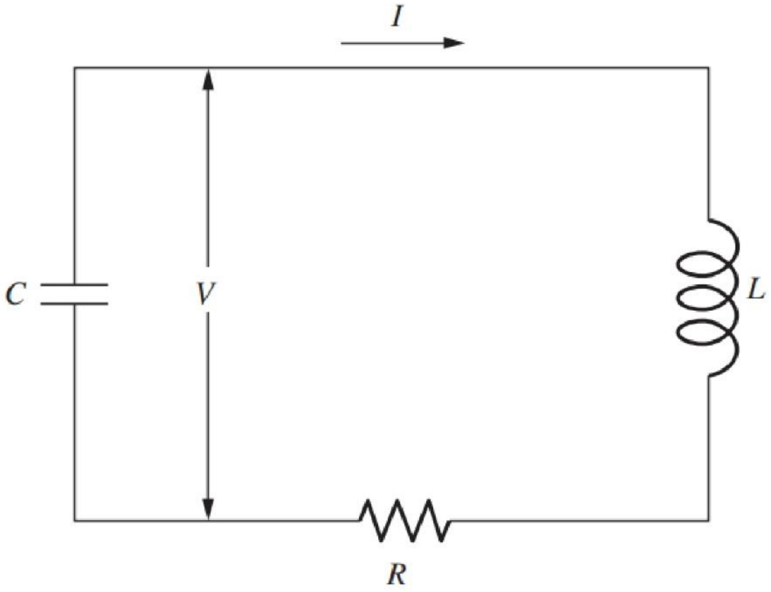
$$\frac{B^2}{2\mu_0} = \frac{(10^{10}\ \mathrm{T})^2}{2\left(4\pi \cdot 10^{-7}\ \frac{\mathrm{kg\,m}}{\mathrm{C}^2}\right)} = 4 \cdot 10^{25}\ \mathrm{J/m^3}.$$

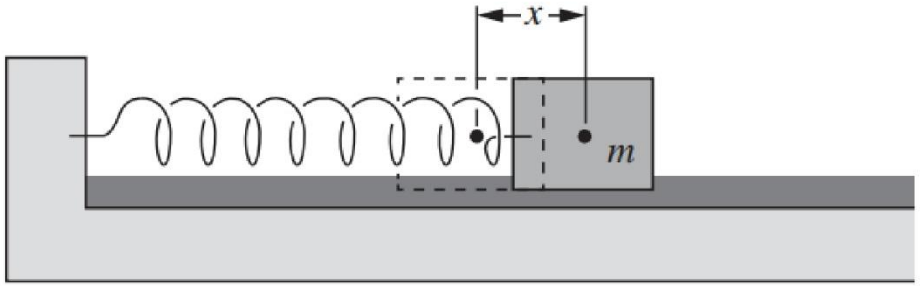
$$\rho = \frac{4 \cdot 10^{25}\ \mathrm{J/m^3}}{9 \cdot 10^{16}\ \mathrm{J/kg}} = 4.4 \cdot 10^{8}\ \mathrm{kg/m^3} = 4.4 \cdot 10^{5}\ \mathrm{g/cm^3}.$$

This is very large. By comparison, the mass density of water is $1\ \mathrm{g/cm^3}$.

Wikipedia: Neutron stars have overall densities of 3.7×10^{17} to 5.9×10^{17} $\mathrm{kg/m^3}$

AC RLC circuits: a quick review





$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0.$$

$$I=-\frac{dQ}{dt},\quad Q=CV,\quad V=L\frac{dI}{dt}+RI.$$

$$\frac{d^2V}{dt^2}+\left(\frac{R}{L}\right)\frac{dV}{dt}+\left(\frac{1}{LC}\right)V=0.$$

$$V(t)=Ae^{-\alpha t}\cos\omega t$$

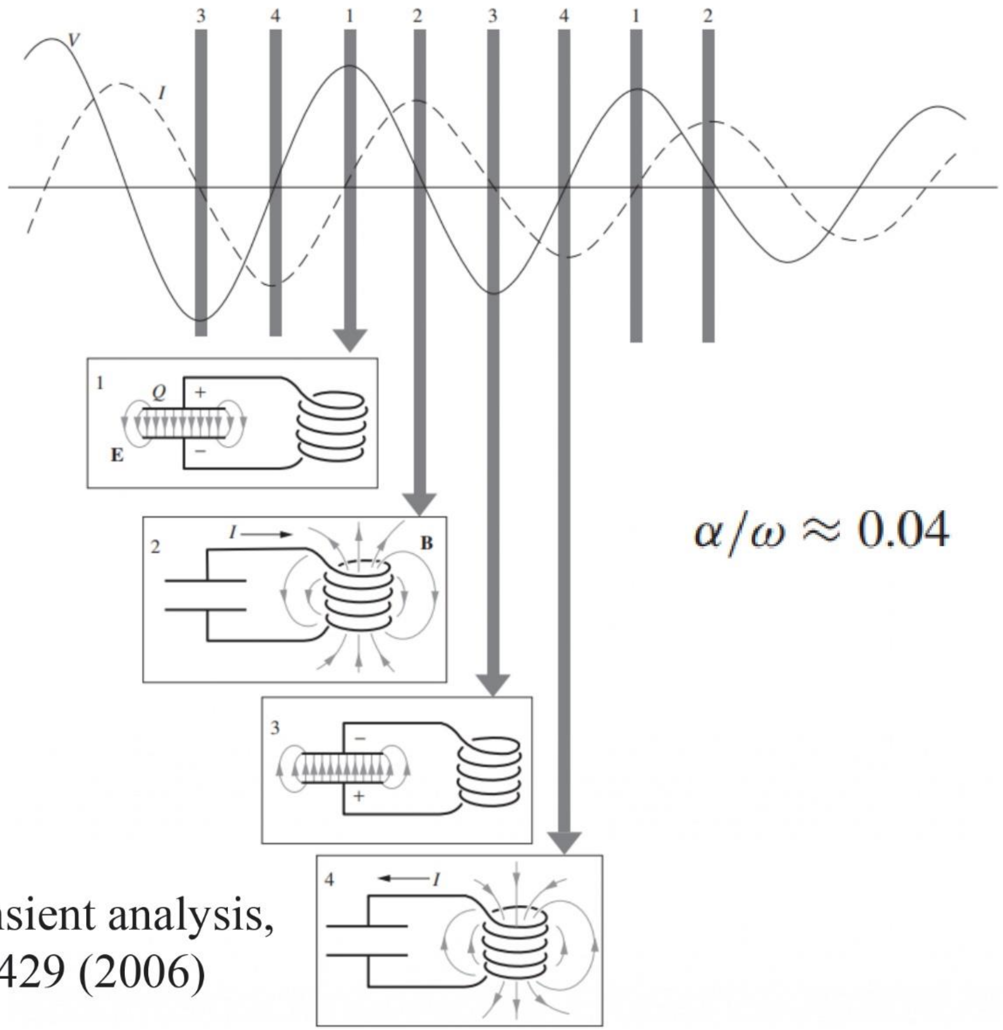
$$\frac{dV}{dt}=Ae^{-\alpha t}\big[-\alpha\cos\omega t-\omega\sin\omega t\big],$$

$$\frac{d^2V}{dt^2}=Ae^{-\alpha t}\big[(\alpha^2-\omega^2)\cos\omega t+2\alpha\omega\sin\omega t\big]$$

$$2\alpha\omega-\frac{R\omega}{L}=0\quad\text{and}\quad\alpha^2-\omega^2-\alpha\frac{R}{L}+\frac{1}{LC}=0.$$

$$\boxed{\alpha=\frac{R}{2L}}\qquad\boxed{\omega^2=\frac{1}{LC}-\frac{R^2}{4L^2}}$$

AC RLC circuits: a quick review



For detailed transient analysis, see *AJP* **74**, 429 (2006)

$$V(t) = Ae^{-\alpha t}\cos\omega t$$

$$I(t) = -C\frac{dV}{dt} = AC\omega\left(\sin\omega t + \frac{\alpha}{\omega}\cos\omega t\right)e^{-\alpha t}$$

The ratio α/ω is a measure of the damping.

$\alpha/\omega \approx 0.04$

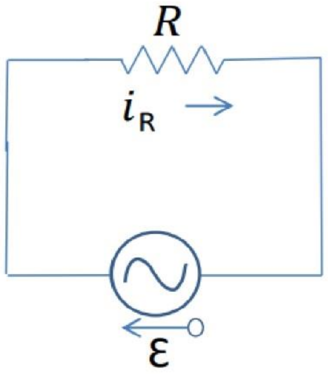
1: energy in E-field (capacitor)
1~2 :energy discharged from capacitor
2: energy in B-field (inductor)

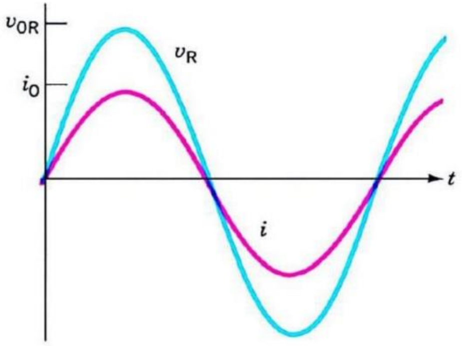
R is taking its toll, and as the oscillation goes on, the energy remaining in the fields gradually diminishes

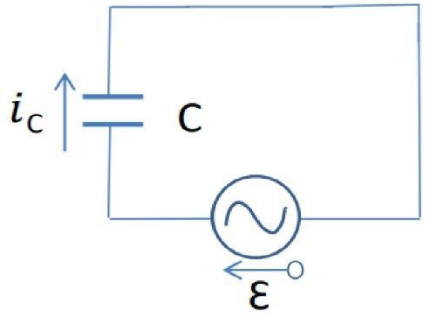
AC RLC circuits: a quick review

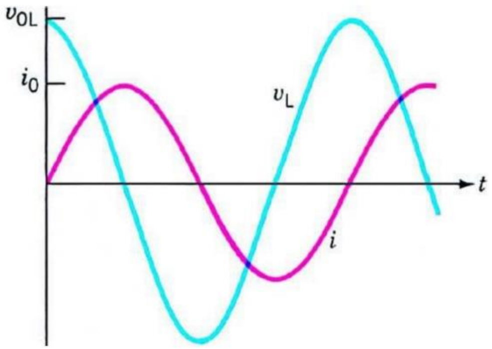
$$\varepsilon = \varepsilon_0 \sin \omega t$$

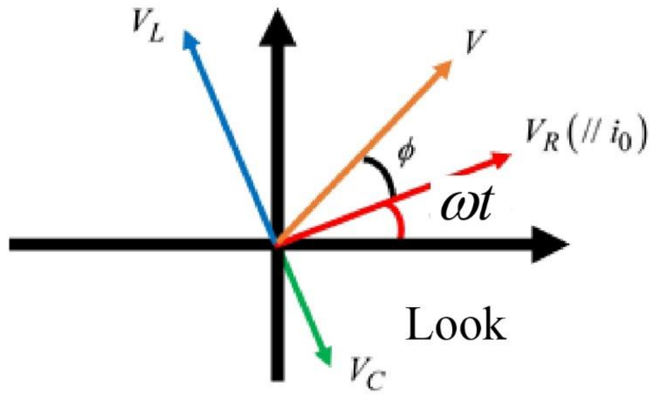
$$i = i_0 \sin \omega t$$







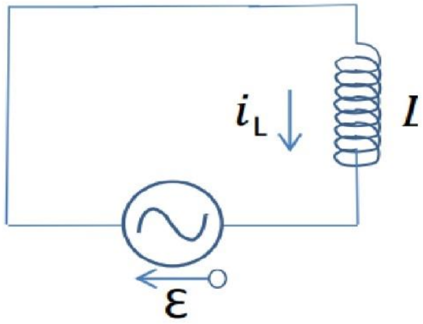


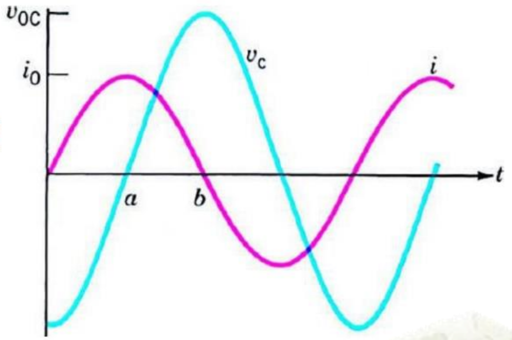


$$v_R = i_R R = i_0 R \sin \omega t \equiv V_R \sin \omega t$$

$$v_L = L\frac{di}{dt} = L i_0 \omega \cos \omega t \equiv i_0 X_L \sin\left(\omega t + \frac{\pi}{2}\right) = V_L \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$v_C = \frac{q}{C} = -\frac{i_0}{\omega C}\cos \omega t \equiv i_0 X_C \sin\left(\omega t - \frac{\pi}{2}\right) = V_C \sin\left(\omega t - \frac{\pi}{2}\right)$$

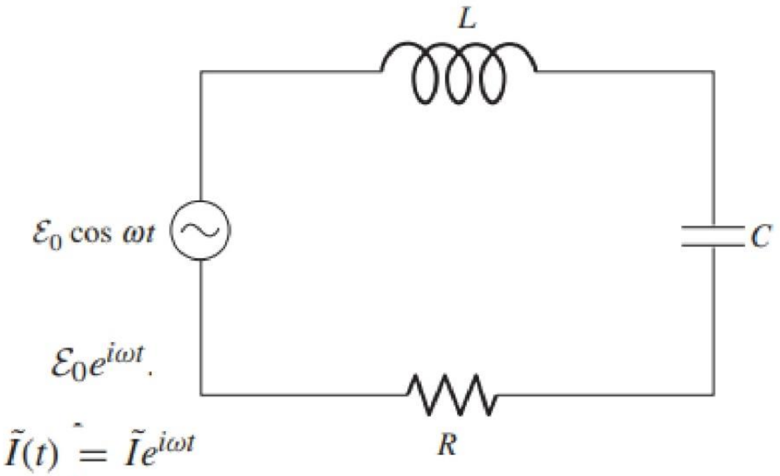




$$\phi = \tan^{-1}\frac{\omega L - 1/\omega C}{R}$$

$$V_0 = i_0\sqrt{R^2 + \left(X_L - X_C\right)^2} \equiv i_0 Z$$

AC RLC circuits: a quick review



$\mathcal{E}_0 e^{i\omega t}$.

$\tilde{I}(t) = \tilde{I} e^{i\omega t}$

$$L\frac{dI(t)}{dt} + RI(t) + \frac{Q(t)}{C} = \mathcal{E}_0 \cos \omega t.$$

$$L\frac{d\tilde{I}(t)}{dt} + R\tilde{I}(t) + \frac{\tilde{Q}(t)}{C} = \mathcal{E}_0 e^{i\omega t}.$$

$$L\frac{d}{dt}\mathrm{Re}[\tilde{I}(t)] + R\,\mathrm{Re}[\tilde{I}(t)] + \frac{1}{C}\int \mathrm{Re}[\tilde{I}(t)]\,dt = \mathcal{E}_0 \cos \omega t.$$

If our differential equation were modified to contain a term that wasn't linear in $I(t)$, for example $RI(t)^2$, then this method wouldn't work, because $\mathrm{Re}[\tilde{I}(t)^2]$ is *not* equal to $\left(\mathrm{Re}[\tilde{I}(t)]\right)^2$

$$Li\omega\tilde{I}e^{i\omega t} + R\tilde{I}e^{i\omega t} + \frac{\tilde{I}e^{i\omega t}}{i\omega C} = \mathcal{E}_0 e^{i\omega t}.$$

$$\tilde{I} = \frac{\mathcal{E}_0}{i\omega L + R + 1/i\omega C} = \frac{\mathcal{E}_0\left[R - i(\omega L - 1/\omega C)\right]}{R^2 + (\omega L - 1/\omega C)^2}.$$

$$\tilde{I} = \frac{\mathcal{E}_0}{R^2 + (\omega L - 1/\omega C)^2} \cdot \sqrt{R^2 + (\omega L - 1/\omega C)^2}\, e^{i\phi}$$

$$= \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}\, e^{i\phi} \equiv I_0 e^{i\phi},$$

AC RLC circuits: a quick review

$$
\begin{tikzcd}
\boxed{\tilde{I}(t)} \arrow[d, "\text{Real part}"'] \arrow[dr, "\text{Magnitude}"] & \boxed{\tilde{I}} \arrow[l, "\text{Times } e^{i\omega t}"'] \arrow[d, "\text{Magnitude}"] \\
\boxed{I(t)} \arrow[r, "\text{Amplitude}"'] & \boxed{I_0}
\end{tikzcd}
$$

$$
\begin{aligned}
\tilde{I} &= \frac{\mathcal{E}_0}{R^2 + (\omega L - 1/\omega C)^2} \cdot \sqrt{R^2 + (\omega L - 1/\omega C)^2}\, e^{i\phi} \\
&= \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}\, e^{i\phi} \equiv I_0 e^{i\phi},
\end{aligned}
$$

where

$$
I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \quad \text{and} \quad \tan\phi = \frac{1}{R\omega C} - \frac{\omega L}{R}.
$$

$$
\begin{aligned}
I(t) = \mathrm{Re}\big[\tilde{I} e^{i\omega t}\big] = \mathrm{Re}\big[I_0 e^{i\phi} e^{i\omega t}\big] &= I_0 \cos(\omega t + \phi) \\
&= \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \cos(\omega t + \phi),
\end{aligned}
$$