Chapter 7: Electrodynamics

7.1 Electromotive Force 7.1.1 Ohm's Law

Pushing on the charges makes a current flow. How fast the charges move depends on the nature of the materials and the forces.

current density $\longrightarrow \mathbf{J} = \rho\mathbf{v} \longleftarrow$ velocity of the charge

$\uparrow$

volume charge density

Ohm's law (an empirical equation): $\mathbf{J} = \sigma\mathbf{E} = \dfrac{1}{\rho}\mathbf{E}$

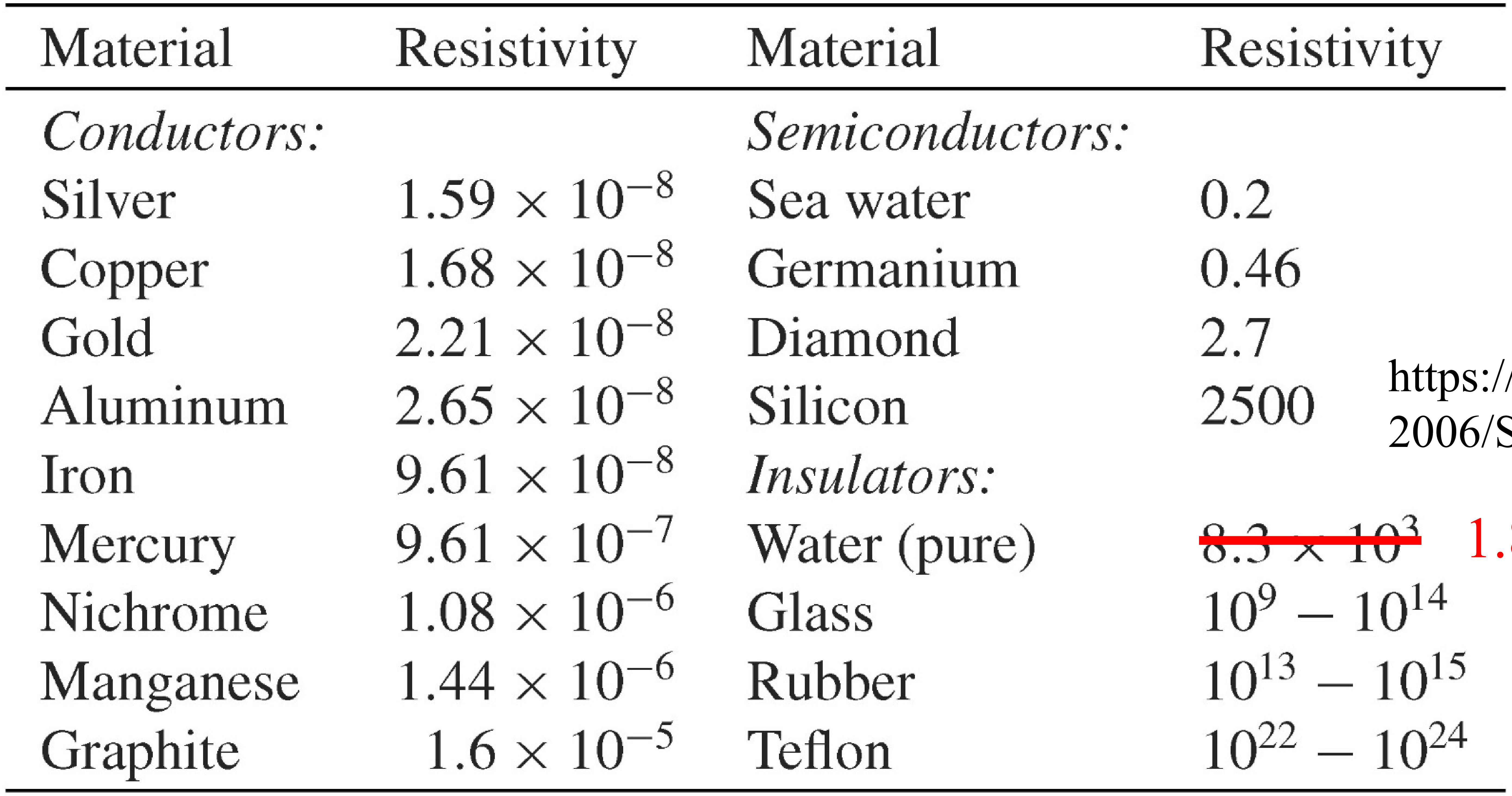
$\uparrow$ conductivity $\nwarrow$ resistivity

ρ: volume charge density or resistivity?
σ: surface charge density or conductivity?

The Lorentz force drives the charges to produce current:

$$\mathbf{J} \propto \mathbf{F} \;\Rightarrow\; \mathbf{J} = \sigma(\mathbf{E} + \cancelto{\approx 0}{\mathbf{v}\times\mathbf{B}})$$

Resistivities (ohm-meters)



Material	Resistivity	Material	Resistivity
Conductors:		*Semiconductors:*	
Silver	1.59×10^{-8}	Sea water	0.2
Copper	1.68×10^{-8}	Germanium	0.46
Gold	2.21×10^{-8}	Diamond	2.7
Aluminum	2.65×10^{-8}	Silicon	2500
Iron	9.61×10^{-8}	*Insulators:*	
Mercury	9.61×10^{-7}	Water (pure)	~~8.3×10^{3}~~ 1.8×10^{5}
Nichrome	1.08×10^{-6}	Glass	$10^{9} - 10^{14}$
Manganese	1.44×10^{-6}	Rubber	$10^{13} - 10^{15}$
Graphite	1.6×10^{-5}	Teflon	$10^{22} - 10^{24}$

https://hypertextbook.com/facts/2006/SamTetruashvili.shtml

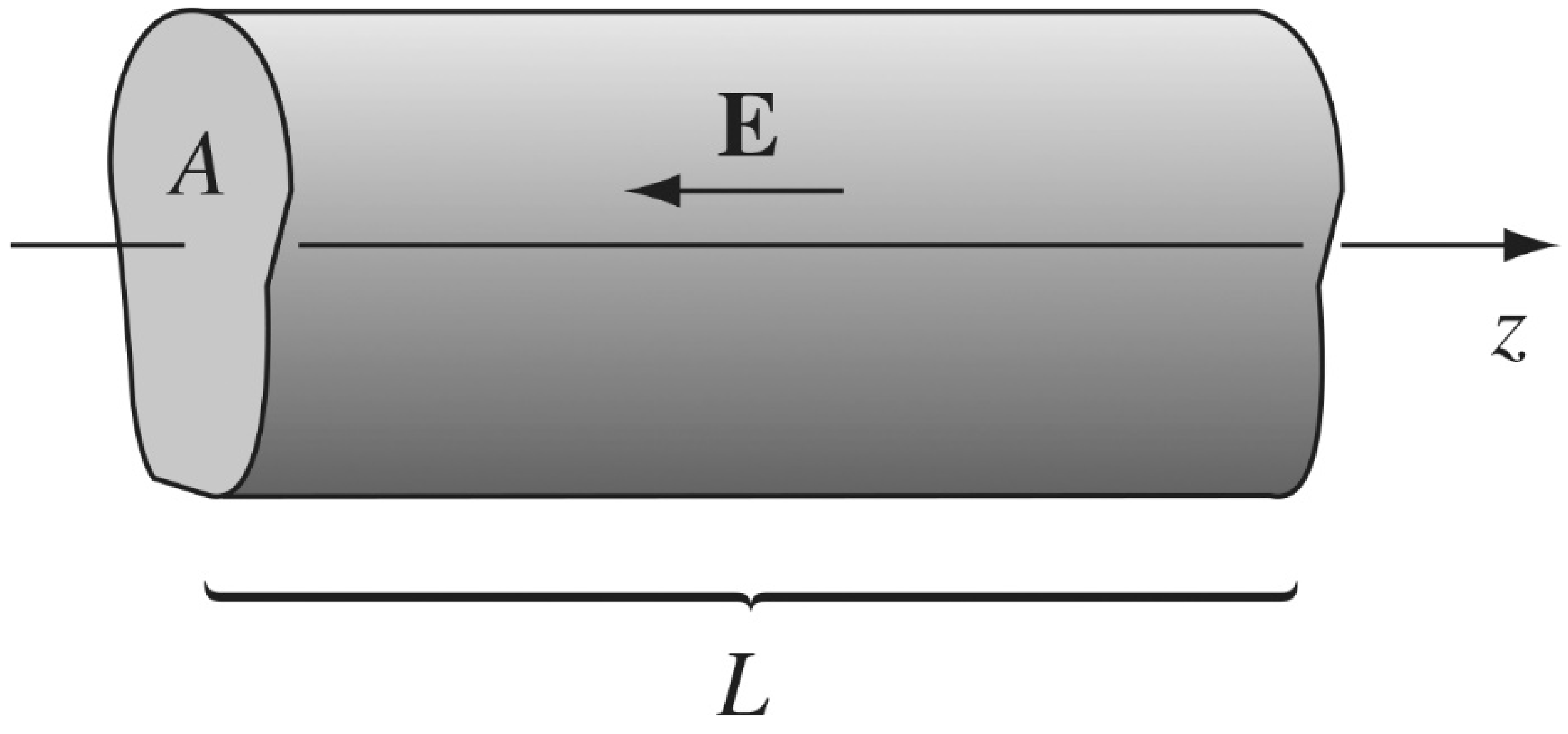
Confusion 1: $\mathbf{E} = 0$ inside a conductor ➔ $\mathbf{J} = 0$?

2: For a perfect conductor $\sigma = \infty$ ➔ $\mathbf{E} = 0$?

Question: Can we treat the connecting wires in electric circuits as equal potentials?

Example 7.1

A cylindrical resistor of cross-sectional area A and length L is made from material with conductivity σ. If the potential is constant over each end, and the potential difference between the ends is V, what current flows?



Sol:

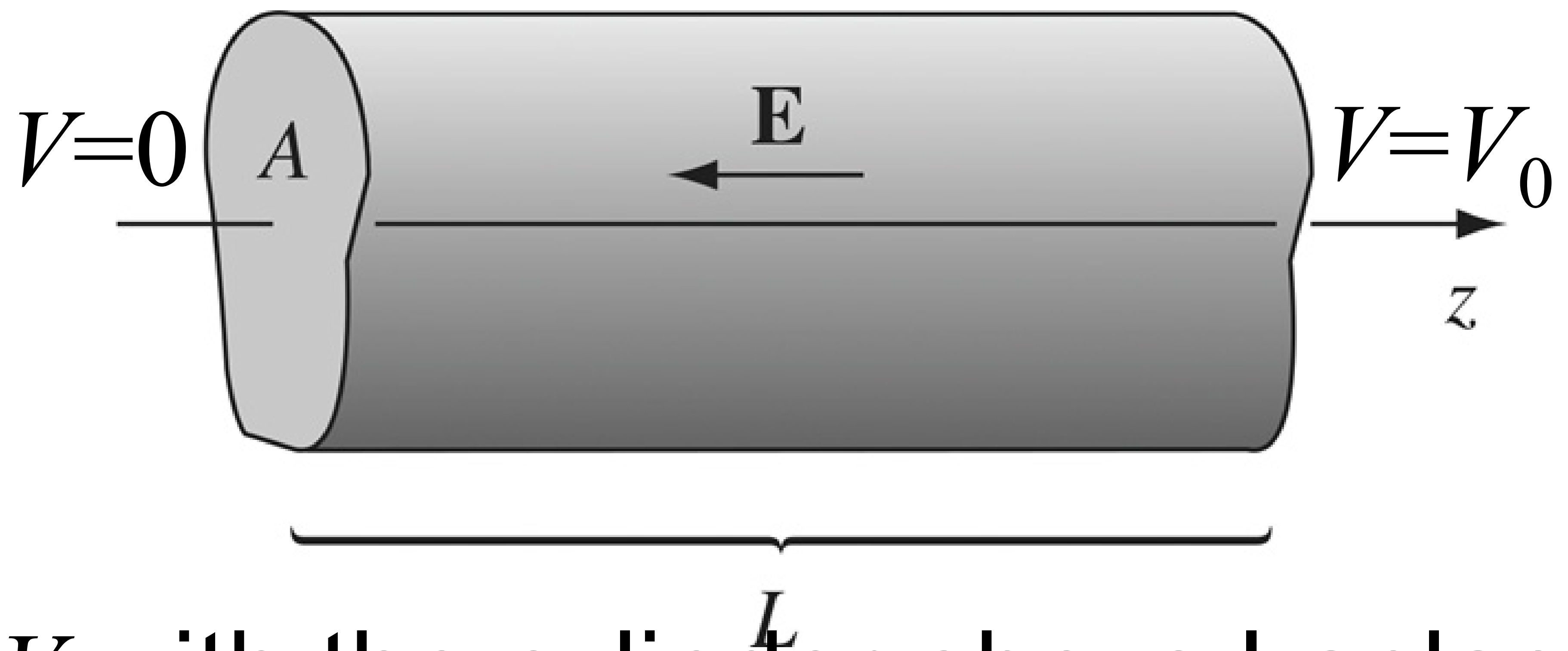
$$I = \mathbf{J} \cdot \mathbf{A} = \sigma E A = \frac{\sigma A}{L} V$$

Question: Is the electric field uniform within the wire?

To be proved in a moment, see Ex. 7.3.

Example 7.3

Prove that the electric field within the wire is uniform.



Sol:

The potential V with the cylinder obeys Laplace's equation.

On the cylinder surface $\mathbf{J}\cdot\mathbf{n}=0 \quad \therefore \mathbf{E}\cdot\mathbf{n}=0,$ and hence $\partial V/\partial n=0$

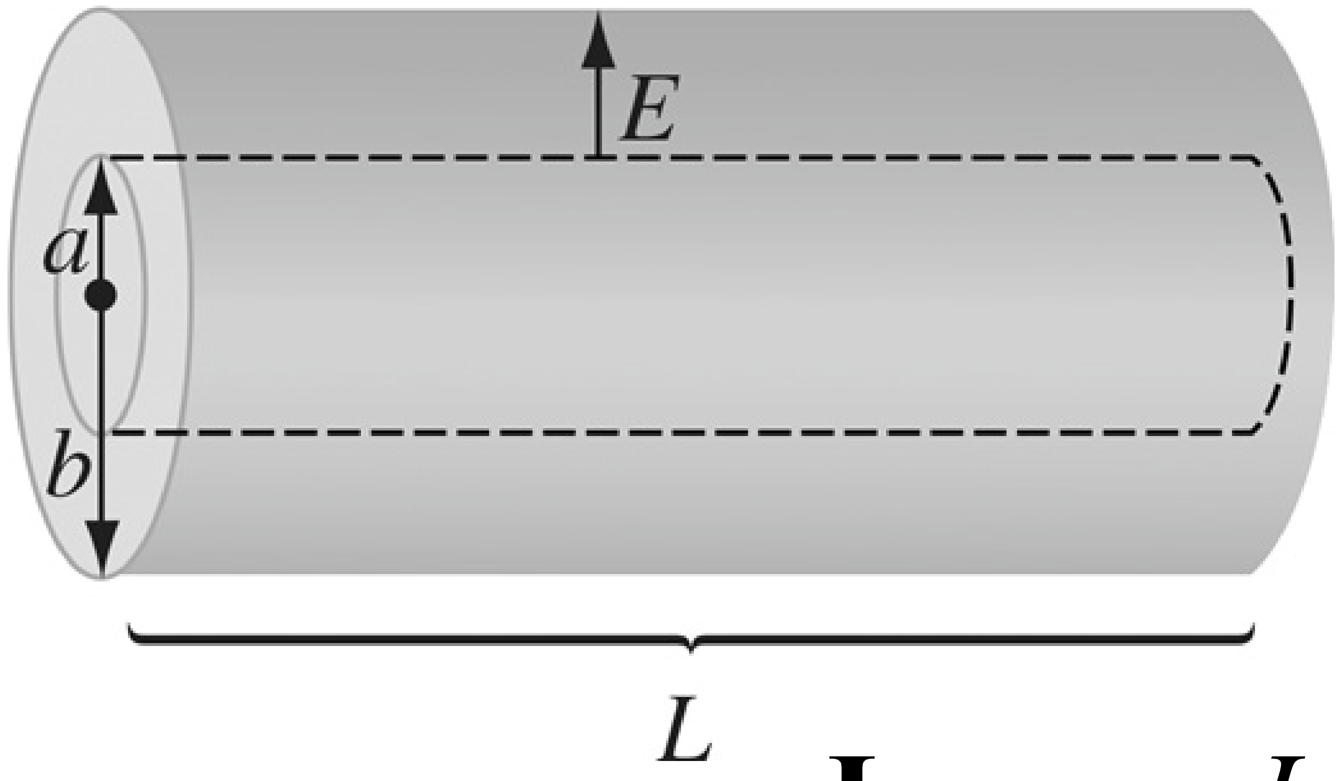
With V or its normal derivate specified on all the surfaces, the potential is uniquely determine (Prob. 3.4).

Guess: A potential obeys Laplace's equation and fits the boundary conditions.

$$V(z)=\frac{V_0 z}{L} \text{ and } \mathbf{E}=-\nabla V=-\frac{V_0}{L}\hat{\mathbf{z}} \longleftarrow$$ the unique solution.

Example 7.2

Two long cylinders (radii a and b) are separated by material of conductivity σ. If they are maintained at a potential different V, what current flows from one to the other, in a length L?



Sol:

$$\mathbf{J} = \frac{\mathbf{I}}{A} = \sigma \mathbf{E} \qquad \mathbf{E} = \frac{\mathbf{I}}{\sigma A} = \frac{I}{\sigma 2\pi s L} \hat{\mathbf{r}}$$

$$V = \int_a^b \mathbf{E} \cdot d\mathbf{s} = \int_a^b \frac{I}{2\pi\sigma L} \frac{1}{s} ds = \frac{I}{2\pi\sigma L} \ln(b/a)$$

$$I = \frac{2\pi\sigma L}{\ln(b/a)} V$$

Ohm's Law

Ex. 7.1 $V = \frac{L}{\sigma A} I$

Ex. 7.2 $V = \frac{\ln(b/a)}{2\pi\sigma L} I$

$$\Longrightarrow \quad V = I\underset{\substack{\uparrow \\ \text{resistance}}}{R}$$

(A more familiar version of Ohm's law.)

The total current flowing from one electrode to the other is proportional to the potential difference between them.

Resistance is measured in ohms (Ω): an ohm is a volt per ampere.

For a steady current and uniform conductivity,

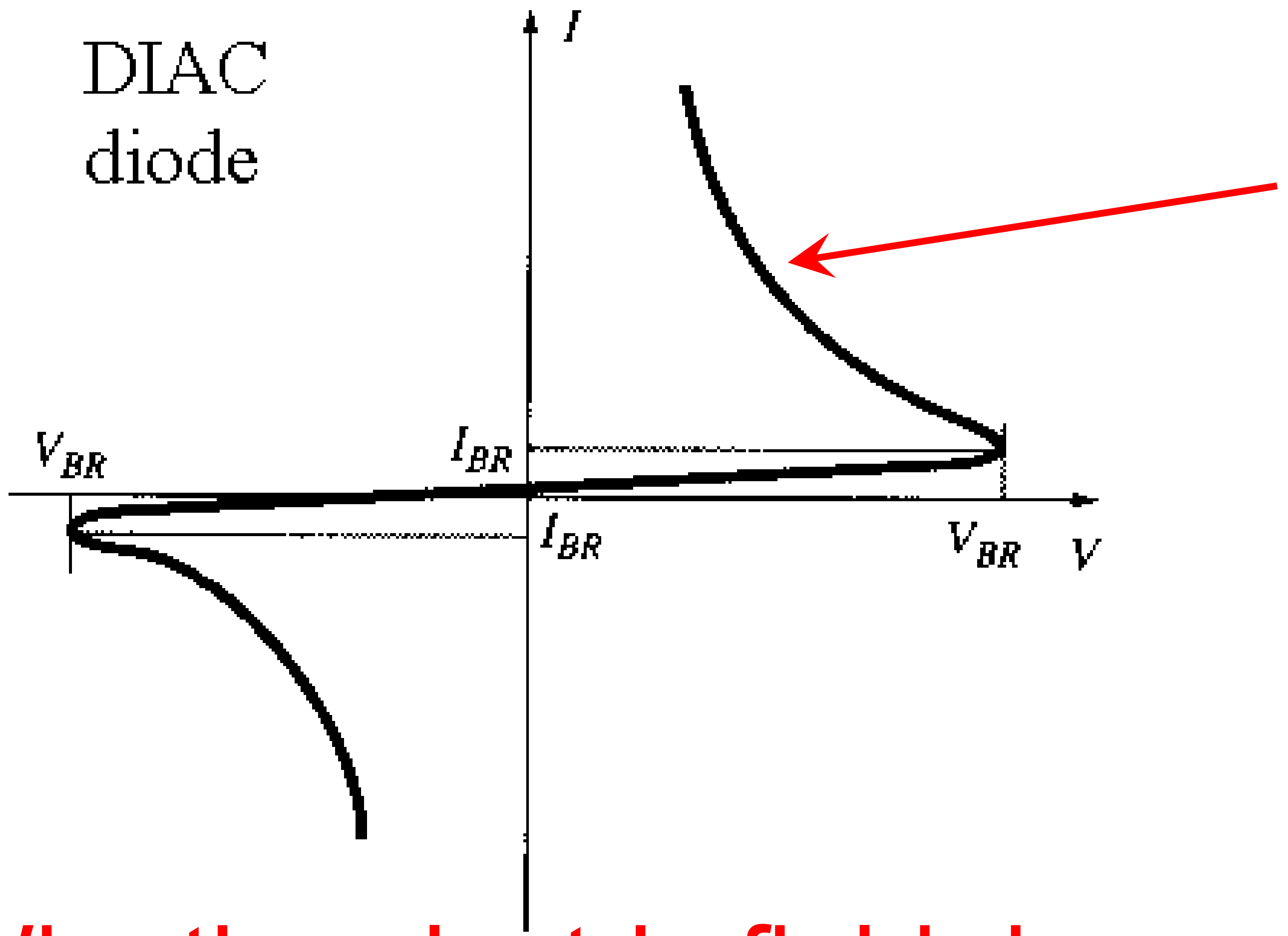
$$\rho = \varepsilon_0 \nabla \cdot \mathbf{E} = \varepsilon_0 \nabla \cdot (\frac{\mathbf{J}}{\sigma}) = \frac{\varepsilon_0}{\sigma} \nabla \cdot \mathbf{J} = 0$$

Any unbalanced charge resides on the surface.

Ohm’s Law (rule of thumb)

Gauss’s law or Ampere’s law is really a true law, but Ohm’s law is an empirical equation.

* Finding an exception won’t win a Nobel prize.



negative resistance

Q1: Why the electric field does not accelerate the charge particle to a very high speed?

Q2: Ohm’s law implies that a constant field produces a constant current, which suggests a constant velocity. Isn’t that a contradiction of Newton’s law.

Ohm’s Law (a naive picture)

A naive picture: Electrons are frequently collided with ions which slow down the acceleration.

Mean free path

$$\lambda = \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2\lambda}{a}}, \text{ where } a = \frac{qE}{m}$$

$$\text{average velocity: } v_{\text{ave}} = \frac{1}{2}at = \sqrt{\frac{\lambda qE}{2m}} \propto \sqrt{E}$$

The velocity is proportional to the square root of the field. That is no good!

Q1: How to explain it correctly?

The charges in practice are already moving quite fast because of their thermal energy.

Ohm’s Law (Drude model)

The net *drift velocity* is a tiny extra bit. The time between collisions is actually much shorter than we supposed.

$$\text{collision time: } t = \frac{\lambda}{v_{\text{thermal}}}$$

$$\text{average velocity: } \mathbf{v}_{\text{ave}} = \frac{1}{2}\mathbf{a}t = \frac{\mathbf{a}\lambda}{2v_{\text{thermal}}}$$

$$\text{acceleration: } \mathbf{a} = \frac{\mathbf{F}}{m} = \frac{q}{m}\mathbf{E}$$

(f : free electrons per molecule)

$$\mathbf{J} = n(fq)\mathbf{v}_{\text{ave}} = nfq\frac{\lambda}{2v_{\text{thermal}}}\frac{q}{m}\mathbf{E} = \left(\frac{nf\lambda q^2}{2mv_{\text{thermal}}}\right)\mathbf{E}$$

(n : molecules per unit volume)

The Joule Heating Law

$$\mathbf{J} = (\frac{nf\lambda q^2}{2mv_{\text{thermal}}})\mathbf{E} = \sigma\mathbf{E}, \quad \text{where } \sigma = \frac{nf\lambda q^2}{2mv_{\text{thermal}}}$$

This equation correctly predicts that conductivity is proportional to the density of the moving charges and *ordinarily* decreases with increasing temperature.

The Joule heating law:

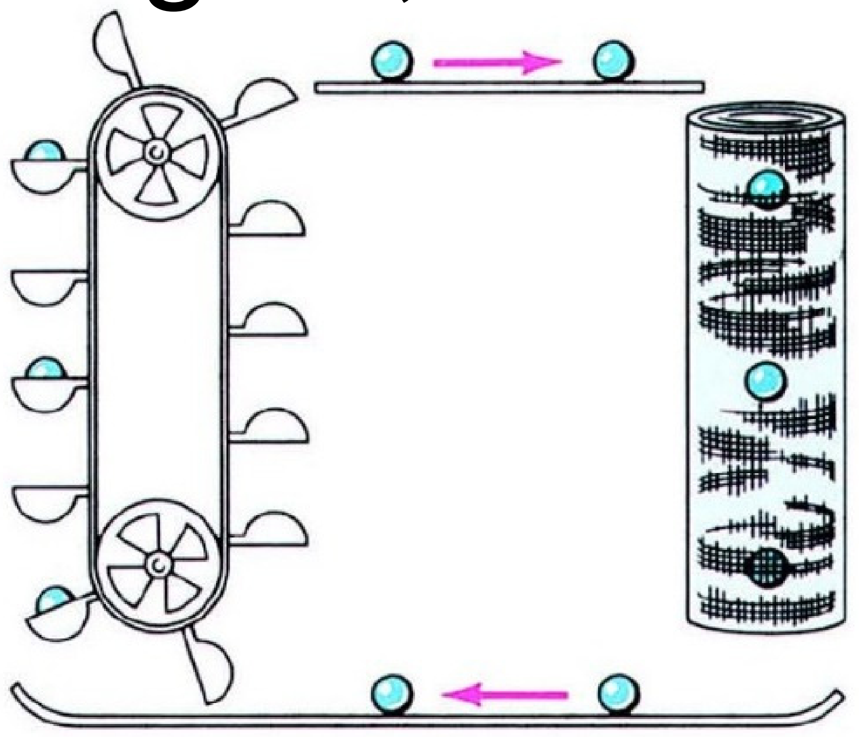
$$P = IV = I^2R = \frac{V^2}{R} \qquad \text{where } \begin{cases} I : \text{amperes} \\ R : \text{ohms} \\ V : \text{volts} \\ P : \text{watts} \end{cases}$$

7.1.2 Electromotive Force (emf)

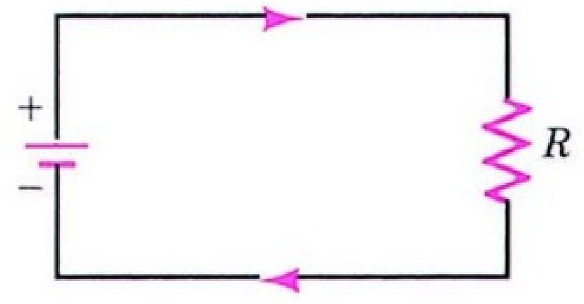
An emf is the work per unit charge done by the source of emf in moving the charge around *a closed loop*.

$$\mathcal{E} = \frac{W_{\text{ne}}}{q}$$

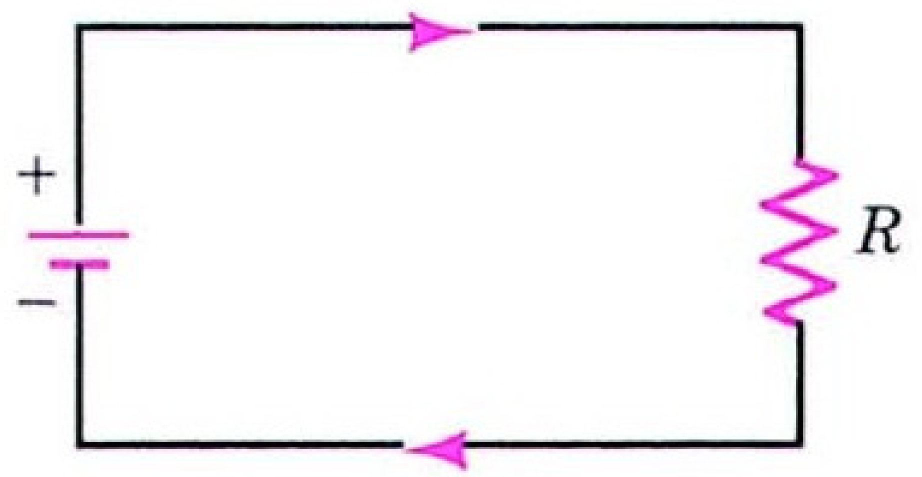
The subscript ”ne” emphasizes that the work is done by some nonelectrostatic agent, such as a battery or an electrical generator.



(a)



(b)

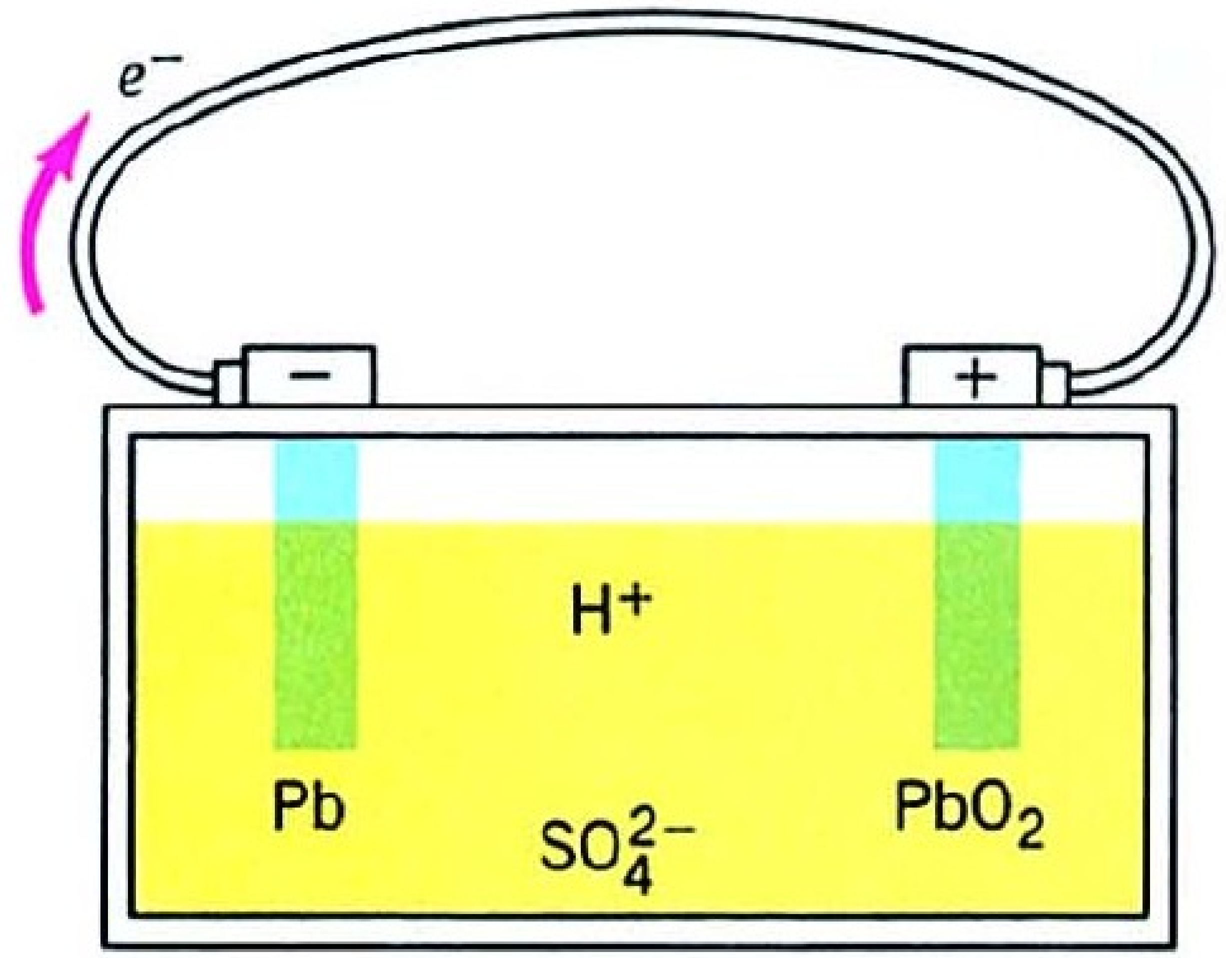


(b)

What is the difference between emf and potential difference?

Electromotive Force: Production of a Current

What is the function of the acid solution in the voltaic pile?



$$Pb + SO_4^{2-} \rightarrow PbSO_4 + 2e^-$$

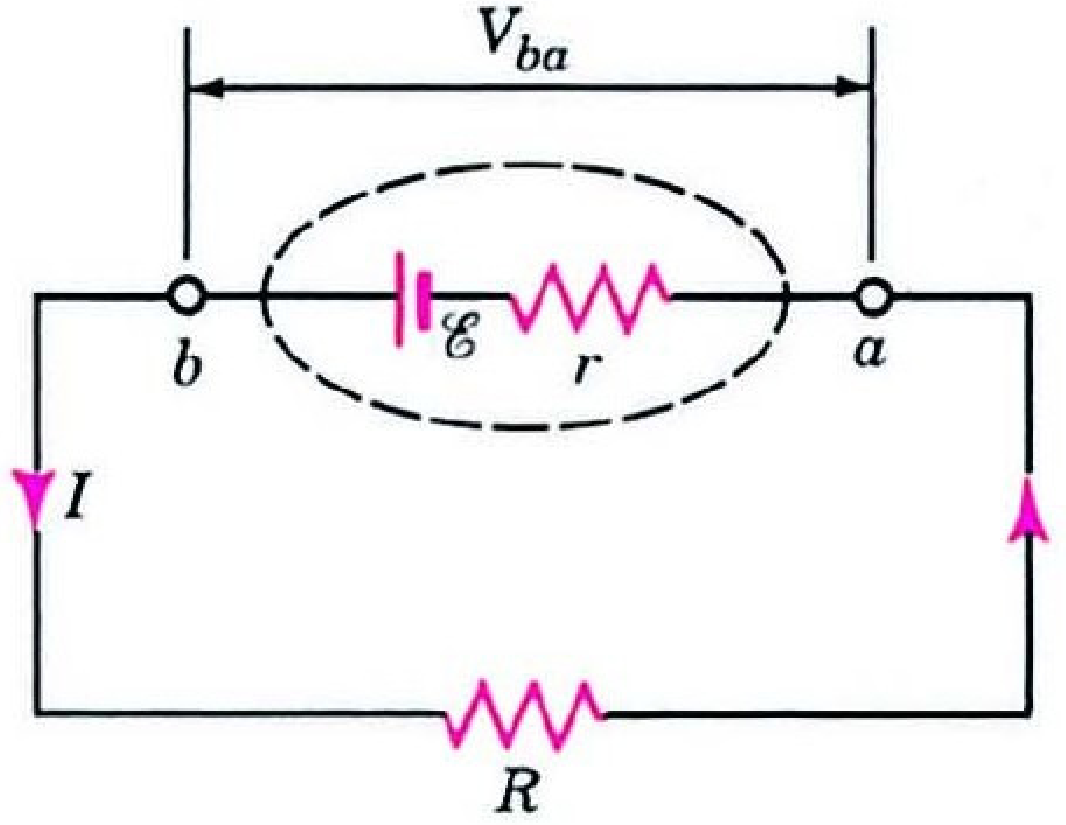
$$PbO_2 + 4H^+ + SO_4^{2-} + 2e^- \rightarrow PbSO_4 + 2H_2O$$

Note that for every electron that leaves the Pb plate, another enters the PbO_2 plate.

Electromotive Force: Terminal Potential Difference

A real source of emf, such as a battery, has *internal resistance* r.

$$V_{ba} = V_b - V_a = \mathcal{E} - Ir$$

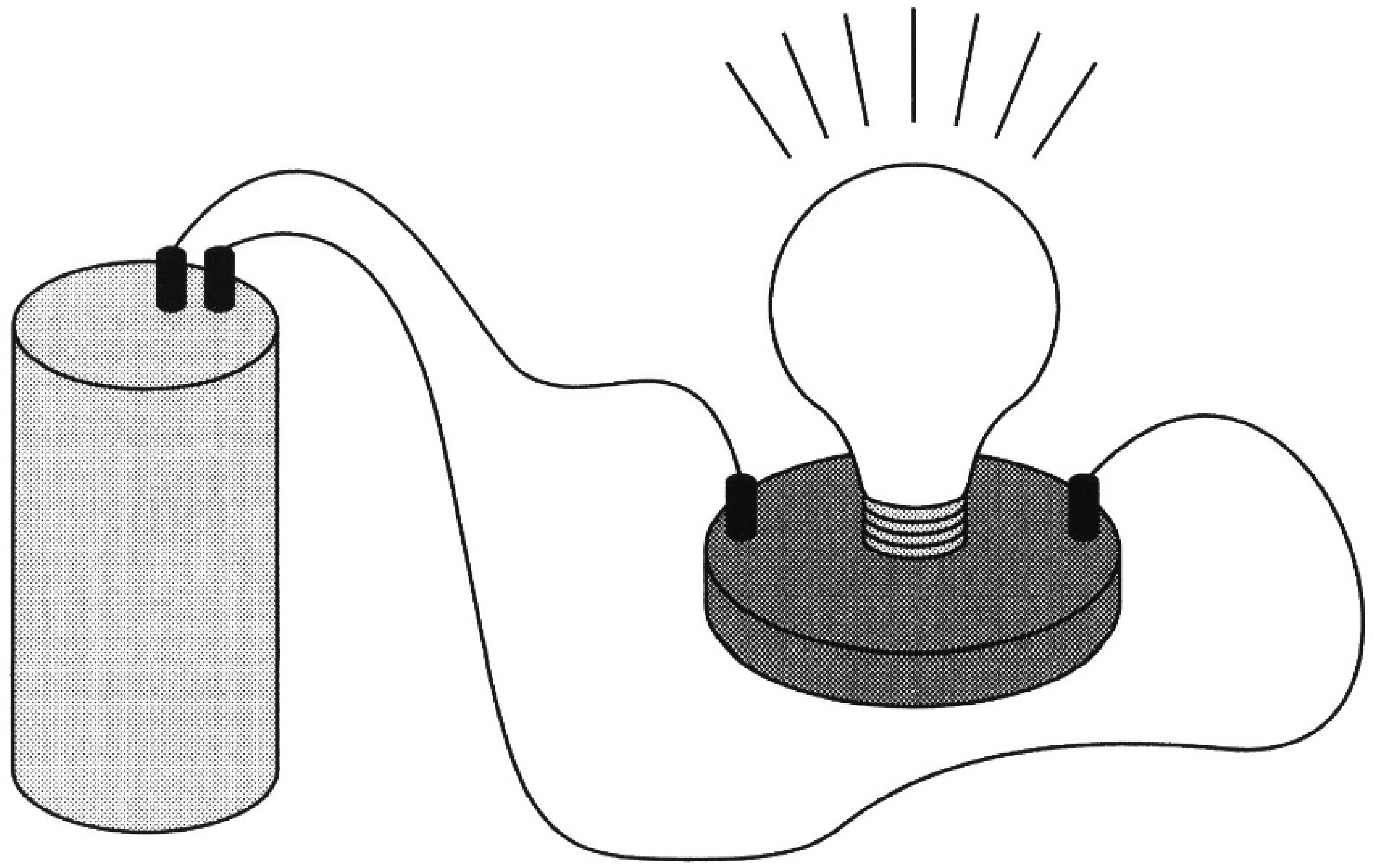


The change in potential is called the **terminal potential difference**.

Unlike the emf, which is a fixed property of the source, the terminal potential difference depends on the current flowing through it.

As a battery ages its internal resistance increases, and so, for a given output current, the terminal potential difference falls.

Electromotive Force Drives the Electrons



Example: A battery is hooked up to a light bulb.

The battery generates the force which drives the electrons move along the loop.

Snail's pace: the charges in a wire move slowly (~0.1 mm/s @ϕ=1mm, 1A, see Prob. 5.19(b)).

Q1: Why does the bulb response so fast when turning it on or off?

Q2: How do all the charges know to start moving at the same instant?

Example: The Snail's Pace

Calculate the average electron drift velocity in a copper wire 1mm in diameter, carrying a current of 1 A.

Sol: $J = \frac{I}{\pi s^2} = \rho v_\mathrm{d} \Rightarrow v_\mathrm{d} = \frac{I}{\pi s^2 \rho}$ (ρ: volume charge density)

$$\rho = \frac{\text{mobile charges}}{\text{volume}} = \frac{\text{charge}}{\text{atom}}\frac{\text{atom}}{\text{mole}}\frac{\text{mole}}{\text{gram}}\frac{\text{gram}}{\text{volume}}$$

$$= (2\times 1.6\times 10^{-19})(6\times 10^{23})(1/64)(9) = 2.7\times 10^4\,\mathrm{C/cm^3}$$

$$v_\mathrm{d} = \frac{I}{\pi s^2 \rho} = \frac{1}{\pi \times 0.05^2 \times 2.7\times 10^4} = 4.7\times 10^{-3}\ \text{(cm/s)}$$

$$@\ 1\text{A},\ \phi = 1\ \text{mm} \Rightarrow v_\mathrm{d} = 0.047\ \text{(mm/s)}$$

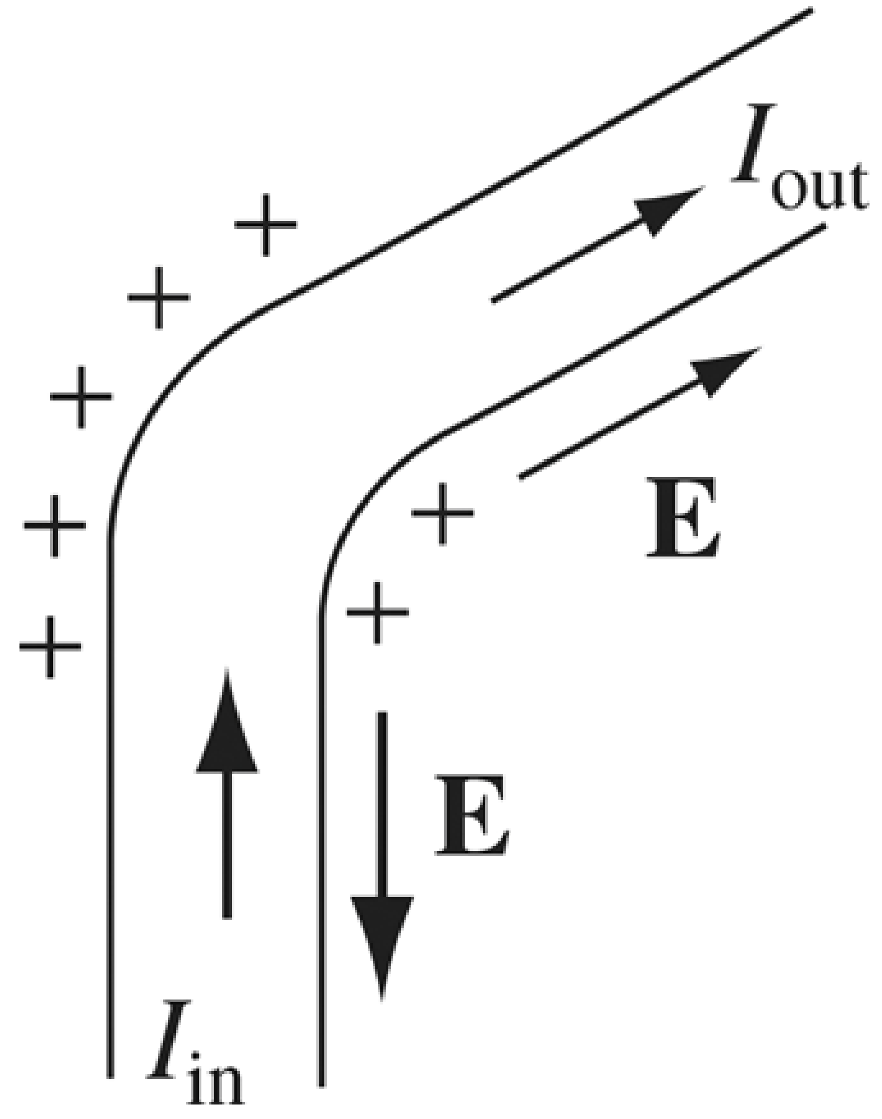
$$@10\text{A},\ \phi = 1\ \text{mm} \Rightarrow v_\mathrm{d} = 0.47\ \text{(mm/s)}\ \ \text{Snail's pace}$$

$$\boxed{\frac{1}{2} m_e v_{thermal}^2 = \frac{3}{2} kT \Rightarrow v_{thermal} = \sqrt{\frac{3kT}{m_e}} \approx 1.2\times 10^5\ \text{(m/s)}\ @T = 300\ \text{K}}$$

Will the Charge Piling Up Somewhere?

If a current is not the same all the way around, then the charge is piling up somewhere, and the electric field of this accumulating charge is in such a direction as to even out the flow.

Charge piling up at the “knee” produces a field aiming away from the kink.



It self-corrects the current flow.

Kirchhoff's current law (KCL): The algebraic sum of currents in a network of conductors meeting at a point is zero.

Forces Involved in Driving Currents Around a Circuit

Two forces involved in driving currents around a circuit.
$\mathbf{f}_s$: ordinarily confined to one portion of the loop (a battery, say).
$\mathbf{E}$: the *electrostatic* force: smooth out the flow and communicate the influence of the source to distant parts of the circuit.

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}$$ Force per unit charge.

What is the physical agency responsible for $\mathbf{f}_s$?

Battery → a chemical force
Piezoelectric crystal → mechanical pressure
Thermal couple → temperature gradient
Photoelectric cell → light

The Electromotive Force 電動勢

The net effect of the electromotive force is determined by the line integral of $\mathbf{f}$ around the circuit:

$= 0$ for electrostatics

$$\mathcal{E} \equiv \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l} + \cancel{\oint \mathbf{E} \cdot d\mathbf{l}} = \oint \mathbf{f}_s \cdot d\mathbf{l}$$

(the electromotive force, emf)

(very bad 糟糕的)

Emf is a lousy term, since it is not a force at all --- it is the integral of a force per unit charge.

$$\mathcal{E} = \frac{W_{\text{ne}}}{q}$$

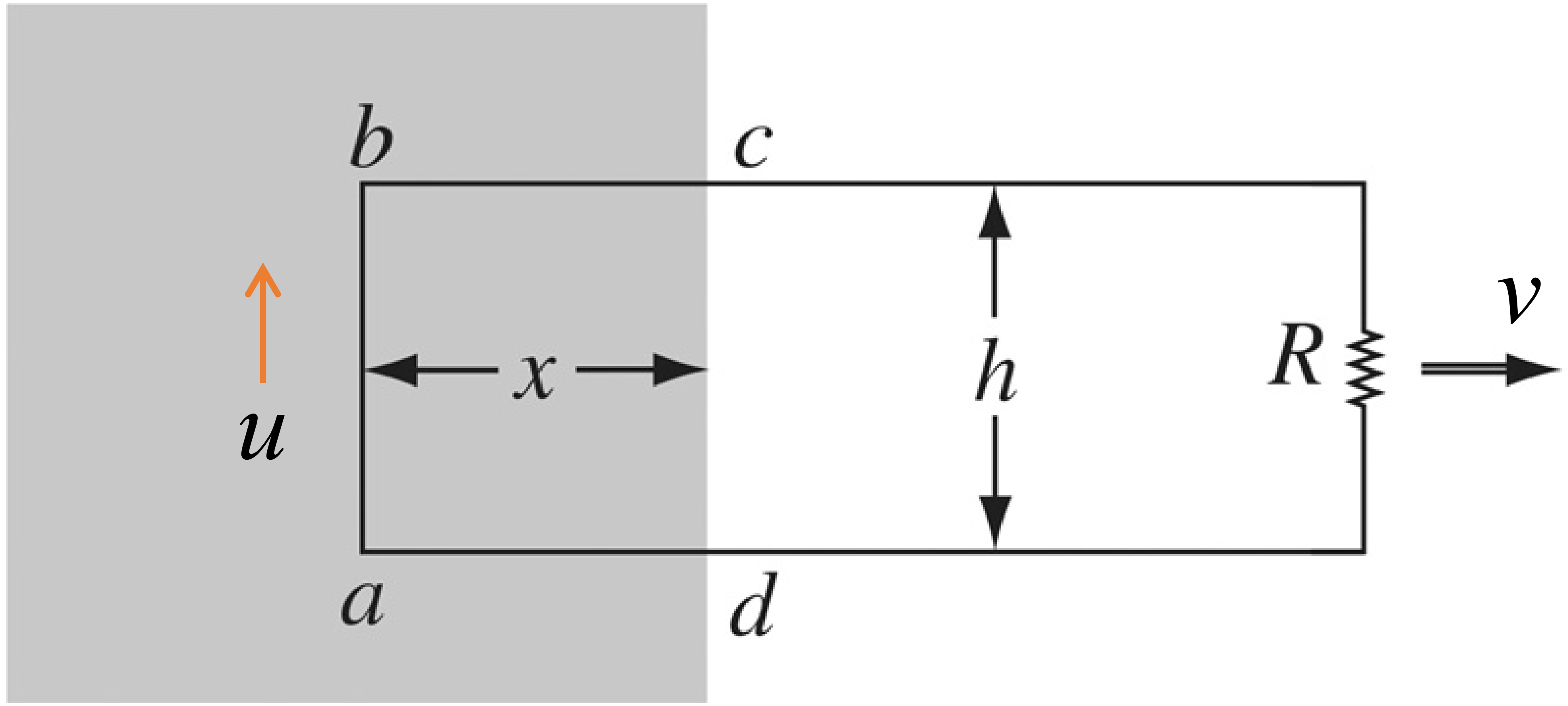
An emf is the work per unit charge done by the source of emf in moving the charge around *a closed loop*.

7.1.3 Motional emf

The most common source of the emf: the generator

Generators exploit motional emf's, which arise when you move a wire through a magnetic field.

A primitive model for a generator



Shaded region: uniform B-field pointing into the page.

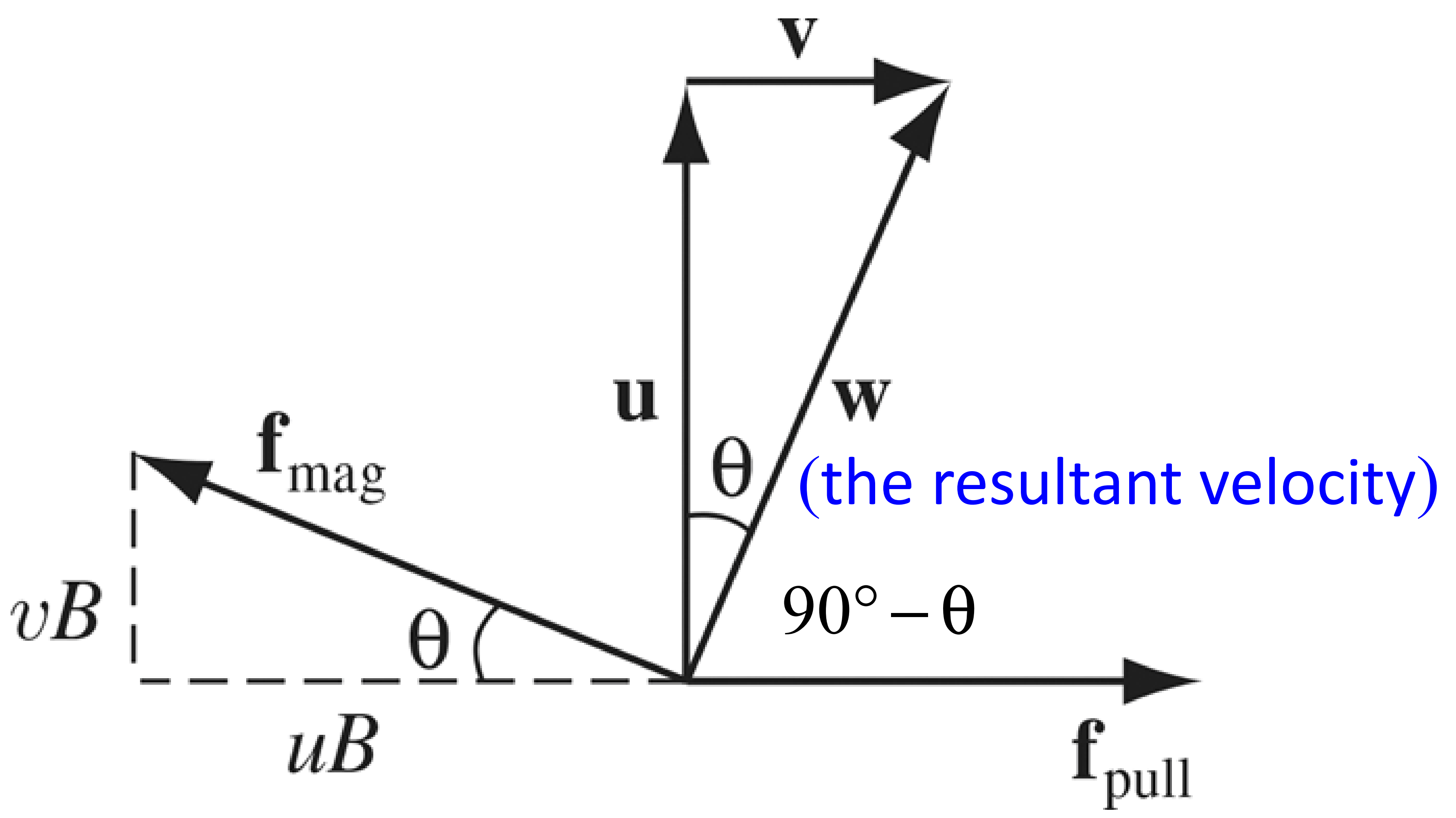
R: whatever it is, we are trying to drive current through.

$$\mathcal{E} = \oint \mathbf{f}_{\mathrm{mag}} \cdot d\mathbf{l} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = vBh$$

Magnetic Force Does No Work

A person exerts a force per unit charge on the wire by pulling it. The force counteracts the force generated by the magnetic force quB.

$$f_{\text{pull}} = uB$$



This force is transmitted to the charge by the structure of the wire.

The work done per unit charge is:

$$\int \mathbf{f}_{\text{pull}} \cdot d\mathbf{l} = (uB)(h\tan\theta) = u\tan\theta Bh = vBh = \mathcal{E}$$

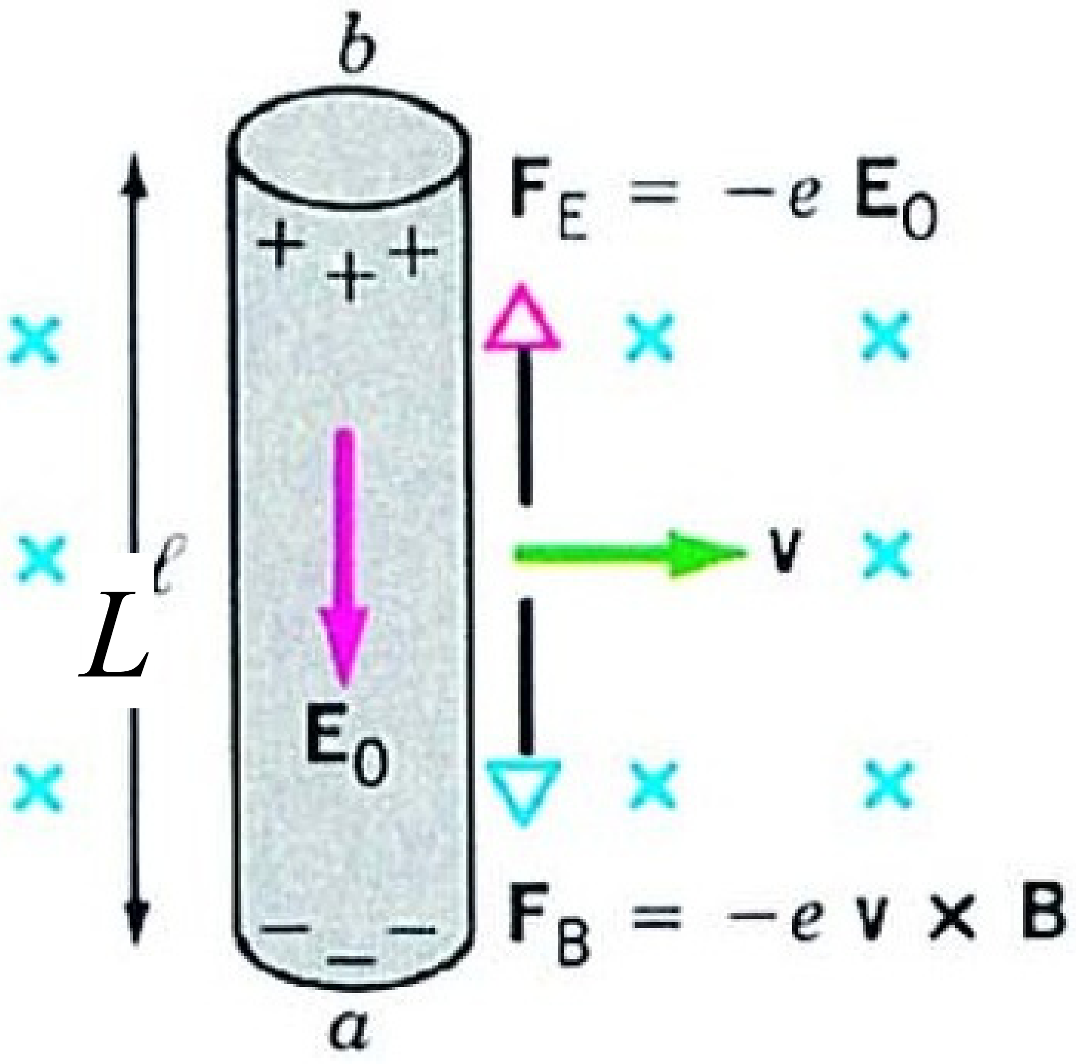
The work done per unit charge is exactly equal to the emf.

Motional emf (another example)

When the magnetic field is constant in time, there is no induced electric field.

When a metal rod moving perpendicular to magnetic field lines, there is a separation of charge and an associated electrostatic potential difference sets up.

The potential difference associated with this electrostatic field is given by $V_b - V_a = E_0 L = BLv$.



$$\mathcal{E} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

Since there is no current flowing, the “terminal potential difference” is equal to the **motional emf**.

Instantaneous emf

$$\mathcal{E} = vBh = Bhv(t) = \mathcal{E}(t)$$

$\mathcal{E}$: carried out at one instant of time – take a “snapshot” of the loop, if you like, and work from that.

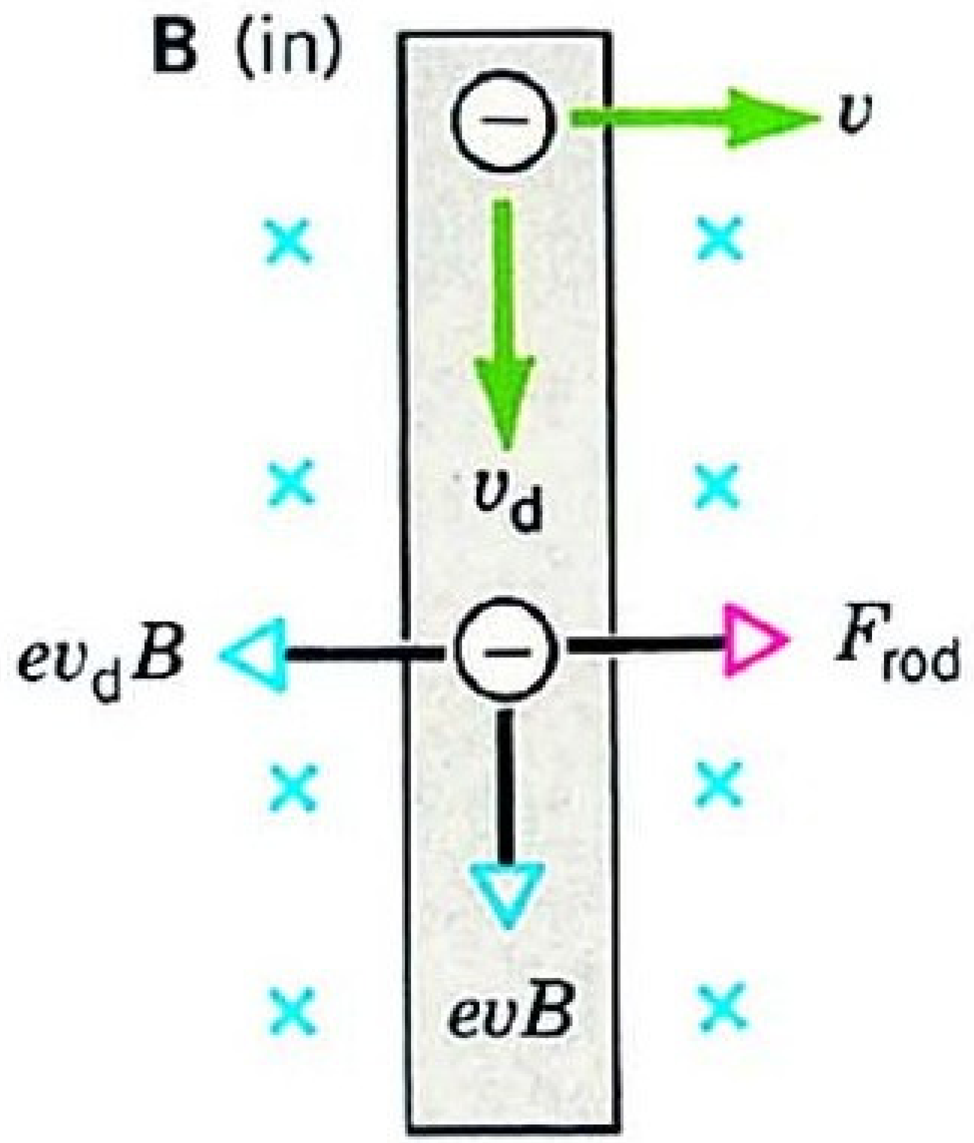
The magnetic force is responsible for establishing the emf and the emf seems to heat the resistor (i.e., do work), but magnetic fields never do work.

Who is supplying the energy that heats the resistor.

The person who’s pulling on the loop!

Magnetic Force Does No Work (II)

In the previous viewgraph, we find a source of emf converts some form of energy into electrostatic energy and does work on charges. Can magnetic forces do work? No.



中介

The magnetic field acts, in a sense, as an intermediary in the transfer of the energy from the external agent to the rod.

The Flux Rule

There is a particular nice way of expressing the emf generated in a moving loop ➔ the flux rule.

Let $\Phi \equiv \int \mathbf{B} \cdot d\mathbf{a}$ the flux of B through the loop

The flux of a rectangular loop $\Phi = Bhx$

The flux change rate $$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv$$

The minus sign accounts for the fact that dx/dt is negative.

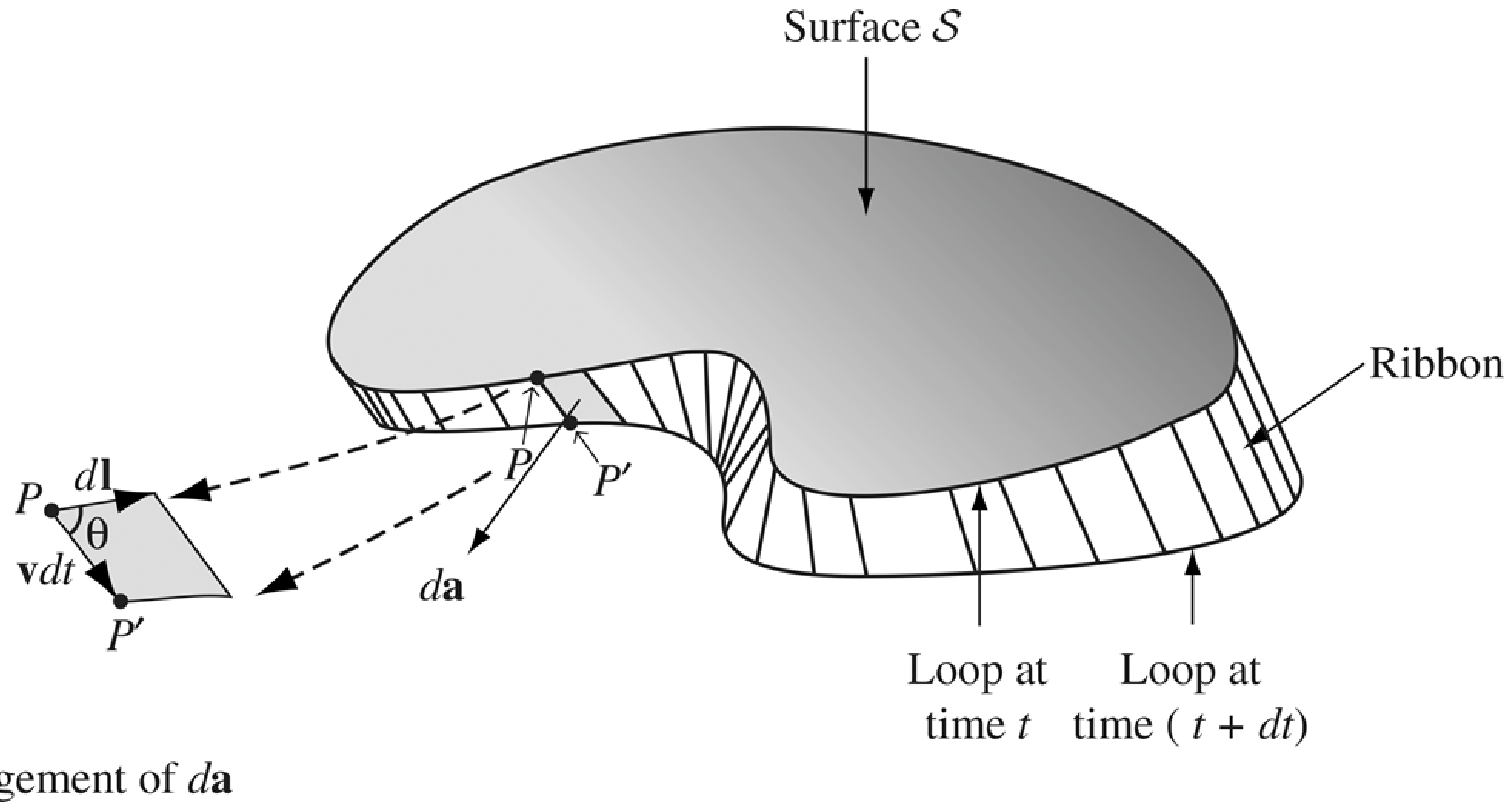
The flux rule for the motional emf: $$-\frac{d\Phi}{dt} = Bhv = \mathcal{E}$$

Next step proves: $$\mathcal{E} = -\frac{d\Phi}{dt}$$

The Flux Rule (Generalized)

The flux rule can be applied to non-rectangular loop through non-uniform magnetic field.

Proof:



Compute the flux at time t using surface S, and the flux at time $t+dt$, using the surface consisting of S plus the “ribbon” that connects the new position of the loop to the old.

The Flux Rule (Generalized II)

The change in flux is

$$d\Phi = \Phi(t+dt) - \Phi(t) = \Phi_{\text{ribbon}} = \int_{\text{ribbon}} \mathbf{B}\cdot d\mathbf{a}$$

The infinitesimal element of area on the ribbon

$$d\mathbf{a} = \mathbf{v}dt \times d\mathbf{l} = (\mathbf{v}\times d\mathbf{l})dt$$

$$d\Phi = \int_{\text{ribbon}} \mathbf{B}\cdot d\mathbf{a} = \int_{\text{ribbon}} \mathbf{B}\cdot(\mathbf{v}\times d\mathbf{l})dt$$

$$\boxed{\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})\\ &= -(\mathbf{B}\times\mathbf{A})\cdot\mathbf{C}\end{aligned}}$$

$$\frac{d\Phi}{dt} = \oint \mathbf{B}\cdot(\mathbf{v}\times d\mathbf{l}) = -\oint(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}$$

magnetic force per unit charge

$$= -\oint[(\mathbf{v}+\mathbf{u})\times\mathbf{B}]\cdot d\mathbf{l} = -\oint\overline{(\mathbf{W}\times\mathbf{B})}\cdot d\mathbf{l}$$

$\mathbf{v}$: velocity of the wire.
$\mathbf{u}$: velocity of a charge down the wire.

$$\frac{d\Phi}{dt} = -\oint \mathbf{f}_{\text{mag}}\cdot d\mathbf{l} = -\mathcal{E} \qquad \text{qed}$$

拉丁片語: "**quod erat demonstrandum**"證明完畢
or 戲稱"**quite easily done**"

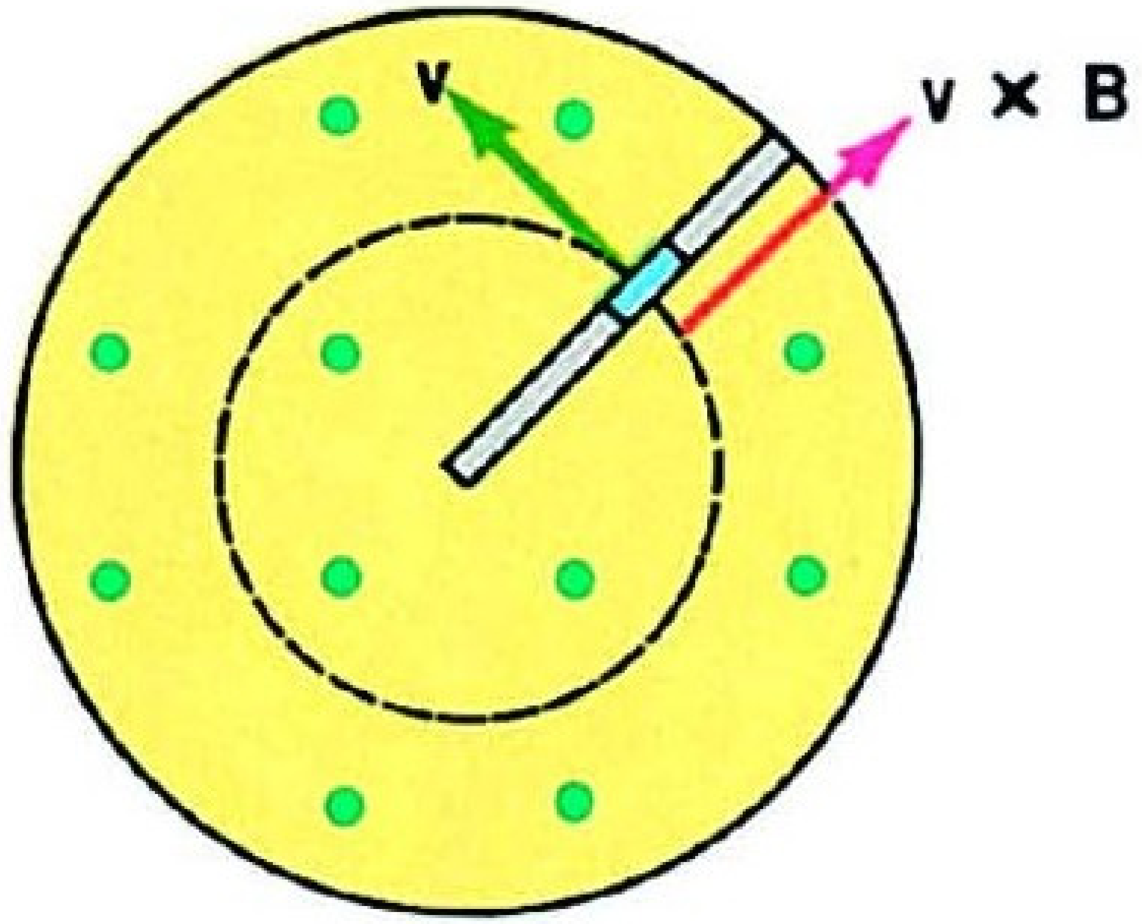
Example 7.4

In a homopolar generator a conducting disk of radius R rotates at angular velocity ω rad/s. Its plane is perpendicular to a uniform and constant magnetic field $\mathbf{B}$. What is the emf generated between the center and the rim?

Solution:

$$\mathcal{E} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = \int_0^R vBdr$$

$$= \int_0^R \omega rBdr = \frac{1}{2}\omega BR^2$$

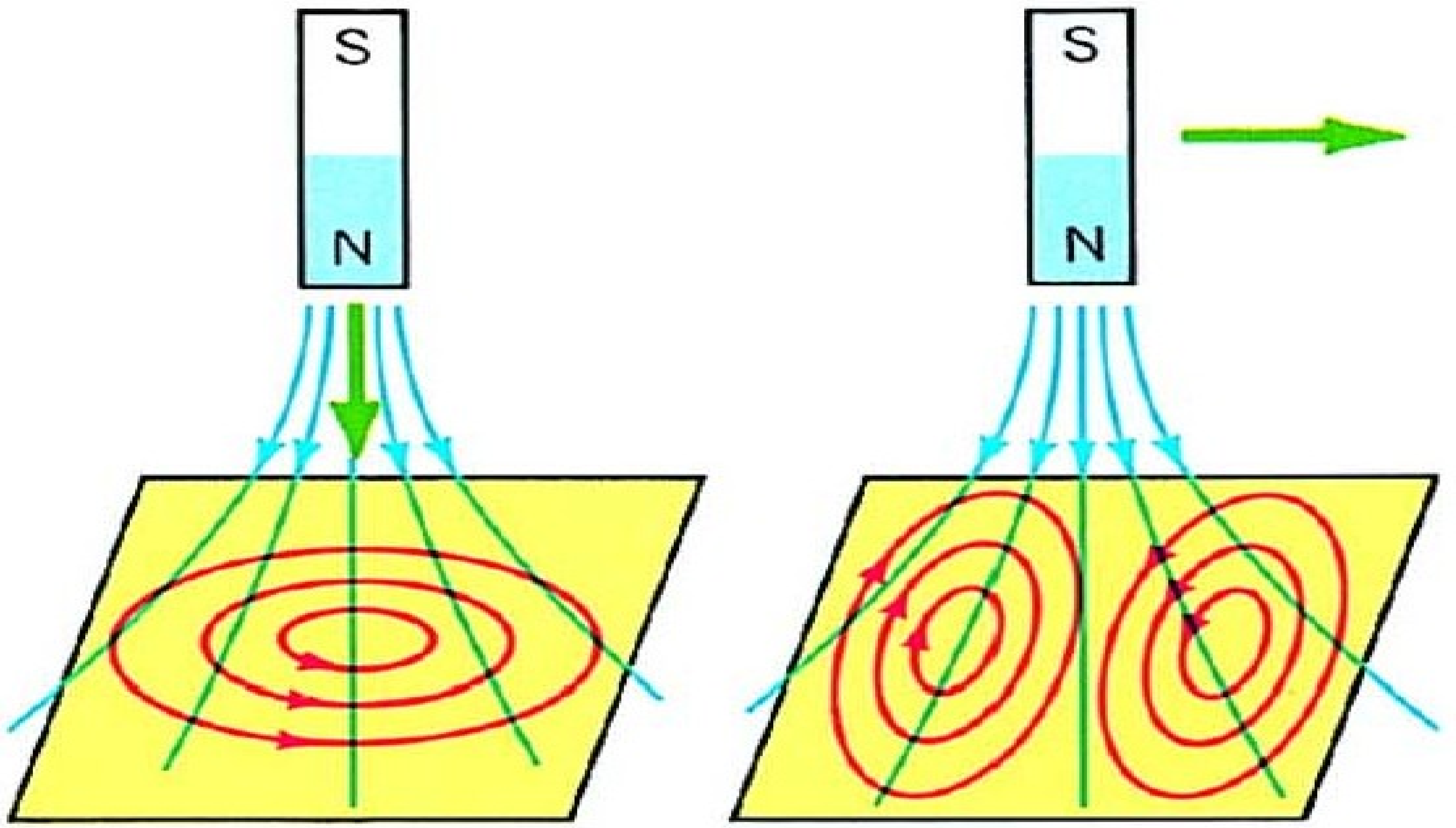


Hint1: How to choose a proper closed loop?

Hint2: The total magnetic flux passing through the disk is constant in time. Where is the induced emf coming from? (Ref. Benson & Feyman)

Eddy Currents (I)

What happens when a bar magnet approaches or moves parallel to a conducting plate? It induces eddy current.

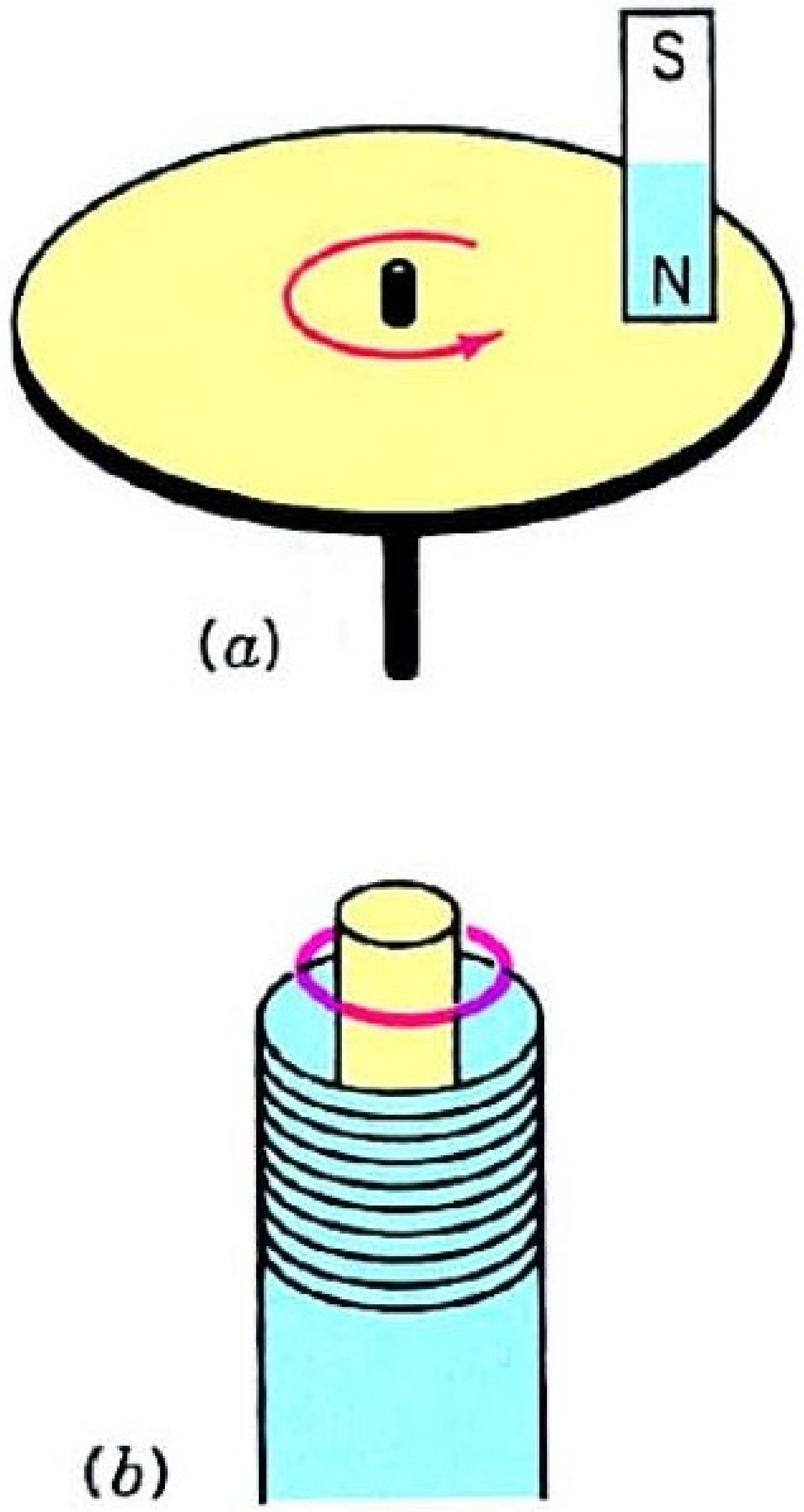


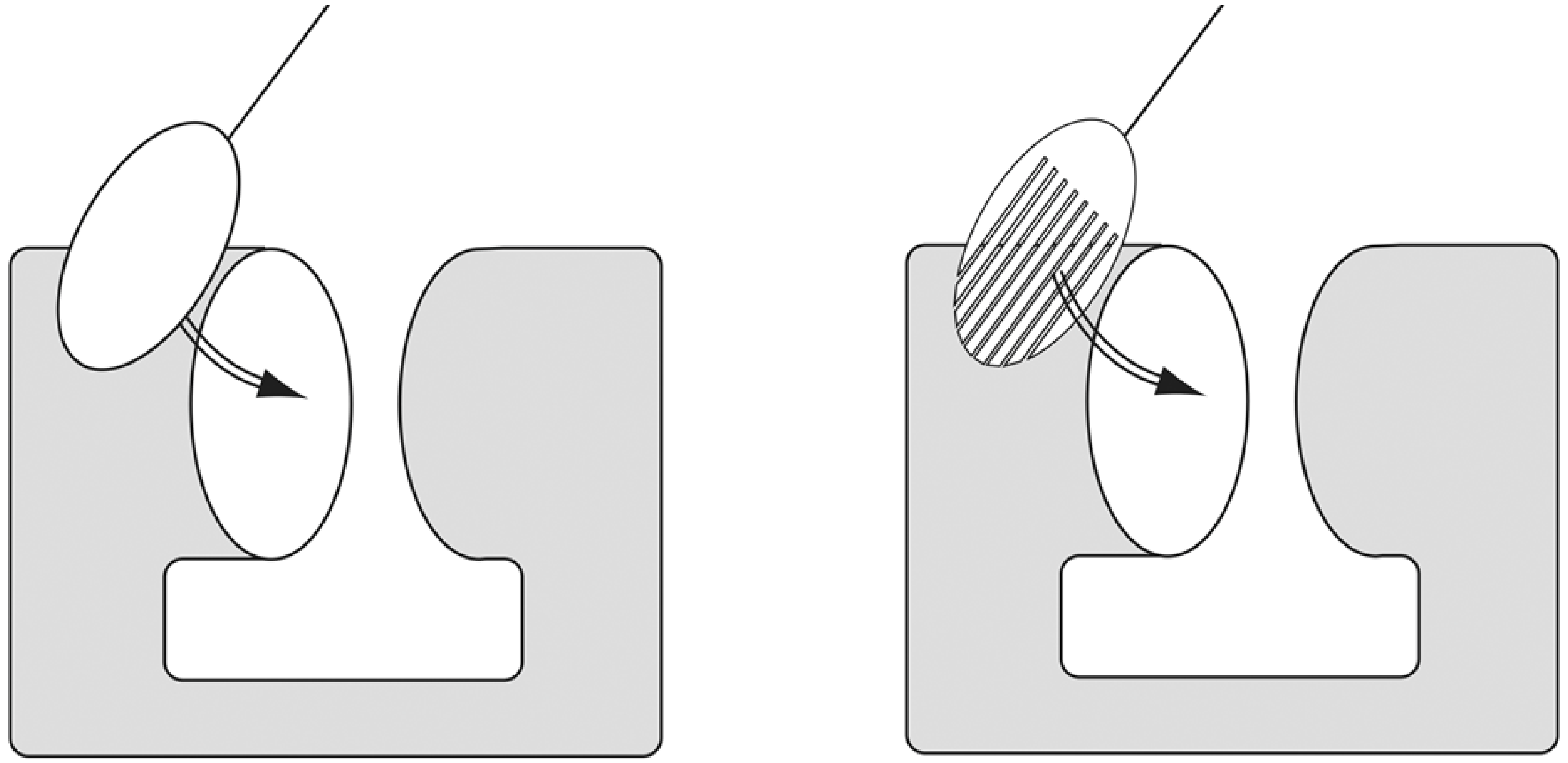
The eddy current is distributed throughout the plate.

Eddy Currents (II)

Applications of the eddy current:

1. The braking system of a train.
2. Eddy current generated in copper pots can also be used for “inductive cooking”.
3. Project a metal ring. The ring gets very hot when projected.

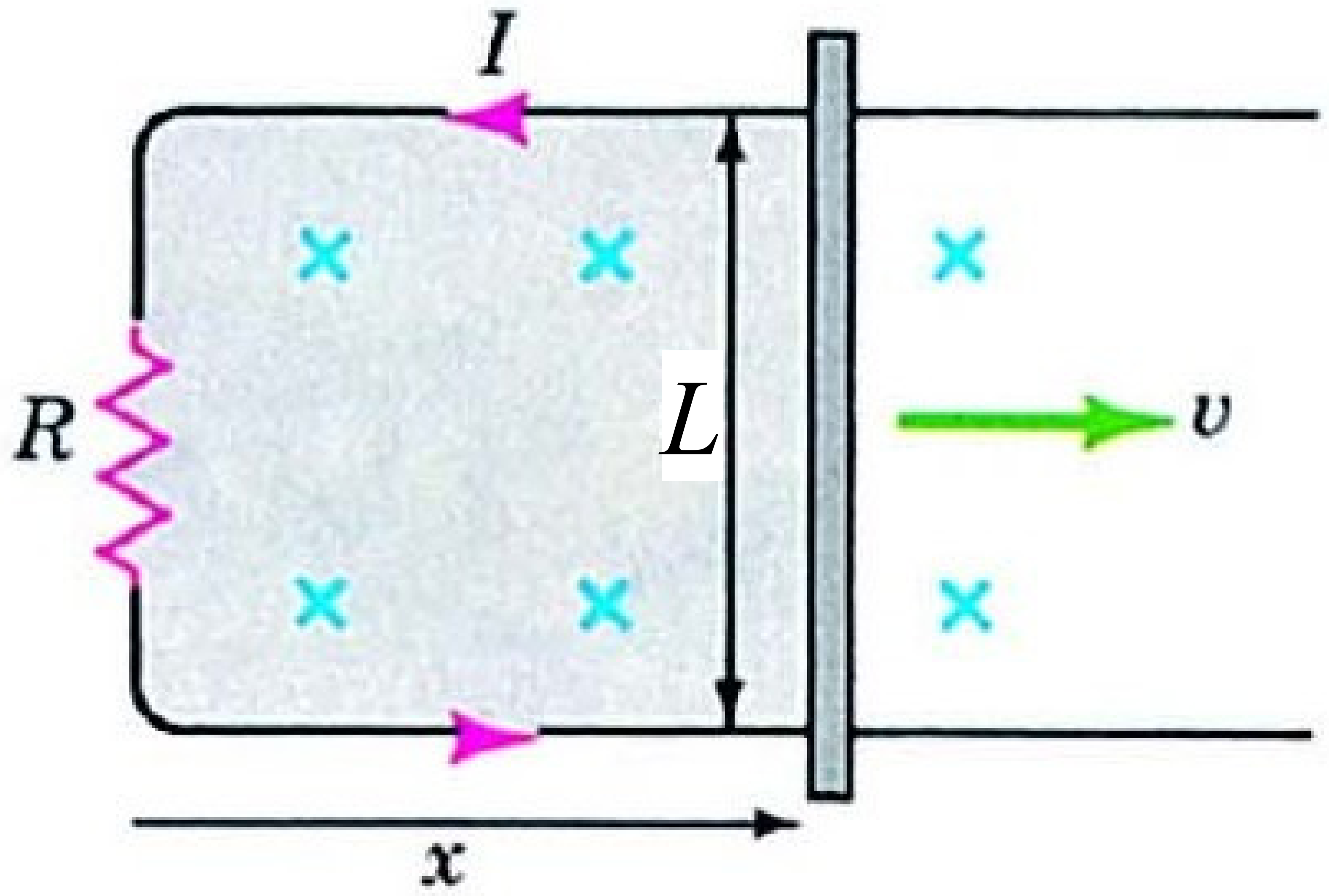



(a) (b)

Example

A metal rod of length L slides at constant velocity v on conducting rails that terminate in a resistor R. There is a uniform and constant magnetic field perpendicular to the plane of the rails. Find: (a) the current in the resistor; (b) the power dissipated in the resistor; (c) the mechanical power needed to pull the rod.

Solution:



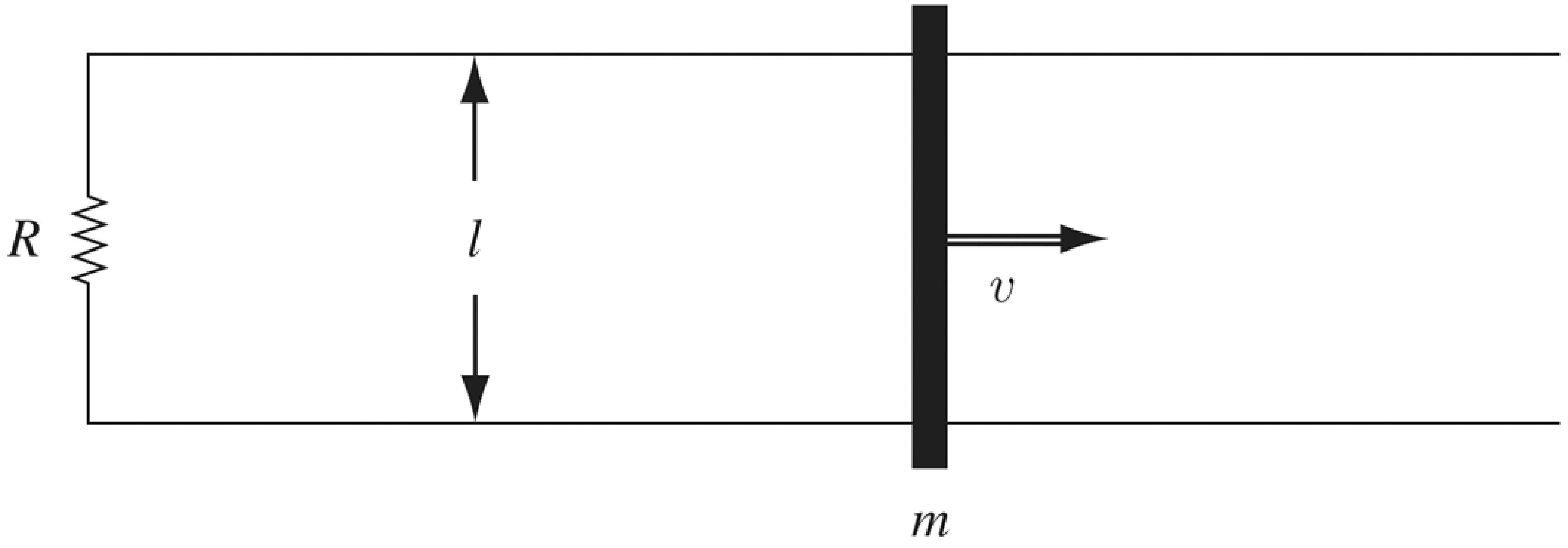
$$\text{(a)}\quad \left|V_{\text{emf}}\right| = \frac{d\Phi}{dt} = B\frac{dA}{dt} = Blv$$

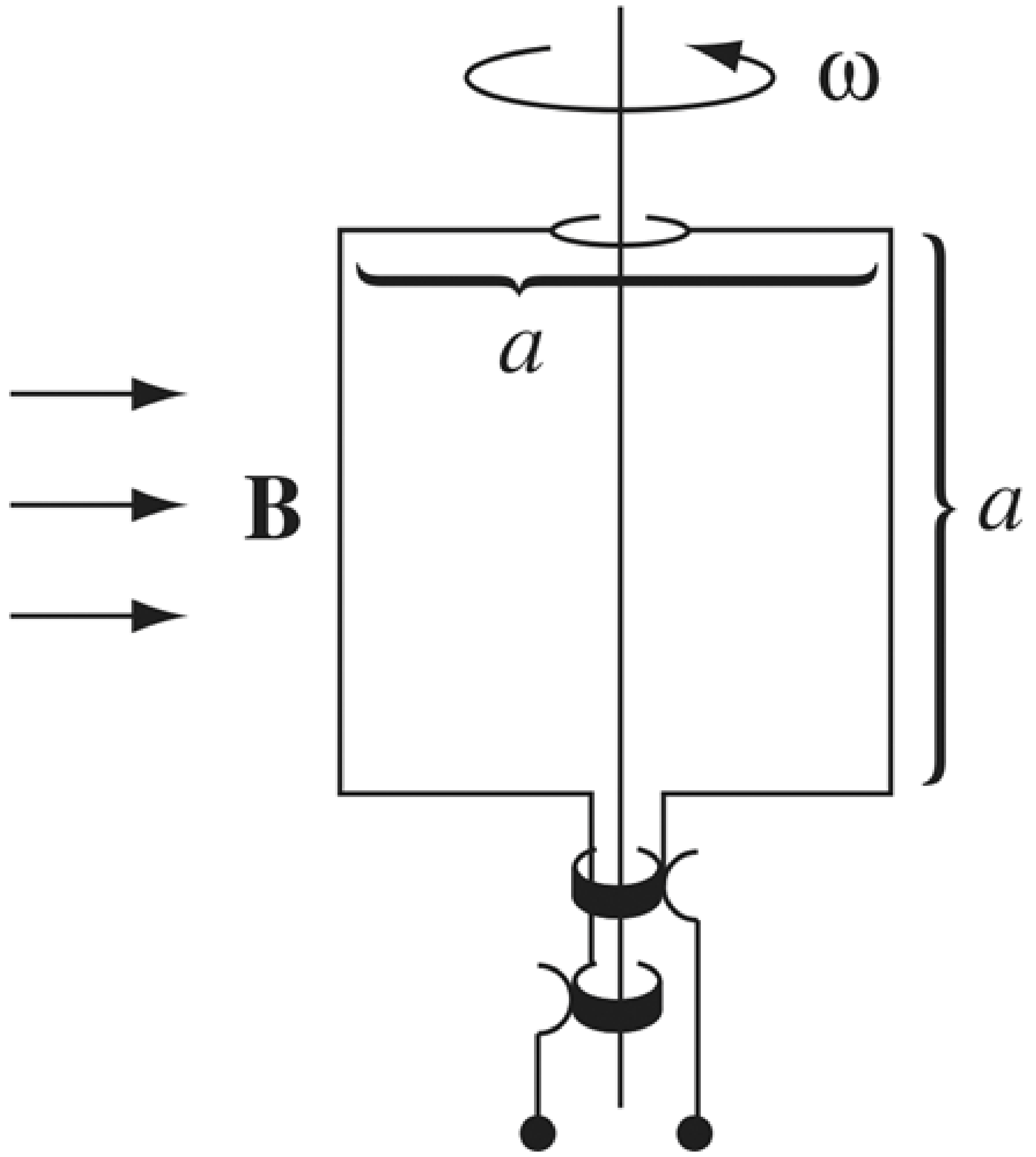
$$I = \frac{\left|V_{\text{emf}}\right|}{R} = \frac{Blv}{R}$$

$$\text{(b)}\quad P_{elec} = I^2 R = \frac{(Blv)^2}{R}$$

$$\text{(c)}\quad P_{mech} = \mathbf{F}_{ext}\cdot\mathbf{v} = \frac{(Blv)^2}{R}$$

More Examples





Homework of Chap. 7 (part I)

Problem 7.2 A capacitor C has been charged up to potential V_0; at time $t = 0$, it is connected to a resistor R, and begins to discharge (Fig. 7.5a).

(a) Determine the charge on the capacitor as a function of time, $Q(t)$. What is the current through the resistor, $I\ (t)$?

(b) What was the original energy stored in the capacitor (Eq. 2.55)? By integrating Eq. 7.7, confirm that the heat delivered to the resistor is equal to the energy lost by the capacitor.

Now imagine *charging up* the capacitor, by connecting it (and the resistor) to a battery of voltage V_0, at time $t = 0$ (Fig. 7.5b).

(c) Again, determine $Q(t)$ and $I(t)$.

(d) Find the total energy output of the battery ($\int V_0 I\ dt$). Determine the heat delivered to the resistor. What is the final energy stored in the capacitor? What fraction of the work done by the battery shows up as energy in the capacitor? [Notice that the answer is independent of R!]

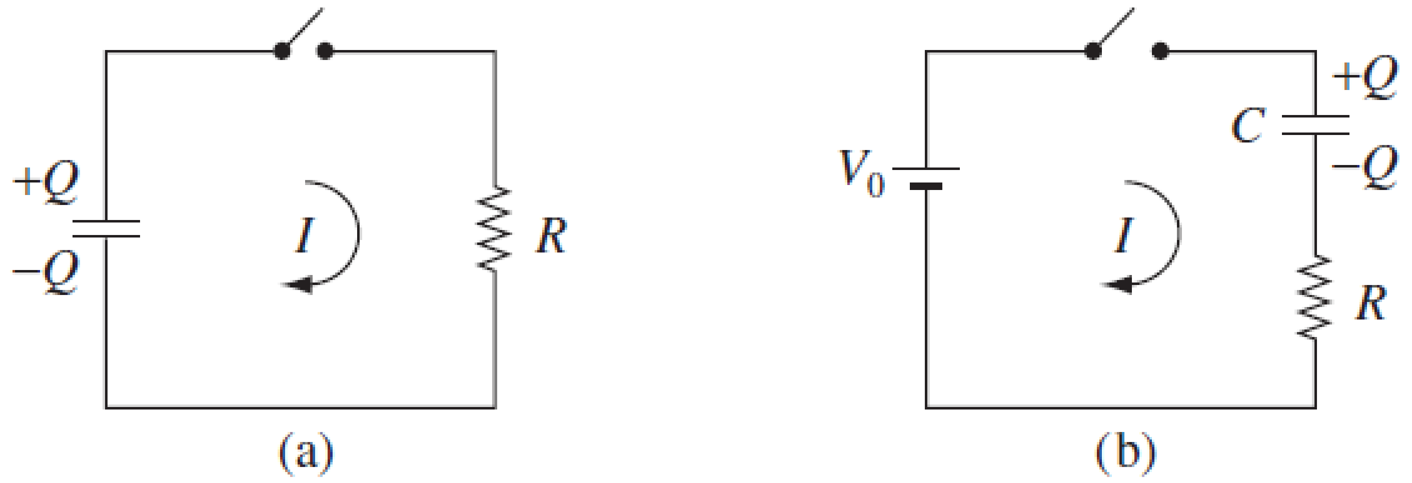


FIGURE 7.5

Homework of Chap. 7 (part I)

Problem 7.6 A rectangular loop of wire is situated so that one end (height *h*) is between the plates of a parallel-plate capacitor (Fig. 7.9), oriented parallel to the field **E**. The other end is way outside, where the field is essentially zero. What is the emf in this loop? If the total resistance is *R*, what current flows? Explain. [*Warning*: This is a trick question, so be careful; if you have invented a perpetual motion machine, there's probably something wrong with it.]

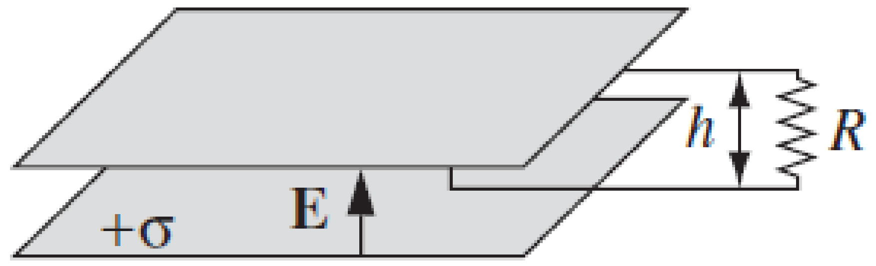


FIGURE 7.9

Problem 7.8 A square loop of wire (side *a*) lies on a table, a distance *s* from a very long straight wire, which carries a current *I* , as shown in Fig. 7.18.

(a) Find the flux of **B** through the loop.

(b) If someone now pulls the loop directly away from the wire, at speed *v*, what emf is generated? In what direction (clockwise or counterclockwise does the current flow?

(c) What if the loop is pulled to the *right* at speed *v*?

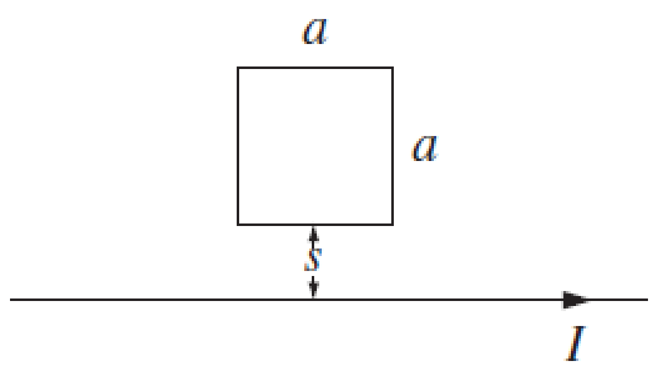


FIGURE 7.18

Homework of Chap. 7 (part I)

Problem 7.11 A square loop is cut out of a thick sheet of aluminum. It is then placed so that the top portion is in a uniform magnetic field **B**, and is allowed to fall under gravity (Fig. 7.20). (In the diagram, shading indicates the field region; **B** points into the page.) If the magnetic field is 1 T (a pretty standard laboratory field), find the terminal velocity of the loop (in m/s). Find the velocity of the loop as a function of time. How long does it take (in seconds) to reach, say, 90% of the terminal velocity? What would happen if you cut a tiny slit in the ring, breaking the circuit? [*Note*: The dimensions of the loop cancel out; determine the actual *numbers*, in the units indicated.]

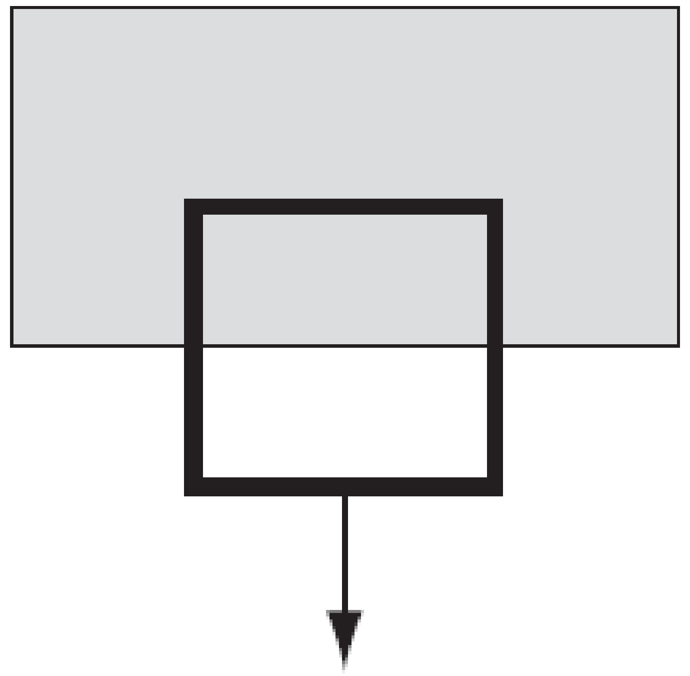


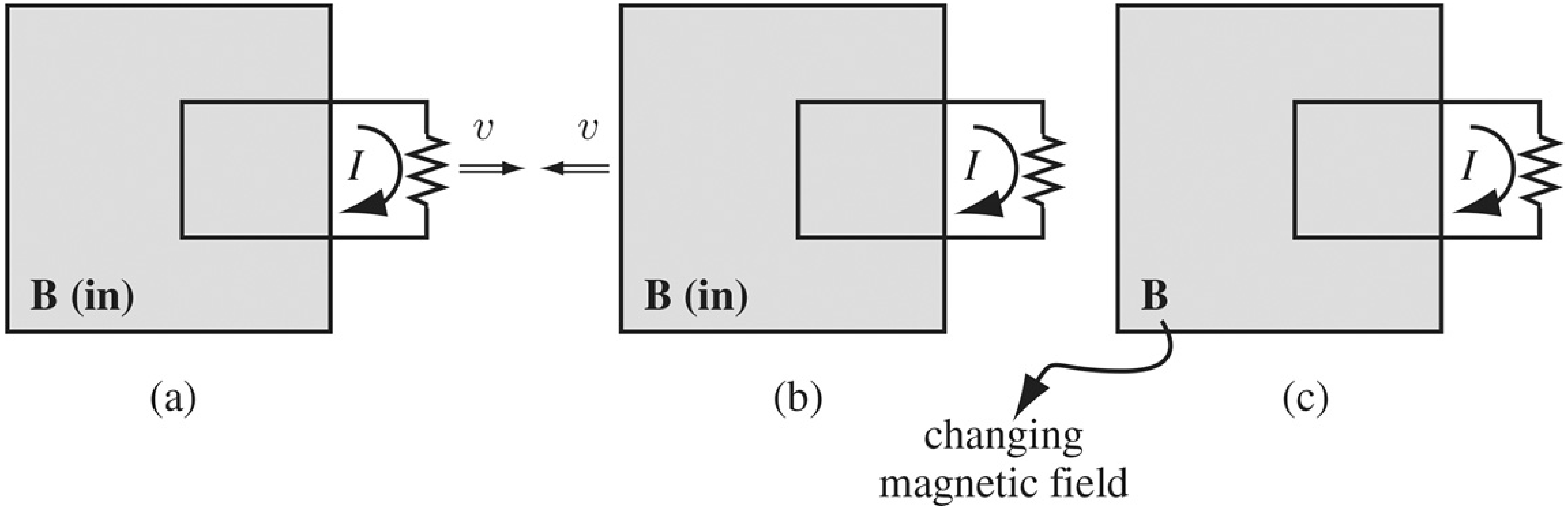
FIGURE 7.20

7.2 Electromagnetic Induction

7.2.1 Faraday's Law

Faraday reported on a series of experiments, including three that can be characterized as follows:

circit moves magnet moves field changes



motional emf

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Faraday's law

Q: Since a stationary charge experiences no magnetic force, what is responsible?

Faraday's Law (at rest)

What sort of field exerts a force on charges *at rest*?

➔Electric field and
a changing magnetic field (Faraday found empirically).

A changing magnetic field induces an electric field.

The emf is equal to the rate of change of flux, when $\mathbf{f}_s = 0$.

$$\mathcal{E} = \oint \mathbf{f}_s \cdot d\mathbf{l} + \oint \mathbf{E} \cdot d\mathbf{l} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{a}$$

Electro-
dynamics

$$\Rightarrow \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{a}$$

← Faraday's law in integral form

$$\left.\begin{array}{l} \oint \mathbf{E} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{E}) \cdot d\mathbf{a} \\ -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{a} = \int (-\frac{\partial \mathbf{B}}{\partial t}) \cdot d\mathbf{a} \end{array}\right\} \Rightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Stokes' theorem

Faraday's law in differential form

Why? See next page.

Faraday's Law (moving at constant velocity)

What sort of field exerts a force on charges *moving at constant velocity*? Hint: Lorentz force

$$\mathcal{E} = \oint \mathbf{f}_s \cdot d\mathbf{l} + \oint \left(\mathbf{E} + (\mathbf{v}\times\mathbf{B})\right)\cdot d\mathbf{l} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a}$$

$$\text{Left:}\quad \oint\left(\mathbf{E}+(\mathbf{v}\times\mathbf{B})\right)\cdot d\mathbf{l} = \int(\nabla\times\mathbf{E})\cdot d\mathbf{a} + \cancel{\int(\nabla\times(\mathbf{v}\times\mathbf{B}))\cdot d\mathbf{a}}$$

$$\text{Right:}\quad -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} = -[\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)]\int\mathbf{B}\cdot d\mathbf{a} = -[\int\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a} + \int \underline{(\mathbf{v}\cdot\nabla)\mathbf{B}}\cdot d\mathbf{a}]$$

$$\boxed{\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - \underline{(\mathbf{A}\cdot\nabla)\mathbf{B}} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})}$$

$$\text{Right:}\quad -\frac{d}{dt}\int\mathbf{B}\cdot d\mathbf{a} = -\int\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a} + \cancel{\int(\nabla\times(\mathbf{v}\times\mathbf{B}))\cdot d\mathbf{a}}$$

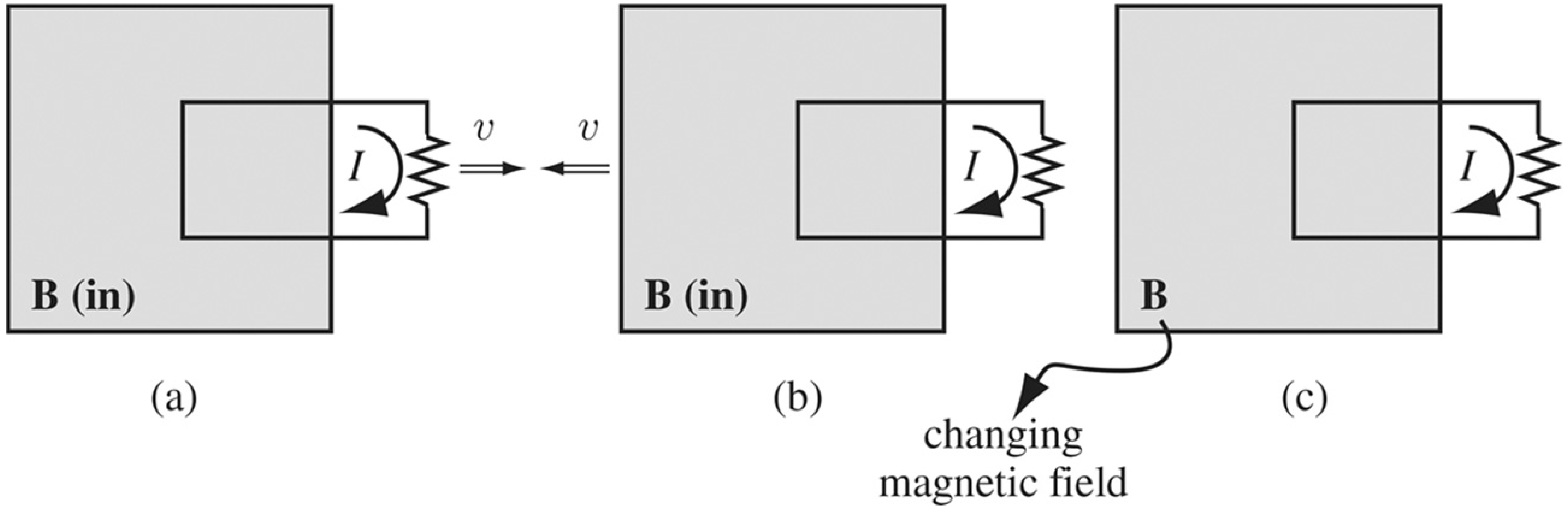
$$-\left[\cancel{\int((\mathbf{B}\cdot\nabla)\mathbf{v} + \mathbf{v}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{v}))\cdot d\mathbf{a}}\right] =0$$

$$\Rightarrow\quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

Also see "The Feymann Lectures on Physics II," Chap.17

Universal Flux Rule

One can subsume all three cases into a kind of universal flux rule:
Whenever the magnetic flux through a loop changes, an emf will appear in the loop.



These three experiments yield the same formula for the emf.

Electric field is induced by changing magnetic field.

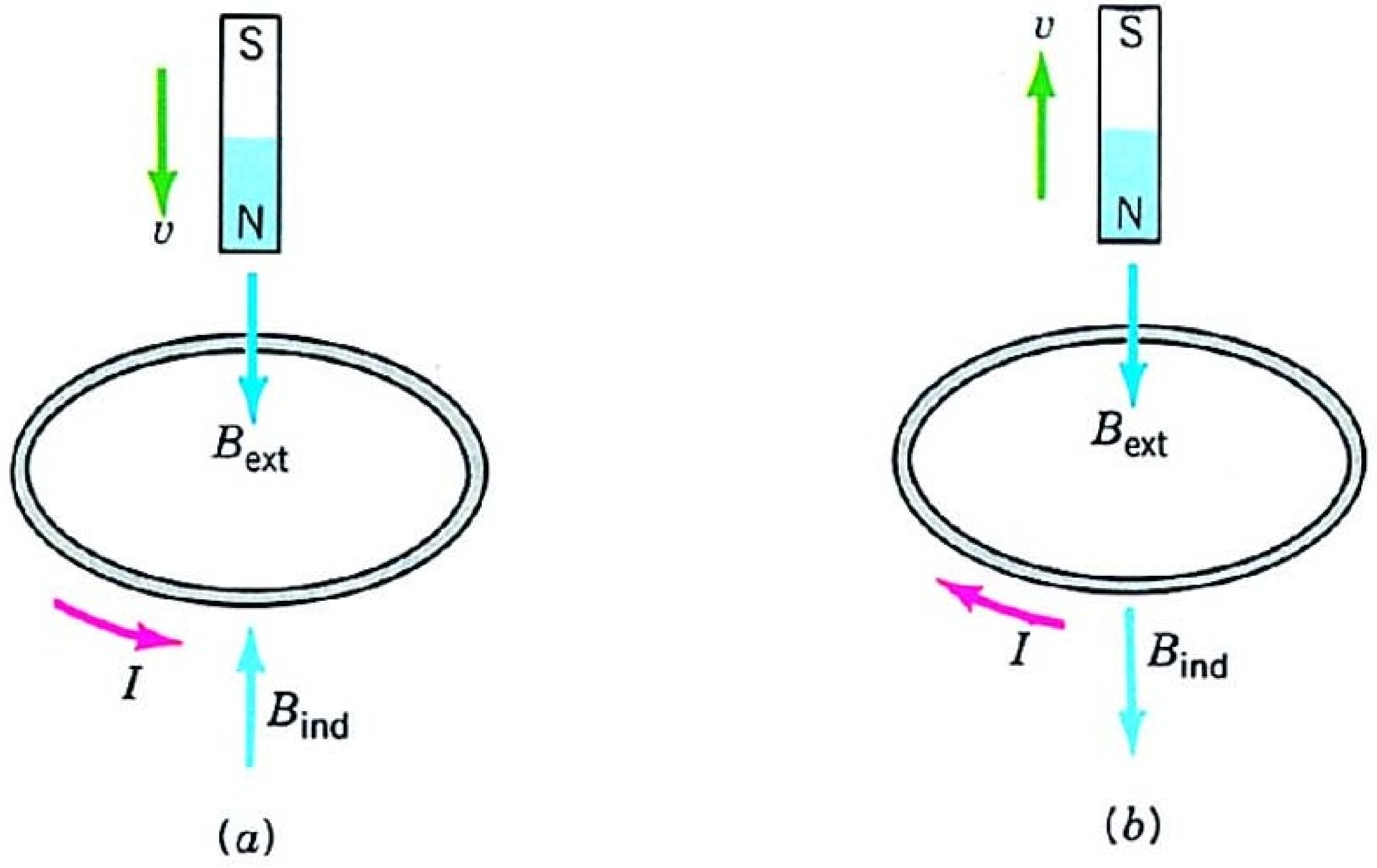
This “coincidence” led Einstein to the special theory of relativity. Chap.12

Lenz’s Law (I)

Lenz’s law is a handy rule, whose sole purpose is to help you get the direction right.

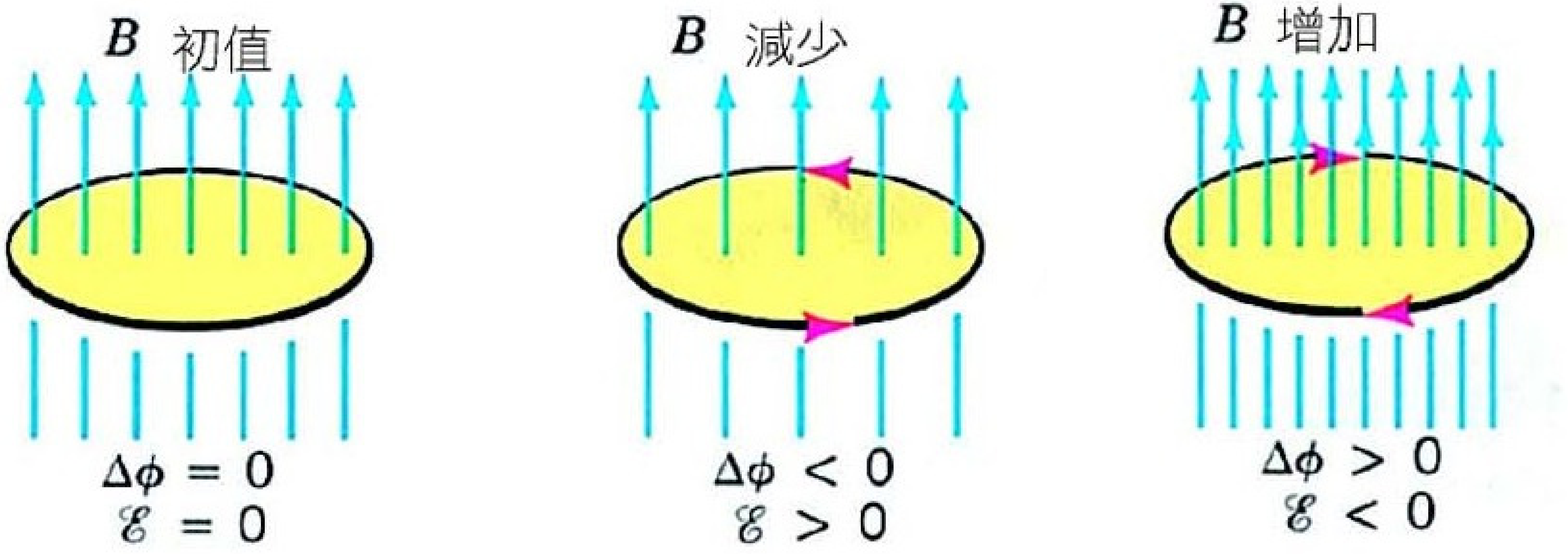
Maxwell restated Lenz’s rule in a more general way:

The effect of the induced emf is such as to oppose the change in flux that produces it.



Lenz’s Law (II)

A sign convention for the induced emf. First we choose the direction of the vector area to make the initial flux positive. The right-hand rule, with the thumb along **B** and the fingers curled around the loop, tells us whether clockwise or counterclockwise is the positive sense.

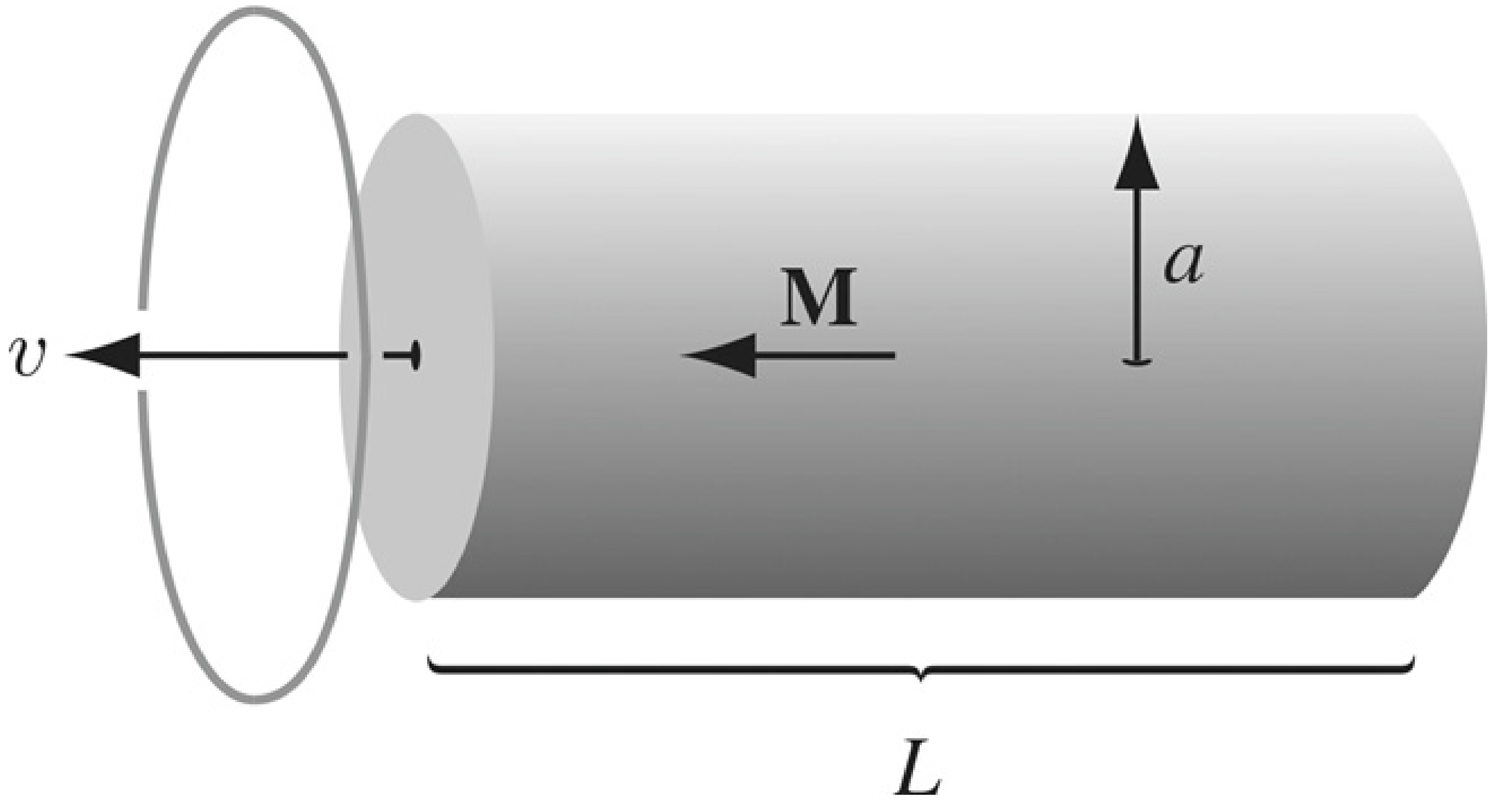


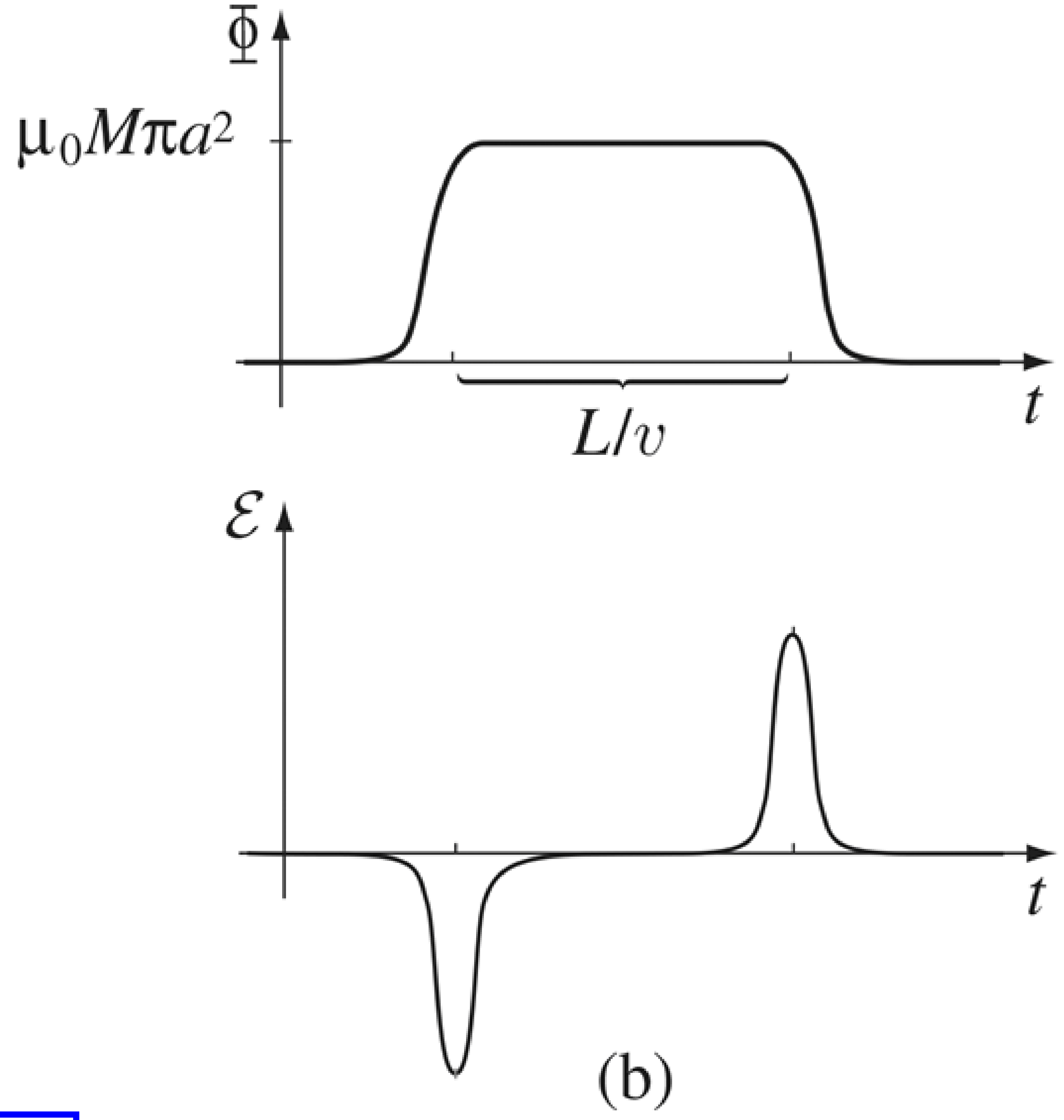
How about superconducting magnet?

Example 7.5

A long cylindrical magnet of length L and radius a, carries a uniform magnetization $\mathbf{M}$ parallel to its axis. It passes at constant velocity v through a circular ring of slightly larger diameter. Graph the emf induced in the ring, as a function of time.

Sol:





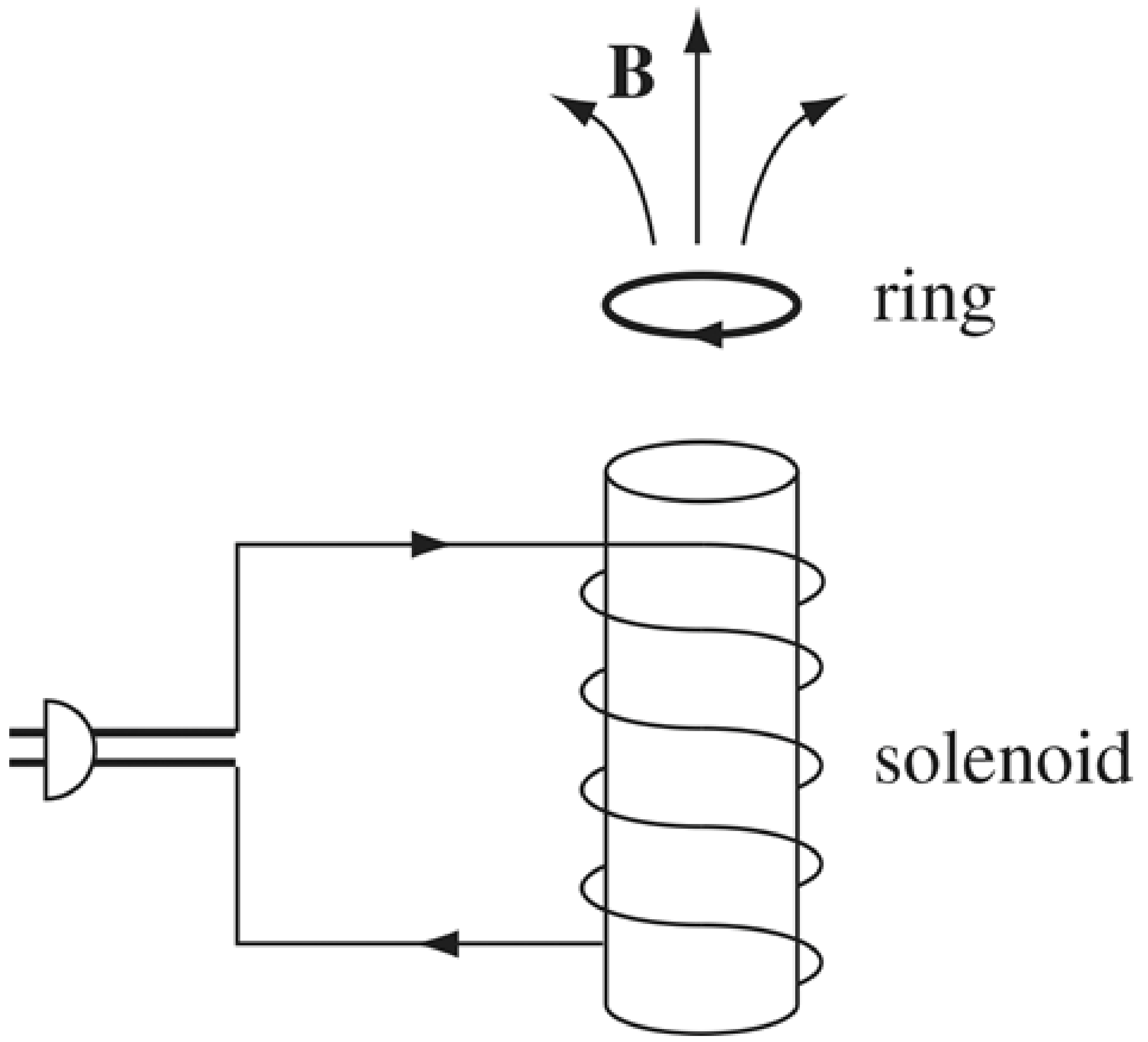
Nature abhors a change in flux.

Ex. Rogowski coils.

Example 7.6

The “jumping ring” demonstration. If you wind a solenoidal coil around an iron core (the iron is there to beef up the magnetic field), place a metal ring on the top, and plug it in, the ring will jump several feet in the air. Why?

Sol: Also see Chap.6 p.5.



7.2.2 The Induced Electric Field

Two distinct kinds of electric fields:

- **E** (in static case): attributed to electric charges, using Coulomb's law.
- **E** (in nonsteady case): associated with changing magnetic field, using Faraday's law.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \longleftarrow$$ (the curl alone is not enough to determine a field)

$$\nabla \cdot \mathbf{E} = 0 \longleftarrow$$ (charge free; then, the electric field is due *exclusively* to a changing **B**)

The Magnetostatic Field

Two distinct kinds of magnetic fields:

$\mathbf{B}$ (in static case): attributed to electric currents, using Ampere’s law.

$\mathbf{B}$ (in nonsteady case): associated with changing electric field, using?

$$\left.\begin{array}{c}\nabla\times\mathbf{B}=\mu_0\mathbf{J}\\ \nabla\cdot\mathbf{B}=0\end{array}\right\}$$

(Faraday-induced electric fields are determined by –($\partial\mathbf{B}/\partial t$) in exactly the same way as magnetostatic fields $\mathbf{B}$ are determined by $\mu_0\mathbf{J}$)

Example 7.7

The current in an ideal solenoid of radius R varies as a function of time. Find the induced electric field at points (a) inside, and (b) outside the solenoid. Express the results in terms of dB/dt.

Sol:

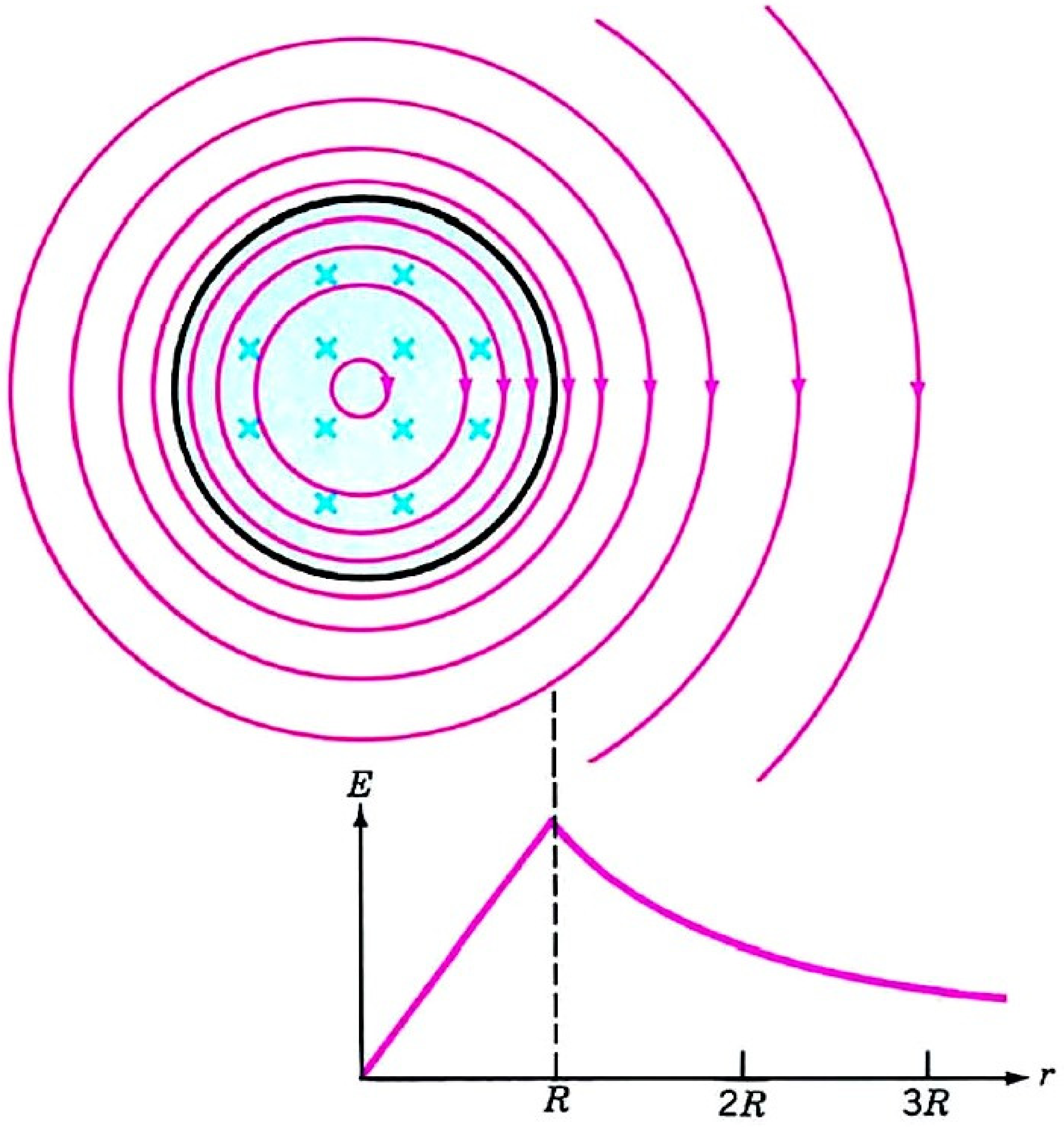
$$\oint \mathbf{E}\cdot d\mathbf{l} = E(2\pi r)$$

$$\text{(a)}\quad E(2\pi r) = -(\pi r^2)\frac{dB}{dt}$$

$$E = -\frac{r}{2}\frac{dB}{dt} \qquad (r < R)$$

$$\text{(b)}\quad E(2\pi r) = -(\pi R^2)\frac{dB}{dt}$$

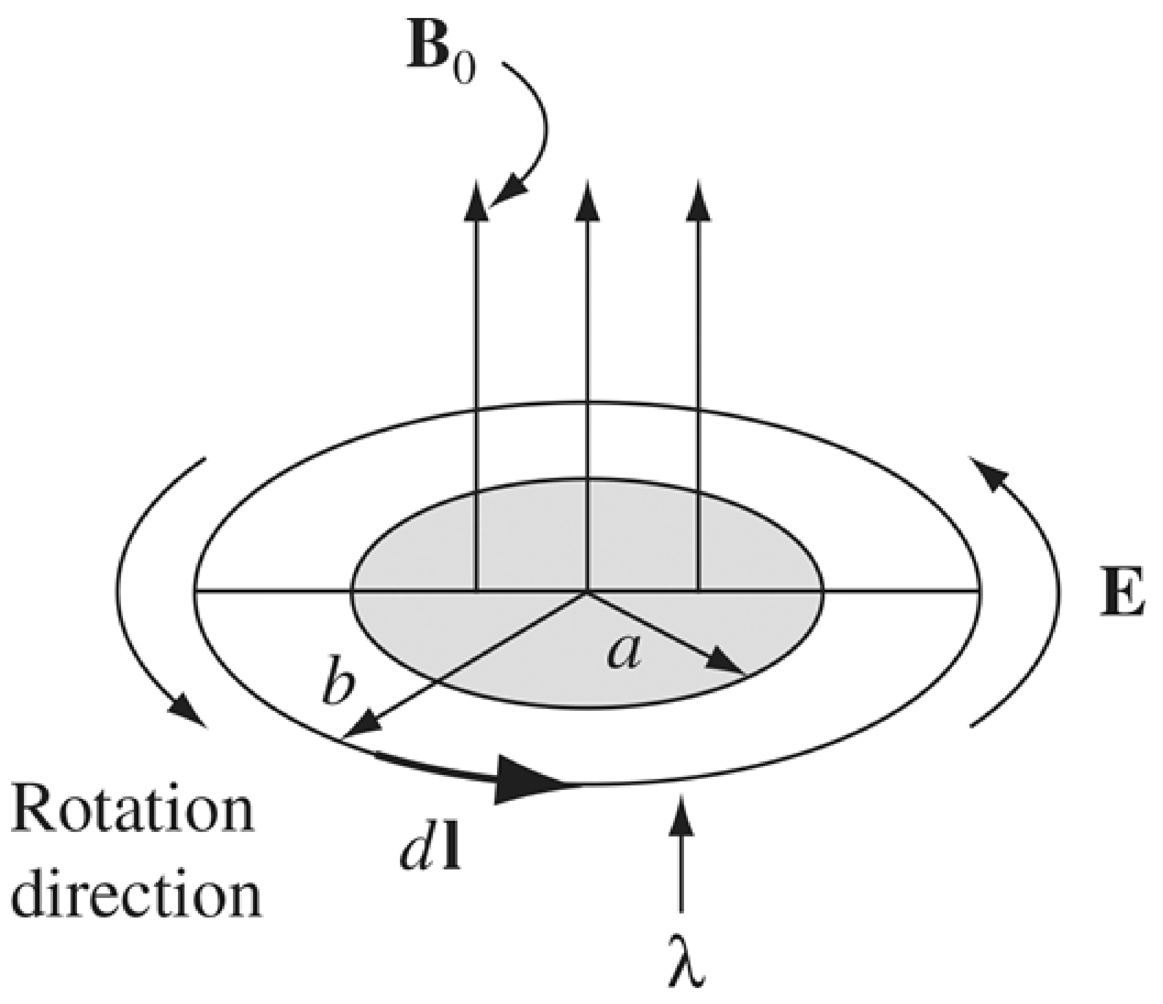
$$E = -\frac{R^2}{2r}\frac{dB}{dt} \qquad (r > R)$$



The electric field is due *exclusively* to a changing $\mathbf{B}$.

Example 7.8

A line charge λ is glued onto the rim of a wheel of radius b, which is then suspended horizontally as shown in the figure, so that it is free to rotate. In the central region, out to radius a, there is a uniform magnetic field B_0, pointing up. Now someone turns the field off. What happens?



Sol: Faraday's law says

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\pi a^2 \frac{dB}{dt}$$

The torque on a segment of length $d\mathbf{l}$ is $\mathbf{r}\times d\mathbf{F}$ or $b\lambda E dl$

The total torque on the wheel is $N = b\lambda \oint \mathbf{E}\cdot d\mathbf{l} = -b\lambda\pi a^2 \dfrac{dB}{dt}$

The angular momentum $L = \int_0^{t_0} N dt = \int_{B_0}^{0} -b\lambda\pi a^2 dB = b\lambda\pi a^2 B_0$

No matter how fast or slow you turn off the field, the ultimate angular velocity of the wheel is the same regardless.

Example 7.8 Quasistatic

$$N = b\lambda \oint \mathbf{E} \cdot d\mathbf{l} = -b\lambda \pi a^2 \frac{dB}{dt}$$

Ampere’s law (magnetostatics)

Faraday’s law (nonsteady)

KK:[rɪˈʒim]

This regime, in which magnetostatic rules can be used to calculate the magnetic field on the right hand side of the Faraday’s law, is called quasistatic.

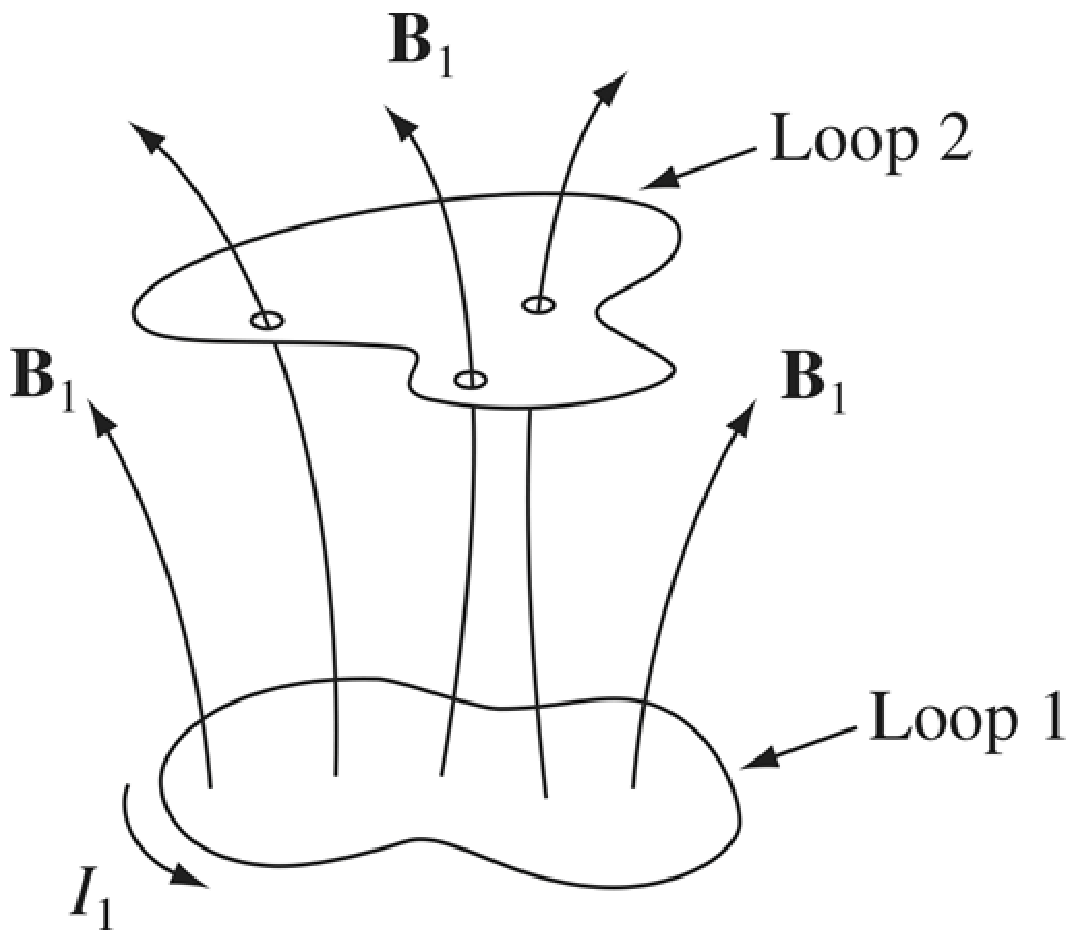
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Biot-Sarvart or Ampere’s laws

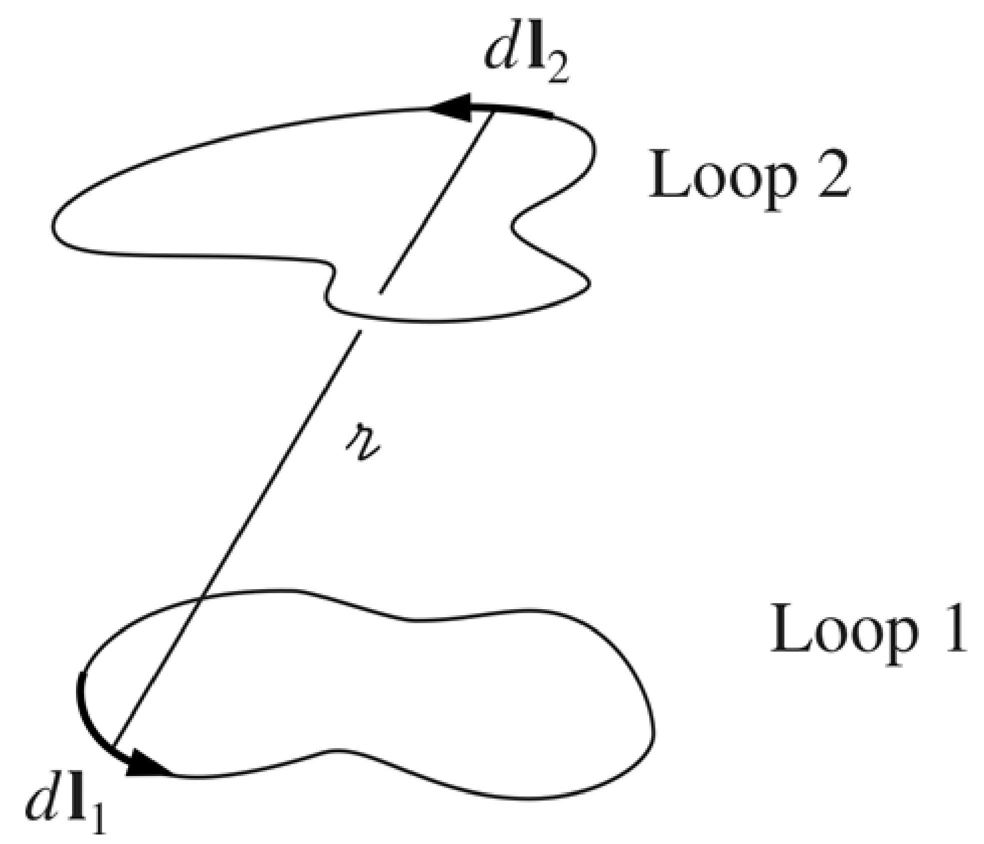
provided the field fluctuation is not extremely rapid

7.2.3 Inductance

Suppose we have two loops of wire at rest.
A steady current I_1 around loop 1➔ $\mathbf{B}_1$
Some $\mathbf{B}_1$ passes through loop 2 ➔ Φ_2



$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a} \quad \text{and} \quad \mathbf{B}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$



$$\Phi_2 = [\frac{\mu_0}{4\pi} \int \oint \frac{d\mathbf{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2} \cdot d\mathbf{a}] I_1 = M_{21} I_1$$

The constant of proportionality:
mutual inductance of the two loops.

Neumann Formula for the Mutual Inductance

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a} = \int (\nabla \times \mathbf{A}_1) \cdot d\mathbf{a} = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2$$

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{\mathscr{r}}$$

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}$$

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}} \quad \Leftarrow \text{ Neumann formula}$$

It involves a double line integral ---
one integration around loop 1, the
other around loop 2.

Important Things about Mutual Inductance

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}} \longleftarrow$$

It is not very useful for practical calculation, but it reveals two important features.

1. M_{21} is purely geometrical quantity, having to do with the size, shape, and relative position.

2. $M_{21} = M_{12}$, so we can drop the subscripts and call them M.

Whatever the shapes and positions of the loops, the flux through 2 when we run current I around 1 is identical to the flux through 1 when we send the same current I around 2.

Advantage of $M_{21} = M_{12}$, see the following examples.

Example (or Ex. 7.10)

A circular coil with a cross-sectional area of 4 cm^2 has 10 turns. It is placed at the center of a long solenoid that has 15 turns/cm and a cross-sectional area of 10 cm^2, as shown below. The axis of the coil coincides with the axis of the solenoid. What is their mutual inductance?

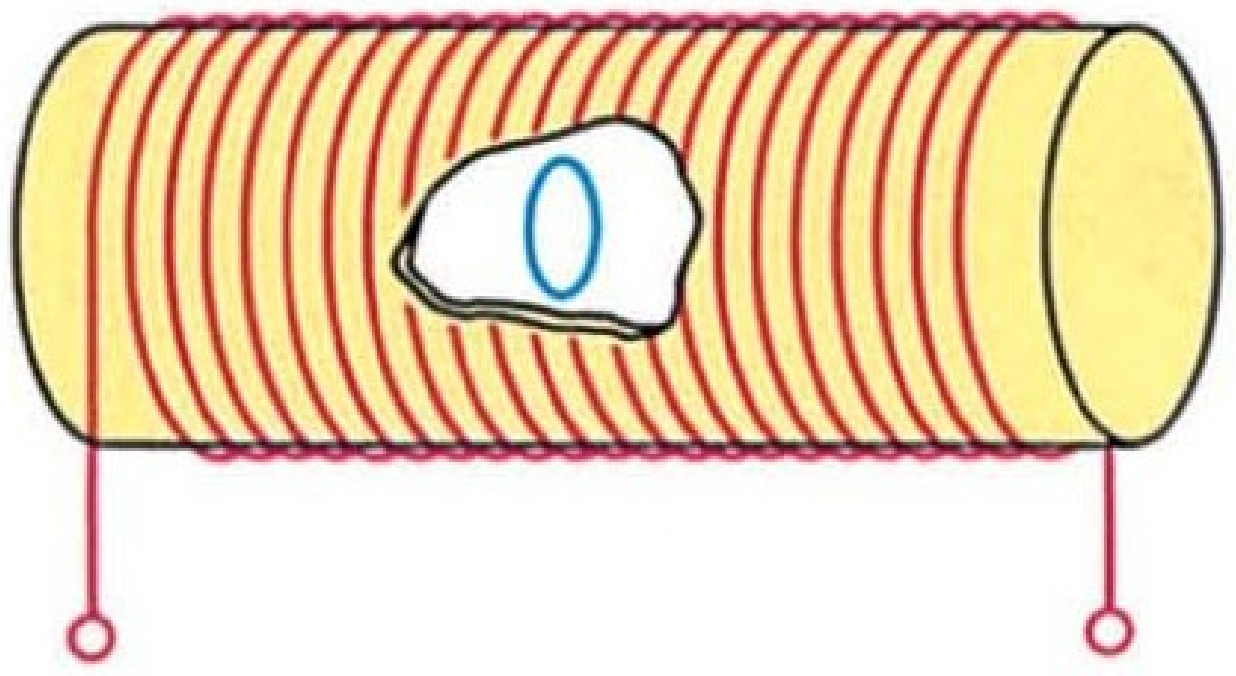
Solution:

$$\Phi_1 = B_2 A_1 = \mu_0 n_2 I_2 A_1$$

$$M = \frac{N_1 \Phi_1}{I_2} = \mu_0 n_2 N_1 A_1$$

$$= (4\pi \times 10^{-7})(1500)(10)(0.0004)$$

$$= 7.54\ \mu\text{H}$$

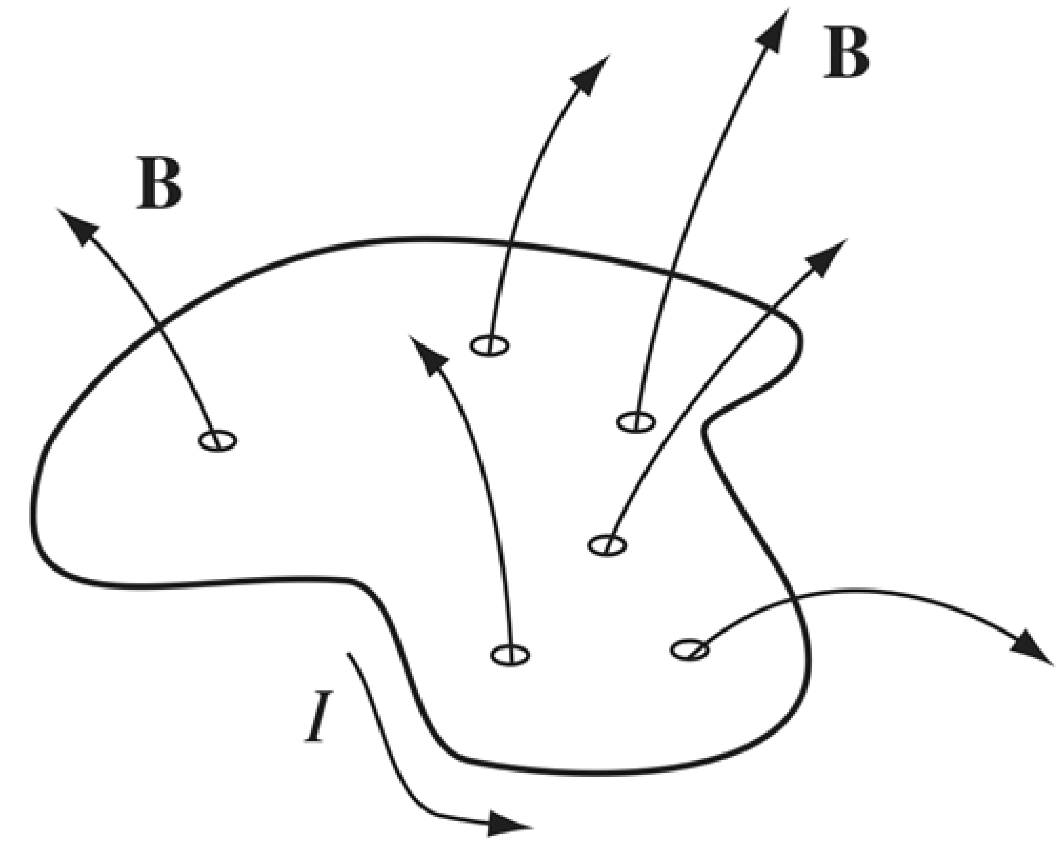


Notice that although $M_{12} = M_{21}$, it would have been much difficult to find Φ_2 because the field due to the coil is quite nonuniform.

Self-Inductance

It is convenient to express the induced emf in terms of a current rather than the magnetic flux through it.

The magnetic flux is directly proportional to the current flowing through it.



$$N_1\Phi_1 = L_1 I_1$$

where L_1 is a constant of proportionality called the self-inductance of coil 1. The SI unit of self-inductance is the henry (H). The self-inductance of a circuit depends on its size and its shape.

The self-induced emf in coil 1 due to changes in I_1 takes the form

$$\mathcal{E}_L = -L_1 \frac{dI_1}{dt}$$

Example 7.11 (toroidal)

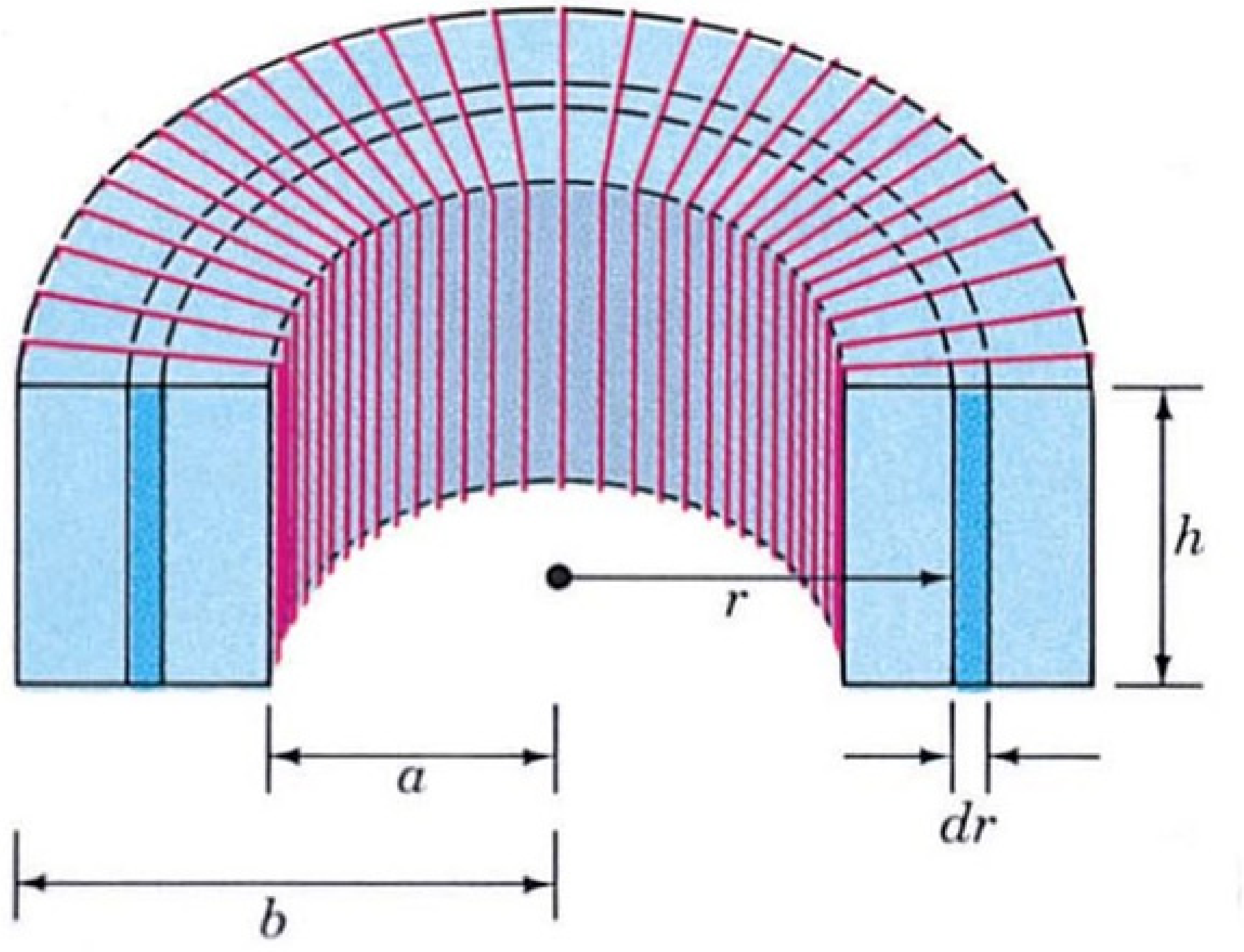
Find the self-inductance of a toroidal coil with rectangular cross section (inner radius a, outer radius b, height h), which carries a total N turns.

Sol: magnetic field inside a toroidal $B = \dfrac{\mu_0 N I}{2\pi s}$

$$L_1 = \frac{N\Phi_1}{I_1} \quad \text{and}$$

$$\Phi_1 = h\int_a^b \frac{\mu_0 N I_1}{2\pi s} ds = \frac{\mu_0 h N I_1}{2\pi} \ln(\frac{b}{a})$$

$$\therefore L_1 = \frac{\mu_0 h N^2}{2\pi} \ln(\frac{b}{a})$$



Example (coaxial, also see Ex. 7.13)

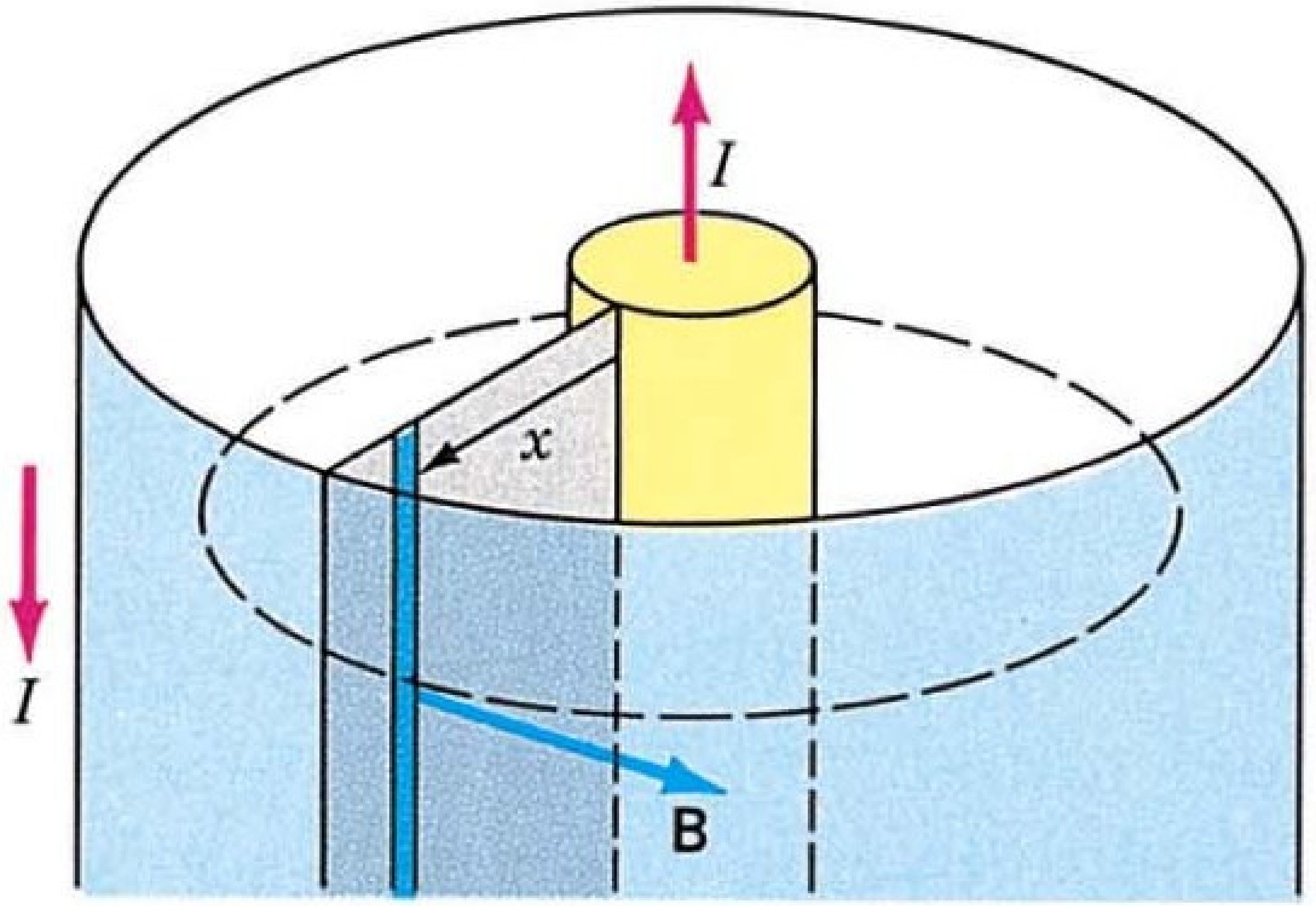
A coaxial cable consists of an inner wire of radius a that carries a current I upward, and an outer cylindrical conductor of radius b that carries the same current downward. Find the self-inductance of a coaxial cable of length ℓ. Ignore the magnetic flux within the inner wire.

Solution:

$$B = \frac{\mu_0 I}{2\pi x}, \qquad d\Phi = B dA = \frac{\mu_0 I}{2\pi x}\ell dx$$

$$\Phi = \int_a^b \frac{\mu_0 I}{2\pi x}\ell dx = \frac{\mu_0 I \ell}{2\pi}\ln\frac{b}{a} = LI$$

$$L = \frac{\mu_0 \ell}{2\pi}\ln\frac{b}{a}$$



Hint1: The direction of the magnetic field.

Hint2: What happens when considers the inner flux?

Example: *LR* Circuits

How does the current rise and fall as a function of time in a circuit containing an inductor and a resistor in series?

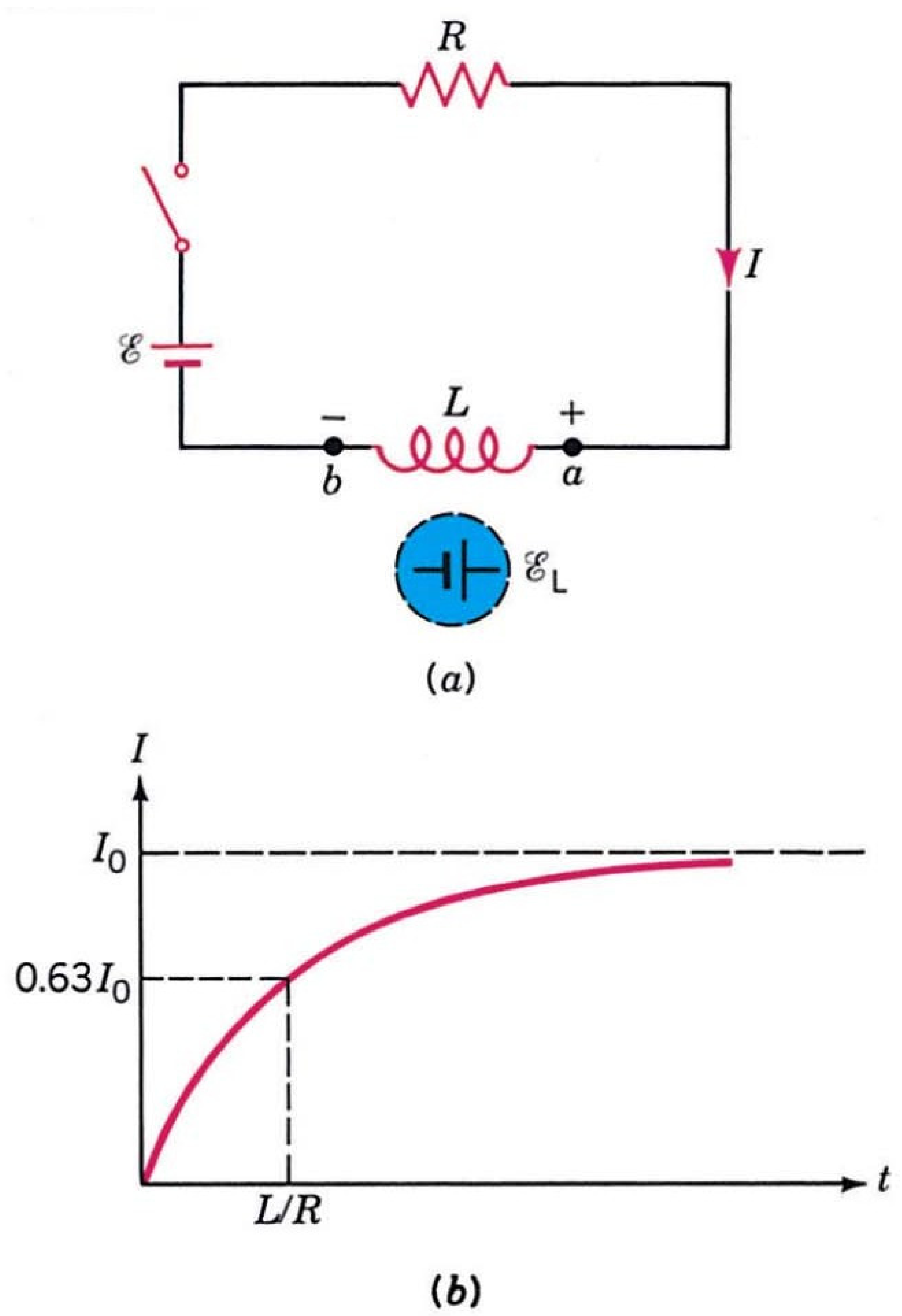
Rise

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

$$\text{Let } I = I_0 e^{-\alpha t} + \beta \Rightarrow \frac{dI}{dt} = -\alpha I_0 e^{-\alpha t}$$

$$\begin{cases} e^{-\alpha t} \quad : \quad \alpha = \dfrac{R}{L} \\ 0 \quad : \quad \mathcal{E} - R\beta = 0 \Rightarrow \beta = \dfrac{\mathcal{E}}{R} \\ t = 0 : \quad I_0 = -\beta = -\dfrac{\mathcal{E}}{R} \end{cases}$$

$$\therefore I = \frac{\mathcal{E}}{R}(1 - e^{-\frac{R}{L}t})$$



(*a*)

(*b*)

The quantity $\tau \equiv L/R$ is called the **time constant**.

Example: *LR* Circuits

Decay

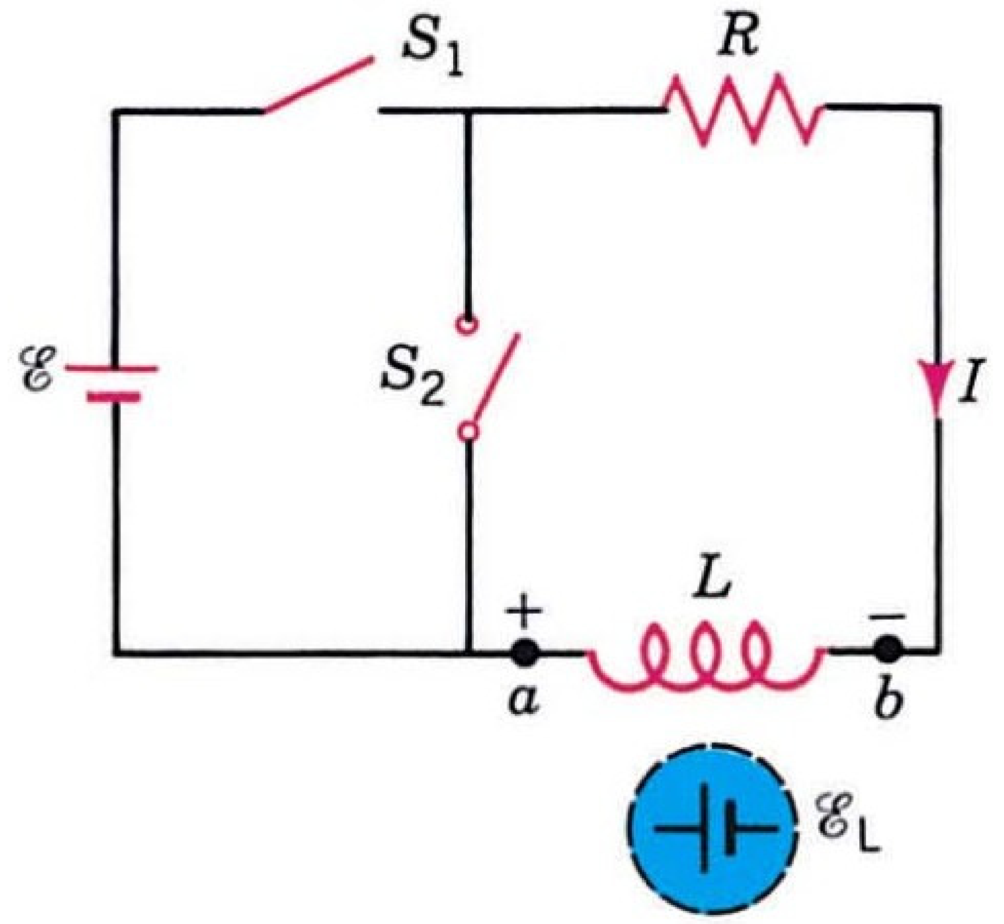
$t \le 0$ S_1 closed and S_2 open
$t > 0$ S_1 open and S_2 closed

$$-IR - L\frac{dI}{dt} = 0$$

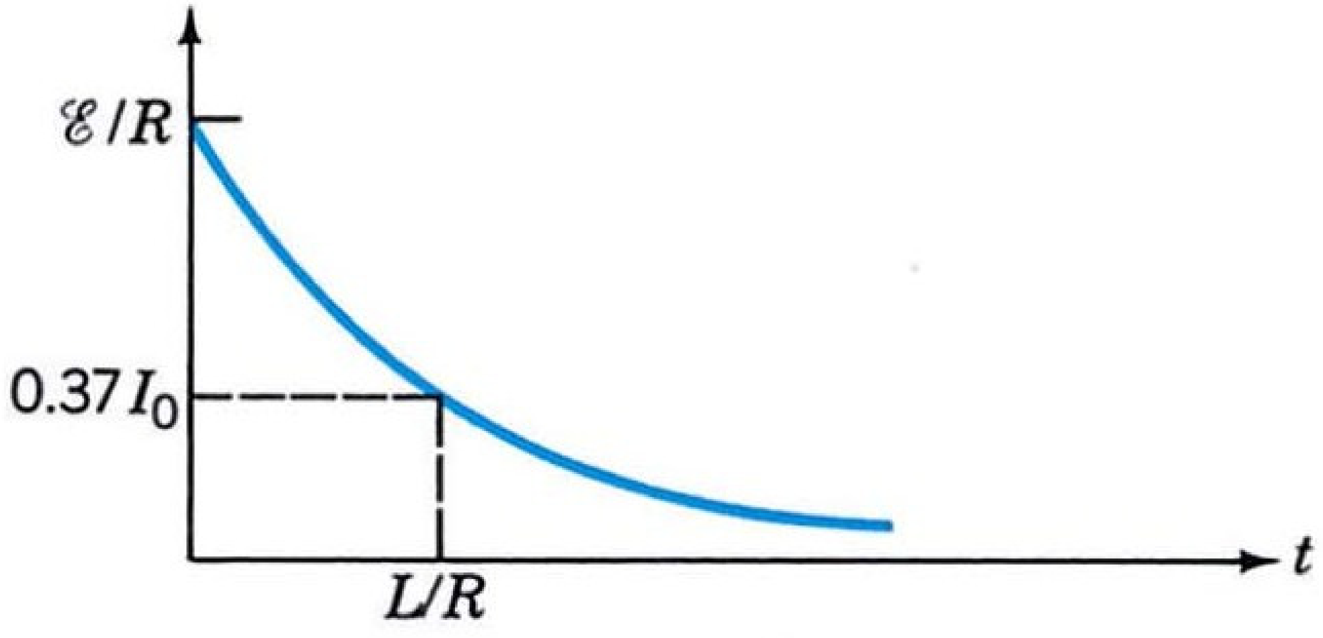
$$\text{Let } I = I_0 e^{-\alpha t} \Rightarrow \frac{dI}{dt} = -\alpha I_0 e^{-\alpha t}$$

$$\begin{cases} e^{-\alpha t} \quad : \quad \alpha = \dfrac{R}{L} \\ t = 0: \quad I_0 = \dfrac{\mathcal{E}}{R} \end{cases}$$

$$\therefore I = \frac{\mathcal{E}}{R} e^{-\frac{R}{L}t} = \frac{\mathcal{E}}{R} e^{-\frac{t}{\tau}}$$



(*a*)



(*b*)

The quantity $\tau \equiv L/R$ is called the **time constant**.

7.2.4 Energy in Magnetic Field

Inductance (like capacitance) is an *intrinsically positive* quantity. Lenz's law dictates that the emf is in such a direction as to oppose any change in current. ➔ **back emf**.

It takes a certain amount of energy to start a current flowing in a circuit.

What we are concerned with are the work you must do against the back emf to get the current going.

Is this a fixed amount? Is it recoverable?

Yes, you get it back when the current is turned off.

It represents energy latent in the circuit or it can be regard as energy stored in the magnetic field.

Energy Stored in an Inductor

The battery that establishes the current in an inductor has to do work against the opposing induced emf. The energy supplied by the battery is stored in the inductor.

In Kirchhoff's voltage law (KVL), we obtain

$$\mathcal{E} = IR + L\frac{dI}{dt}$$

$$I\mathcal{E} = I^2R + LI\frac{dI}{dt}$$

$$I\mathcal{E} = I^2R + \frac{dU_L}{dt}, \quad \text{where } U_L = \frac{1}{2}LI^2$$

power supplied by the battery

power dissipated in the resistor

energy change rate in the inductor

The Power

The work done on a unit charge, against the back emf, in one trip around the circuit is $-\mathcal{E}$.

the work done by you against the emf.

The total work done per unit time is

$$\frac{dW}{dt} = \frac{d(-\mathcal{E}Q)}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}$$

The total work is $$W = \int_0^{I_0} LI\,dI = \frac{1}{2}LI_0^2$$

Depends only on the geometry of the loop (in the form of L) and the final current I_0.

Energy Density of the Magnetic Field

We have expressed the total energy stored in the inductor in terms of the current and we know the magnetic field is proportional to the current. Can we express the total magnetic energy in terms of the B-field? Yes.

Let's consider the case of solenoid.

$$\overbrace{n\ell}^{N}\overbrace{\mu_0 nIA}^{\Phi} = LI \quad \Rightarrow \quad L = \mu_0 n^2 A\ell$$

$$U_L = \frac{1}{2}LI^2 = \frac{1}{2\mu_0}(\mu_0 nI)^2 A\ell = \frac{B^2}{2\mu_0}A\ell$$

$$u_B = \frac{B^2}{2\mu_0} \text{(The energy density of a magnetic field in free space)}$$

Although this relation has been obtained from a special case, the expression is valid for any magnetic field.

Generalized Total Energy

There is a nicer way to write the total magnetic energy W.

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{a} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint_P \mathbf{A}\cdot d\mathbf{l} = LI$$

S: surface bounded by P P: perimeter of the loop

$$W = \frac{1}{2}LI^2 = \frac{1}{2}\Phi I = \frac{1}{2}I\oint_P \mathbf{A}\cdot d\mathbf{l} = \frac{1}{2}\oint_P (\mathbf{A}\cdot\mathbf{I})dl$$

generalize to the volume current

$$W = \frac{1}{2}\oint_P (\mathbf{A}\cdot\mathbf{I})dl = \frac{1}{2}\int_V (\mathbf{A}\cdot\mathbf{J})d\tau, \quad \text{where } \mathbf{J} = \frac{1}{\mu_0}\nabla\times\mathbf{B}$$

$$W = \frac{1}{2}\int_V (\mathbf{A}\cdot\mathbf{J})d\tau = \frac{1}{2\mu_0}\int_V [\mathbf{A}\cdot(\nabla\times\mathbf{B})]d\tau$$

Generalized Total Energy II

Product rule 6, $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

$$\mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot \underbrace{(\nabla \times \mathbf{A})}_{\mathbf{B}} - \nabla \cdot (\mathbf{A} \times \mathbf{B})$$

$$W = \frac{1}{2\mu_0} \int_V [\mathbf{A} \cdot (\nabla \times \mathbf{B})] d\tau = \frac{1}{2\mu_0} \int_V [\mathbf{B} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{B})] d\tau$$

divergence theorem

$$= \frac{1}{2\mu_0} \int_V B^2 d\tau - \frac{1}{2\mu_0} \oint_S (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a}$$

$$V \to \text{all space} \quad \frac{1}{2\mu_0} \oint_S (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a} \to 0$$

$$W = \frac{1}{2\mu_0} \int_{\text{all space}} B^2 d\tau$$

Electric and Magnetic Field Energy

Electric field energy

energy density

$$W_{\text{elec}} = \frac{1}{2}\int (V\rho)d\tau = \frac{\varepsilon_0}{2}\int E^2 d\tau, \qquad u_E = \frac{\varepsilon_0}{2}E^2$$

Magnetic field energy

$$W_{\text{mag}} = \frac{1}{2}\int (\mathbf{A}\cdot\mathbf{J})d\tau = \frac{1}{2\mu_0}\int B^2 d\tau, \quad u_B = \frac{1}{2\mu_0}B^2$$

Magnetic fields themselves do no work. Where does the energy come from?

A changing magnetic field induces an electric field which can do work.

Example

The breakdown electric field strength of air is 3x10^6 V/m. A very large magnetic field strength is 20 T. Compare the energy densities of the field.

Solution:

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = (0.5)(8.85\times10^{-12})(3\times10^6)^2$$

$$= 40 \text{ J/m}^3$$

$$u_B = \frac{1}{2\mu_0}B^2 = \frac{20^2}{2\times4\pi\times10^{-7}}$$

$$= 3.2\times10^8 \text{ J/m}^3$$

Magnetic fields are an effective means of storing energy without breakdown of the air. However, it is difficult to produce such large fields over large regions.

Example (toroidal, Ex. 7.11)

Use the expression for the energy density of the magnetic field to calculate the self-inductance of a toroid with a rectangular cross section.

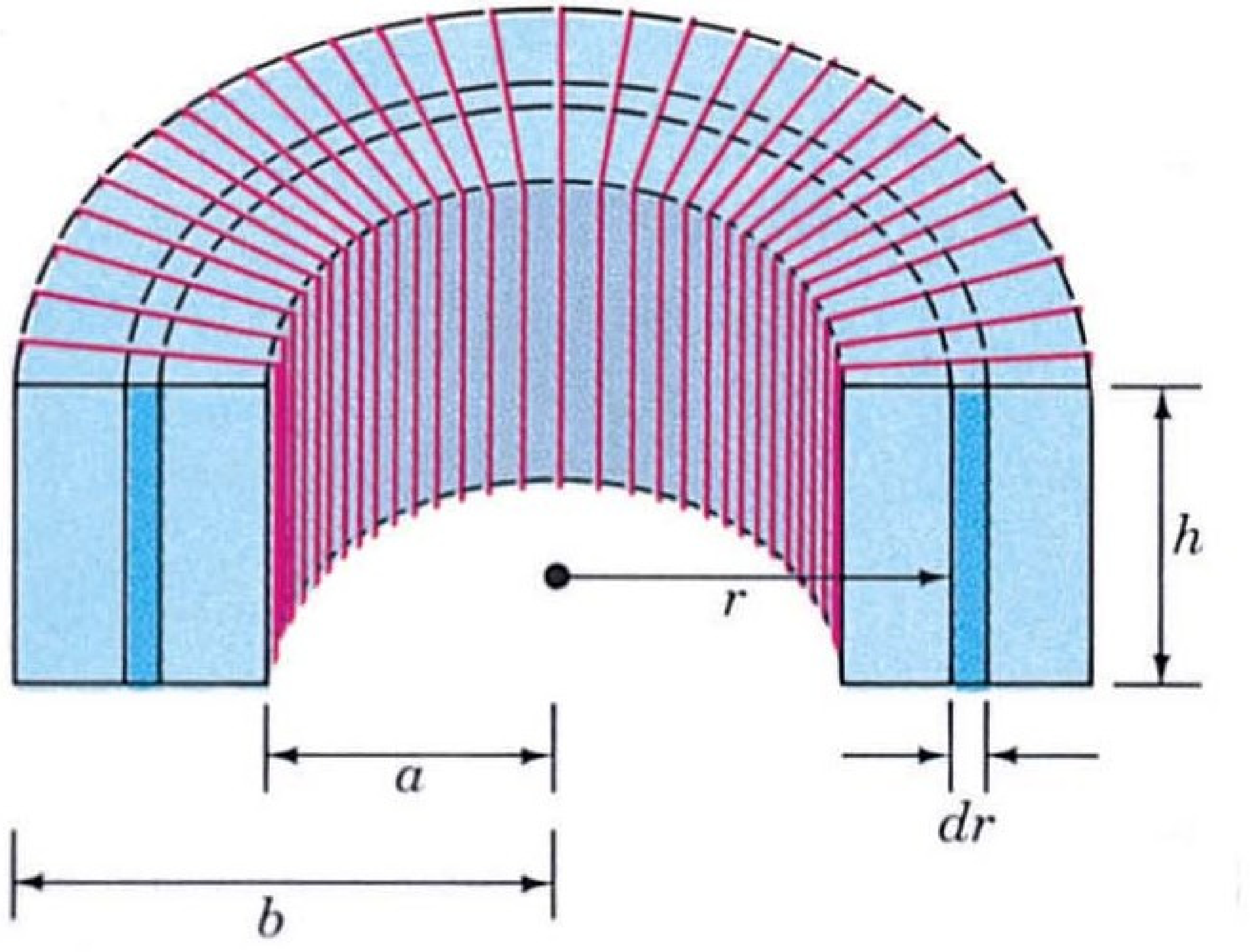
Solution:

$$B = \frac{\mu_0 NI}{2\pi r}$$

$$dU_B = \frac{B^2}{2\mu_0} d\tau = \frac{B^2}{2\mu_0} h(2\pi r dr) = \frac{\mu_0 h(NI)^2}{4\pi r} dr$$

$$U_B = \int_a^b \frac{\mu_0 h(NI)^2}{4\pi r} dr = \frac{\mu_0 h N^2 I^2}{4\pi} \ln(\frac{b}{a}) = \frac{1}{2} LI^2$$

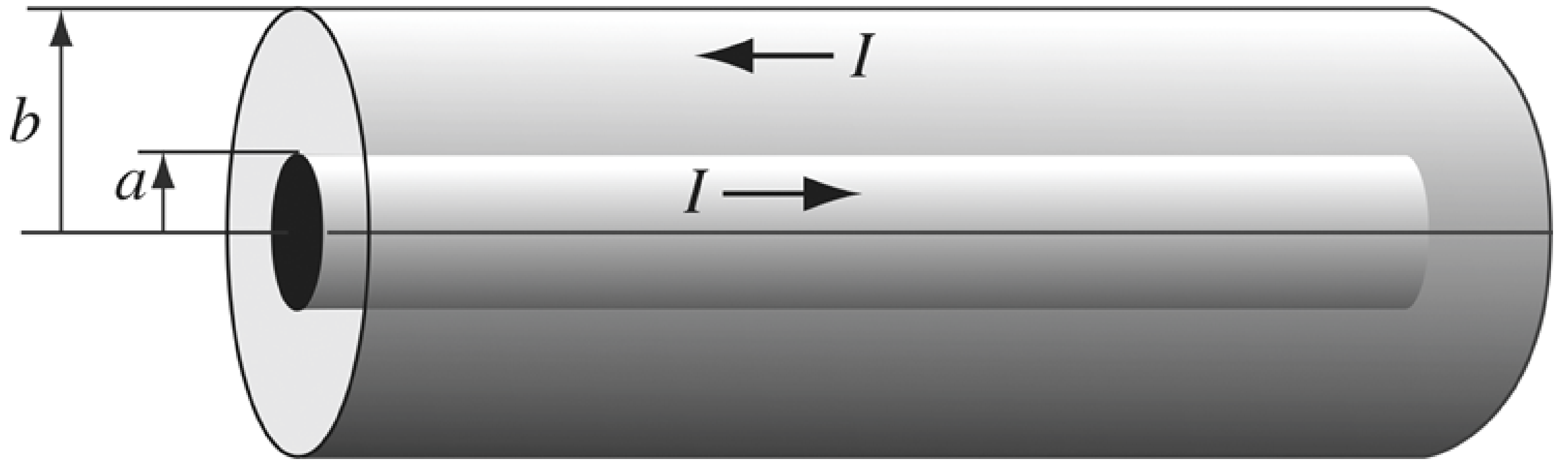
$$L = \frac{\mu_0 N^2 h}{2\pi} \ln(\frac{b}{a})$$



Can we use the concept of magnetic flux to derive the self-inductance? See Ex. 7.11.

Example 7.13 (coaxial)

A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, radius a, and back along the outer cylinder, radius b) as shown in the figure. Find the magnetic energy stored in a section of length ℓ.



Sol:

magnetic field $\mathbf{B} = \dfrac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$

energy density $u_B = \dfrac{1}{2\mu_0}B^2 = \dfrac{\mu_0 I^2}{8\pi^2 s^2}$

magnetic energy $W_B = \int_V u_B d\tau = \int_V \dfrac{\mu_0 I^2}{8\pi^2 s^2}\ell 2\pi s ds = \dfrac{\mu_0 I^2}{4\pi}\ell \ln(\dfrac{b}{a})$

self-inductance $W_B = \dfrac{1}{2}LI^2 \quad \Rightarrow \quad L = \dfrac{\mu_0 \ell}{2\pi}\ln(\dfrac{b}{a})$

Homework of Chap.7 (part II)

Problem 7.18 A square loop, side a, resistance R, lies a distance s from an infinite straight wire that carries current I (Fig. 7.29). Now someone cuts the wire, so I drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during the time this current flows? If you don't like the scissors model, turn the current down *gradually*:

$$I(t)=\begin{cases}(1-\alpha t)I, & \text{for } 0 \le t \le 1/\alpha, \\ 0, & \text{for } \ t \le 1/\alpha.\end{cases}$$

Problem 7.24 Find the self-inductance per unit length of a long solenoid, of radius R, carrying n turns per unit length.

Problem 7.27 A capacitor C is charged up to a voltage V and connected to an inductor L, as shown schematically in Fig. 7.39. At time $t = 0$, the switch S is closed. Find the current in the circuit as a function of time. How does your answer change if a resistor R is included in series with C and L?

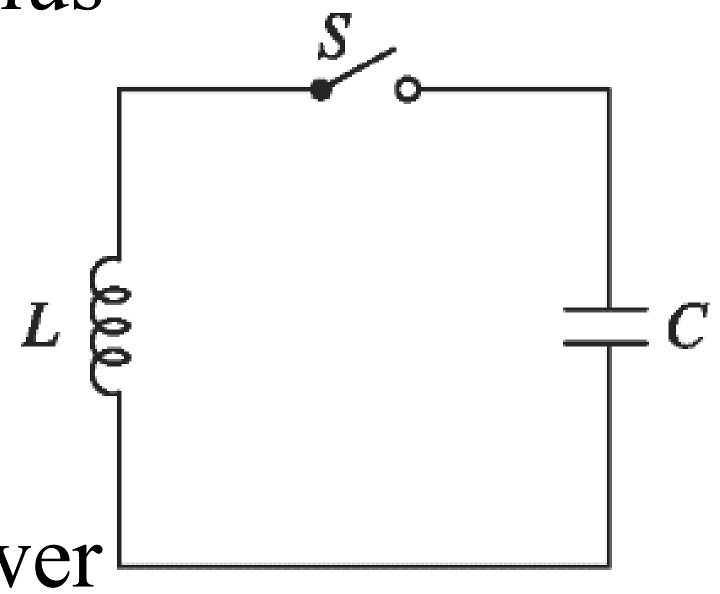


FIGURE 7.39

Problem 7.28 Find the energy stored in a section of length l of a long solenoid (radius R, current I, n turns per unit length), (a) using Eq. 7.30 (you found L in Prob. 7.24); (b) using Eq. 7.31 (we worked out **A** in Ex. 5.12); (c) using Eq. 7.35; (d) using Eq. 7.34 (take as your volume the cylindrical tube from radius $a < R$ out to radius $b > R$).

7.3 Maxwell’s Equations

7.3.1 Electrodynamics before Maxwell

$$\left\{\begin{aligned} &\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho && \text{(Gauss's law)} \\ &\nabla\cdot\mathbf{B} = 0 && \text{(no name)} \\ &\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} && \text{(Faraday's law)} \\ &\nabla\times\mathbf{B} = \mu_0\mathbf{J} && \text{(Ampere's law)} \end{aligned}\right.$$

electromagnetic theory
over a century ago

A fatal inconsistency in Ampere’s law

$$\underset{\substack{\downarrow \\ =0}}{\nabla\cdot(\nabla\times\mathbf{B})} = \mu_0 \underset{\substack{\downarrow \\ \neq 0}}{\nabla\cdot\mathbf{J}}$$

Ampere’s law is incorrect for the nonsteady current.

The Electric and Magnetic Fields

Two distinct kinds of electric fields:

E (in static case): attributed to electric charges, using Coulomb's law.

E (in nonsteady case): associated with changing magnetic field, using Faraday's law.

Two distinct kinds of magnetic fields:

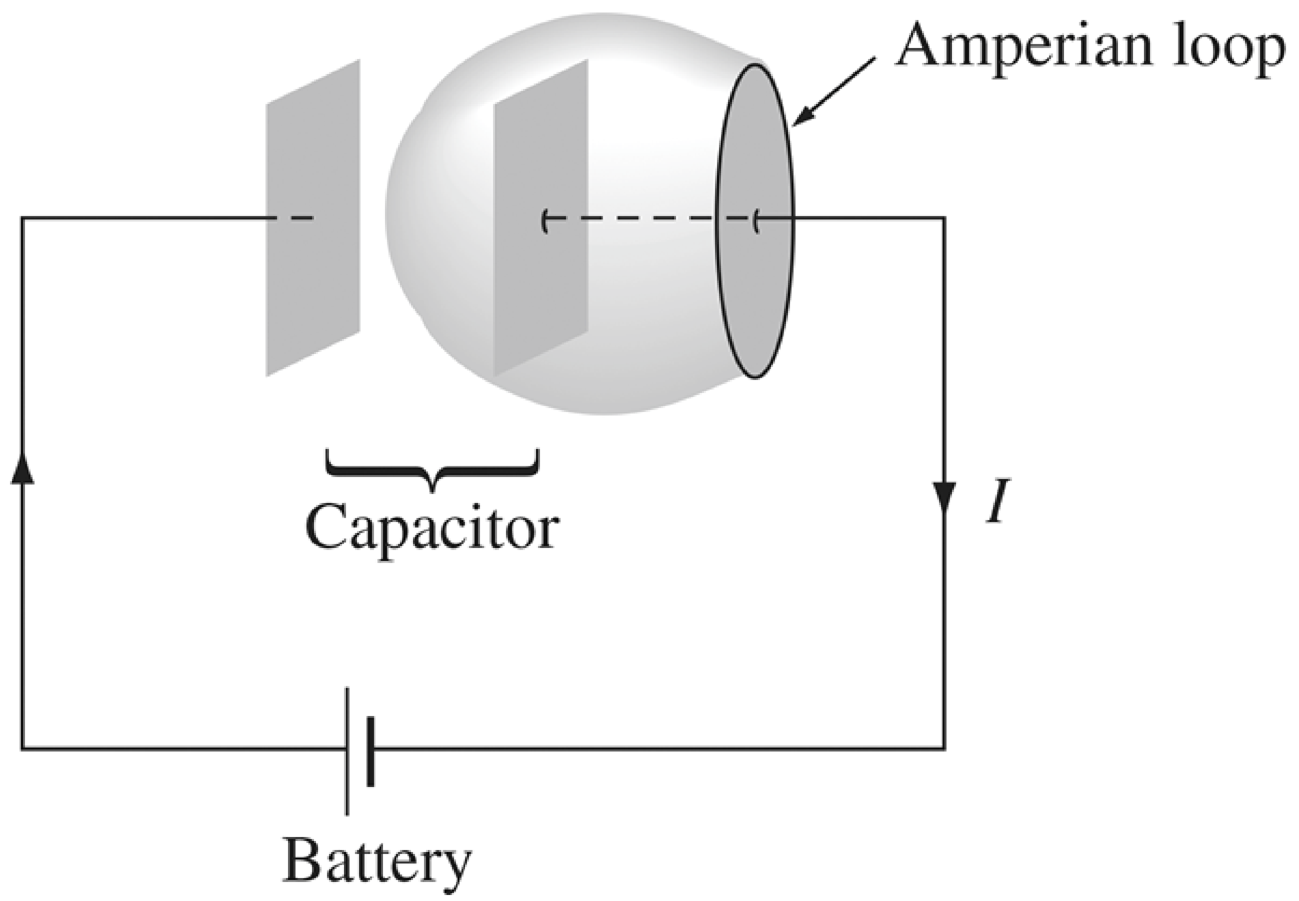
B (in static case): attributed to electric currents, using Ampere's law.

B (in nonsteady case): associated with changing electric field, using?

Another Inconsistency of Ampere’s Law

How do we determine the enclosed current I_{enc}?

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$



- * The simplest surface---the wire puncture this surface so $I_{\text{enc}} = I$ 🡐 Ampere’s law is ok.
- * A balloon-shaped surface---no current passes through this surface. so $I_{\text{enc}} = 0$ 🡐 Ampere’s law is not valid!

For nonsteady current, “the current enclosed by a loop” is ill-defined.

How Maxwell Fixed Ampere's Law

Applying the continuity equation and Gauss's law, the offending term can be rewritten:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial(\varepsilon_0 \nabla \cdot \mathbf{E})}{\partial t} = \nabla \cdot (-\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

$$\text{A new current } \mathbf{J}' = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \leftarrow \text{ kills off the extra divergence}$$

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 (\nabla \cdot \mathbf{J}') = \mu_0 \nabla \cdot (\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}) = 0$$

When $\mathbf{E}$ is constant (electrostatic+magnetostatic), we will have $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$.

$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ plays a crucial role in the EM wave propagation.

Electric Analogy of Faraday's Law

Maxwell's term cures the defect in Ampere's law,
and moreover, it has a certain aesthetic appeal.

Faraday's law ↘

A changing magnetic field induces an electric field.

A changing electric field induces a magnetic field.

Maxwell called this extra term "the displacement current".

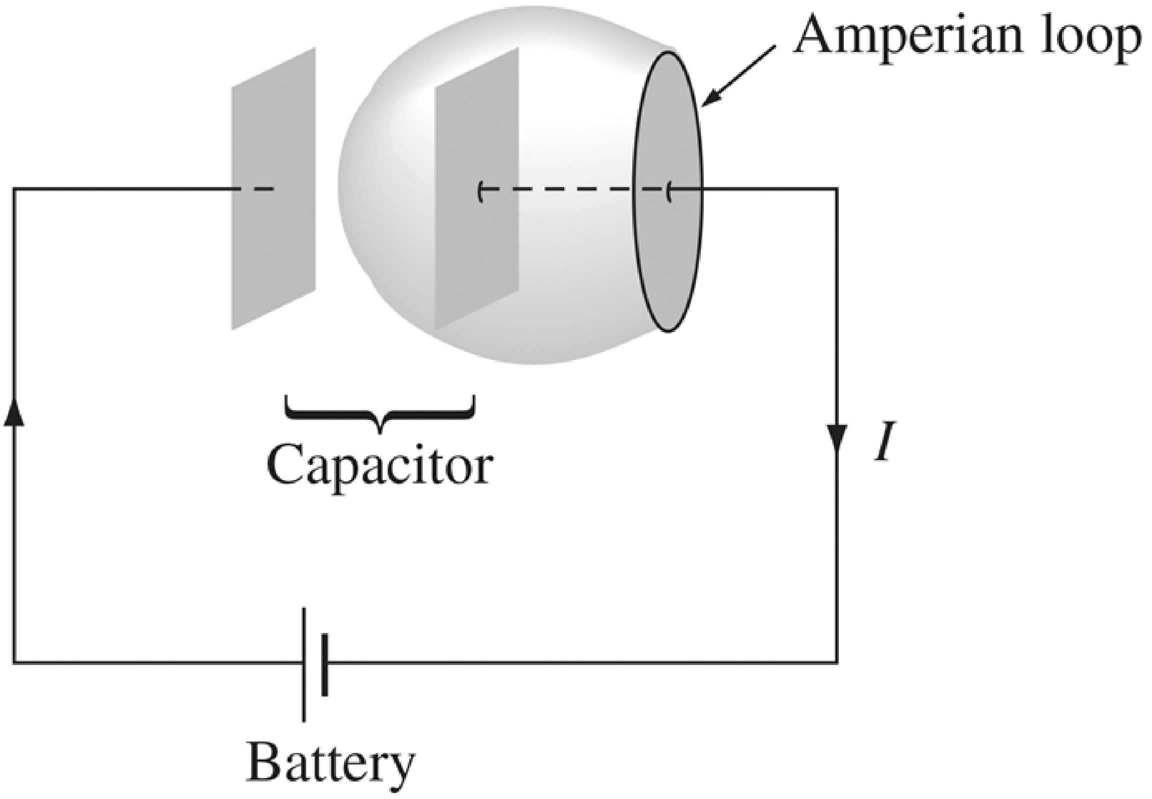
↑

$$\mathbf{J}_d \equiv \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

a misleading name,
nothing to do with current

The Displacement Current

How the displacement current resolves the paradox of the charging capacitor.



The electric field between the two capacitor plates is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{Q}{A}$$

← the charge on the plate

← the area of the plate

$$\varepsilon_0 \frac{\partial E}{\partial t} = \frac{1}{A}\frac{\partial Q}{\partial t} = \frac{I}{A} = J$$

$$\mathbf{J}_{tot} = \mathbf{J} + \mathbf{J}_d \quad \begin{cases} |\mathbf{J}| = \mathrm{J},\ |\mathbf{J}_d| = 0 \text{ at the flat surface} \\ |\mathbf{J}| = 0,\ |\mathbf{J}_d| = \mathrm{J} \text{ at the balloon-shaped surface} \end{cases}$$

7.3.3 Maxwell's Equations

Maxwell's equations in the traditional way.

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\varepsilon_0}\rho & \text{(Gauss's law)} \\ \nabla \cdot \mathbf{B} = 0 & \text{(no name)} \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{(Faraday's law)} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} & \text{(Ampere's law with Maxwell's correction)} \end{cases}$$

Lorentz force law $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

Continuity equation $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t}$

Maxwell's Equations (II)

Another expression of the Maxwell equations.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

The fields ($\mathbf{E}$ and $\mathbf{B}$) on the left
and the sources (ρ and $\mathbf{J}$) on the right.

Maxwell's equations tell you how sources produce fields; reciprocally, the Lorentz force law tells you how fields affect sources. ← **A nonlinear feedback**

Optional

7.3.4 Magnetic Charge

If there is a magnetic “charge” ρ_m and the corresponding current of the magnetic “current” $\mathbf{J}_m$, the Maxwell’s equations read

Why?

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\varepsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \mathbf{J}_m$$

$$\nabla \cdot \mathbf{B} = \mu_0 \rho_m \qquad \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}_e$$

A symmetric between **E** and **B**

$\mathbf{E} \rightarrow \mathbf{B}$

$\mathbf{B} \rightarrow -\mu_0 \varepsilon_0 \mathbf{E}$

Both charges would be conserved:

$$\nabla \cdot \mathbf{J}_e = -\frac{\partial \rho_e}{\partial t}, \text{ and } \nabla \cdot \mathbf{J}_m = -\frac{\partial \rho_m}{\partial t}$$

Q: Has anyone ever found the magnetic charge?

No.

7.3.5 Maxwell’s Equations in Matter

When working with materials that are subject to electric and magnetic polarization, there is a more convenient way to write the Maxwell equations.

Static case:

An electric polarization produces a bound charge: $\rho_b = -\nabla \cdot \mathbf{P}$

A magnetic polarization results in a bound current: $\mathbf{J}_b = \nabla \times \mathbf{M}$

Nonstatic case:

Any change in the electric polarization involves a flow of bound charge.

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp \quad \text{where } \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$$

polarization current
(nothing to do with the *bound* current).

Polarization and Bound Currents

Bound current $\mathbf{J}_b$: magnetization of the material involving the spin and orbital motion of electrons.

Polarization current $\mathbf{J}_p$: the linear motion of charge when the electric polarization changes.

Now $\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}$

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$$

Gauss's law: $\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}(\rho_f - \nabla \cdot \mathbf{P}) \quad \Rightarrow \quad \nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$

Ampere's law: $\nabla \times \mathbf{B} = \mu_0 (\mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

$$\Rightarrow \nabla \times (\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}) = \mathbf{J}_f + \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E} + \mathbf{P})$$

Maxwell’s Equations in Matter

In terms of free charges and currents, Maxwell’s equations read

$$\nabla \cdot \mathbf{D} = \rho_f \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_f$$

The constitutive relations:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

$$\mathbf{M} = \chi_m \mathbf{H}$$

So

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad \Rightarrow \quad \mathbf{B} = \mu_0 (1 + \chi_m) \mathbf{H} = \mu \mathbf{H}$$

7.3.6 Boundary Conditions (I)

Differential form

$$\nabla \cdot \mathbf{D} = \rho_f$$

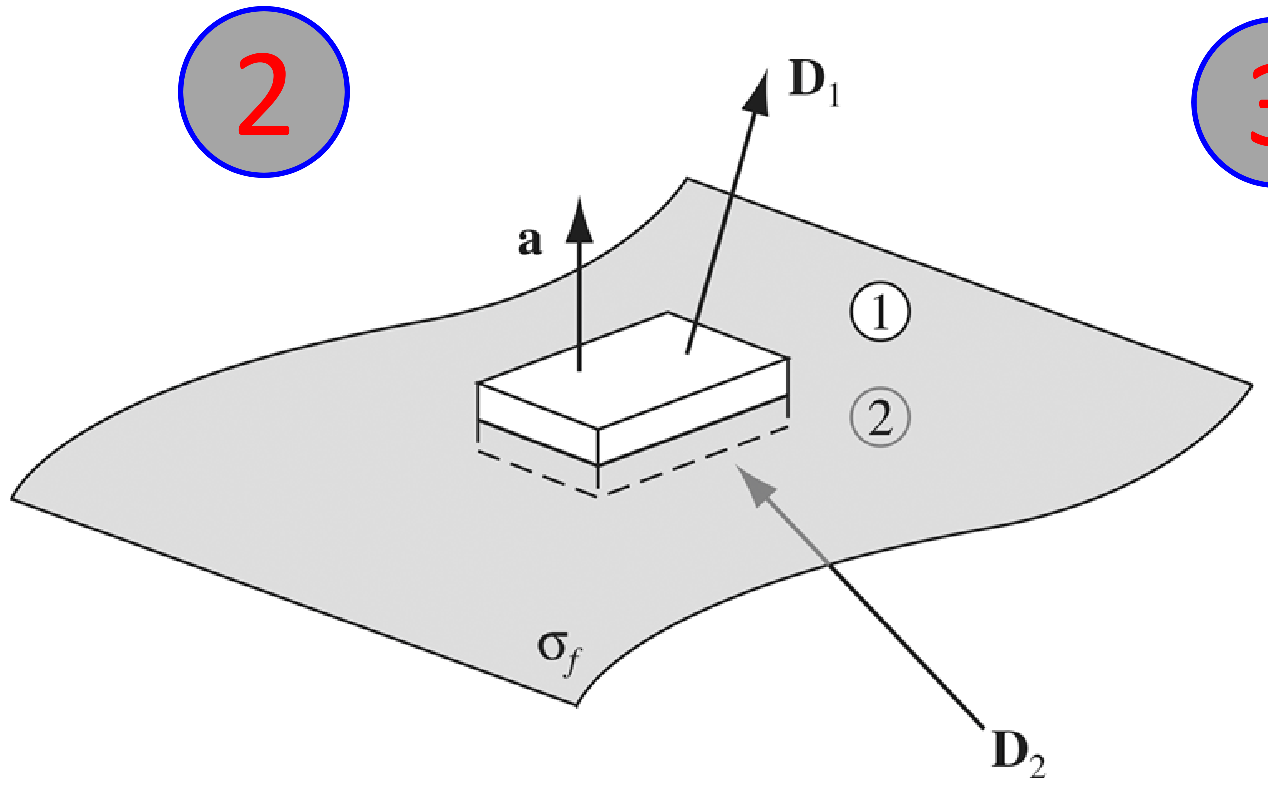
$$\nabla \cdot \mathbf{B} = 0$$

Integral form

①

$$\left.\begin{aligned} \oint_S \mathbf{D} \cdot d\mathbf{a} &= q_f \\ \oint_S \mathbf{B} \cdot d\mathbf{a} &= 0 \end{aligned}\right\} \text{ over any enclosed surface } S.$$

②



wafer thin
Gaussian pillbox

③ The edge of the wafer contributes nothing in the limit as the thickness goes to zero.

④ ⑤

$$\mathbf{D}_1 \cdot \mathbf{a} - \mathbf{D}_2 \cdot \mathbf{a} = \sigma_f a \quad \Rightarrow \quad \underline{\underline{D_1^\perp - D_2^\perp = \sigma_f}} \ \sharp$$

④ ⑤

$$\mathbf{B}_1 \cdot \mathbf{a} - \mathbf{B}_2 \cdot \mathbf{a} = 0 \quad \Rightarrow \quad \underline{\underline{B_1^\perp - B_2^\perp = 0}} \ \sharp$$

Boundary Conditions (II)

Differential form

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$$

$$\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=\mathbf{J}_f$$

Integral form

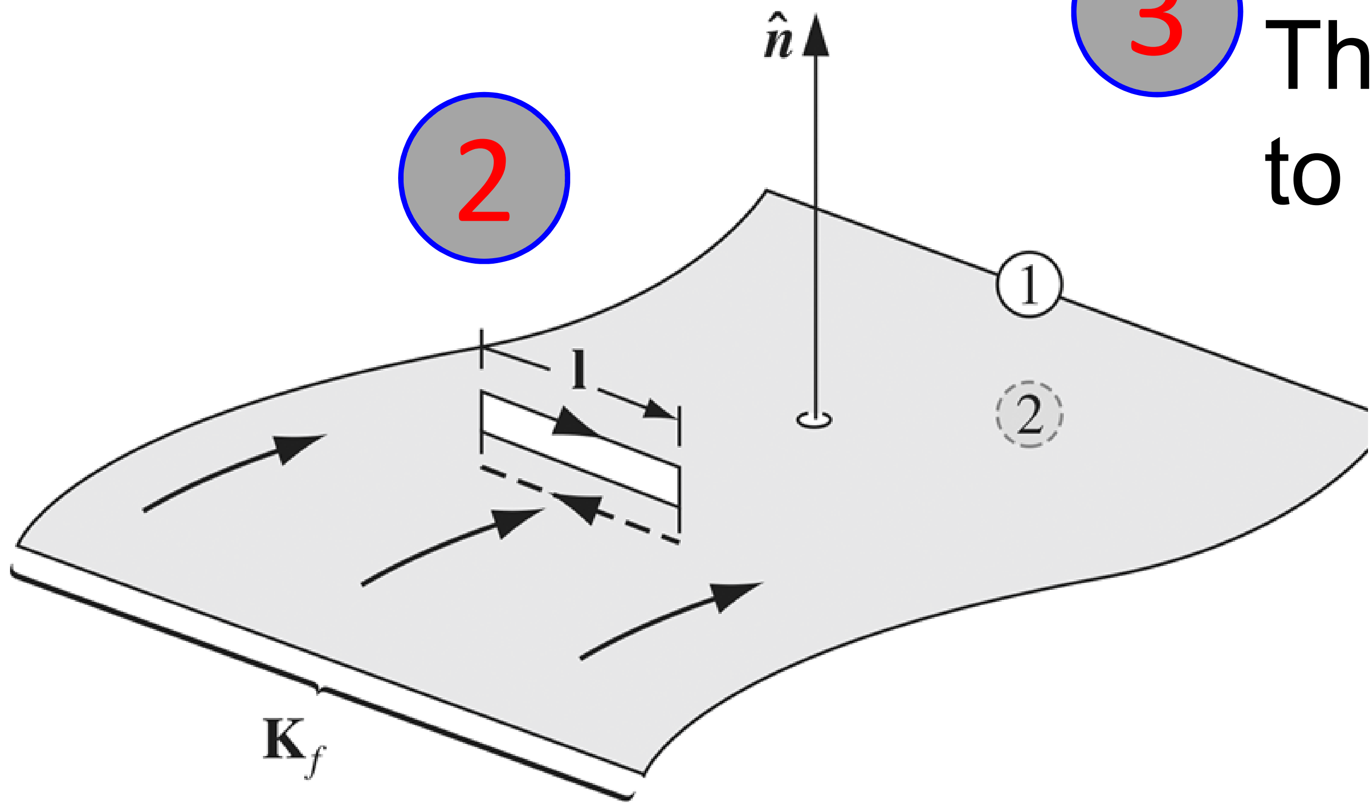
①

$$\oint_P \mathbf{E}\cdot d\mathbf{l}=-\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{a}$$

$$\oint_P \mathbf{H}\cdot d\mathbf{l}=I_f+\frac{\partial}{\partial t}\int_S \mathbf{D}\cdot d\mathbf{a}$$

for any surface S bounded by the closed loop P.

②



very thin Amperian loop straddling the surface

③ The side of the very thin Amperian loop contributes nothing. The flux vanishes in the limit as the area of the loop goes to zero.

④ ⑤

$$\mathbf{E}_1\cdot\mathbf{l}-\mathbf{E}_2\cdot\mathbf{l}=-\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{a}\ \Rightarrow\ \underline{\underline{\mathbf{E}_1^{//}-\mathbf{E}_2^{//}=0}}\ \sharp$$

④ ⑤

$$\mathbf{H}_1\cdot\mathbf{l}-\mathbf{H}_2\cdot\mathbf{l}=I_{f_{\text{enc}}}+\frac{\partial}{\partial t}\int_S \mathbf{D}\cdot d\mathbf{a}\ \Rightarrow(\mathbf{H}_1^{//}-\mathbf{H}_2^{//})\cdot\mathbf{l}=I_{f_{\text{enc}}}$$

$$I_{f_{\text{enc}}}=\mathbf{K}_f\cdot(\hat{\mathbf{n}}\times\mathbf{l})=(\mathbf{K}_f\times\hat{\mathbf{n}})\cdot\mathbf{l}\ \Rightarrow\underline{\underline{\mathbf{H}_1^{//}-\mathbf{H}_2^{//}=(\mathbf{K}_f\times\hat{\mathbf{n}})}}\ \sharp$$

⑥ ⑦

Boundary Conditions in Linear Media

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \qquad \mathbf{E}_1^{//} - \mathbf{E}_2^{//} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \mathbf{H}_1^{//} - \mathbf{H}_2^{//} = (\mathbf{K}_f \times \hat{\mathbf{n}})$$

In case of linear media, **D** and **H** can be expressed in terms of **E** and **B**.

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f \qquad \mathbf{E}_1^{//} - \mathbf{E}_2^{//} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\mathbf{B}_1^{//} - \frac{1}{\mu_2}\mathbf{B}_2^{//} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

If there is no free charge or free current at the interface, then

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = 0 \qquad \mathbf{E}_1^{//} - \mathbf{E}_2^{//} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\mathbf{B}_1^{//} - \frac{1}{\mu_2}\mathbf{B}_2^{//} = 0$$

Homework of Chap.7 (part III)

Problem 7.31 Suppose the circuit in Fig. 7.41 has been connected for a long time when suddenly, at time $t = 0$, switch S is thrown from A to B, bypassing the battery.

(a) What is the current at any subsequent time t ?

(b) What is the total energy delivered to the resistor?

(c) Show that this is equal to the energy originally stored in the inductor.

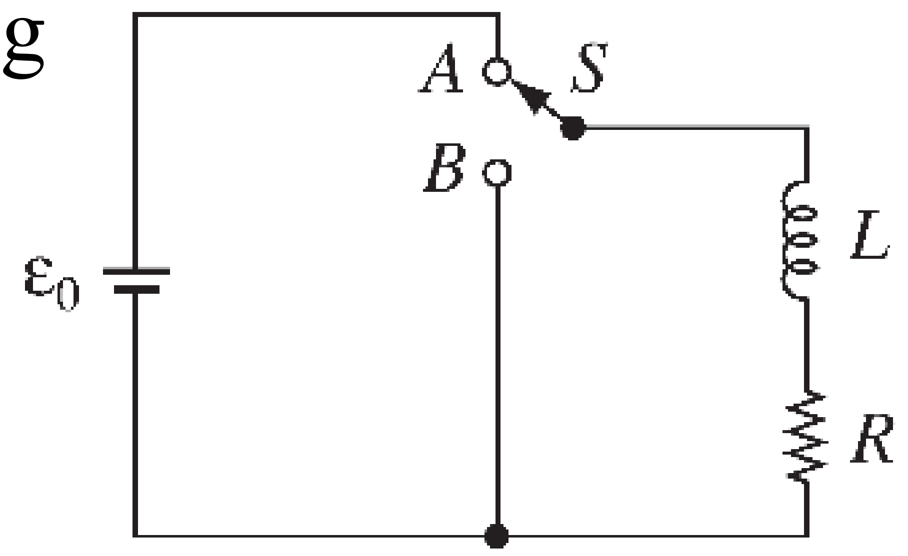


FIGURE 7.41

Problem 7.40 Sea water at frequency $\nu = 4\times10^8$ Hz has permittivity $\varepsilon = 81\varepsilon_0$, permeability $\mu = \mu_0$, and resistivity $\rho = 0.23\Omega\cdot$ m. What is the ratio of conduction current to displacement current? [*Hint*: Consider a parallel-plate capacitor immersed in sea water and driven by a voltage $V_0 \cos(2\pi\nu t)$.]

Problem 7.42 A rare case in which the electrostatic field **E** for a circuit can actually be *calculated* is the following:[28] Imagine an infinitely long cylindrical sheet, of uniform resistivity and radius a. A slot (corresponding to the battery) is maintained at $\pm V_0/2$, *at* $\phi = \pm\pi$, and a steady current flows over the surface, as indicated in Fig.7.51. According to Ohm's law, then,

$$V(a,\phi) = \frac{V_0\phi}{2\pi}, \quad (-\pi < \phi < +\pi).$$

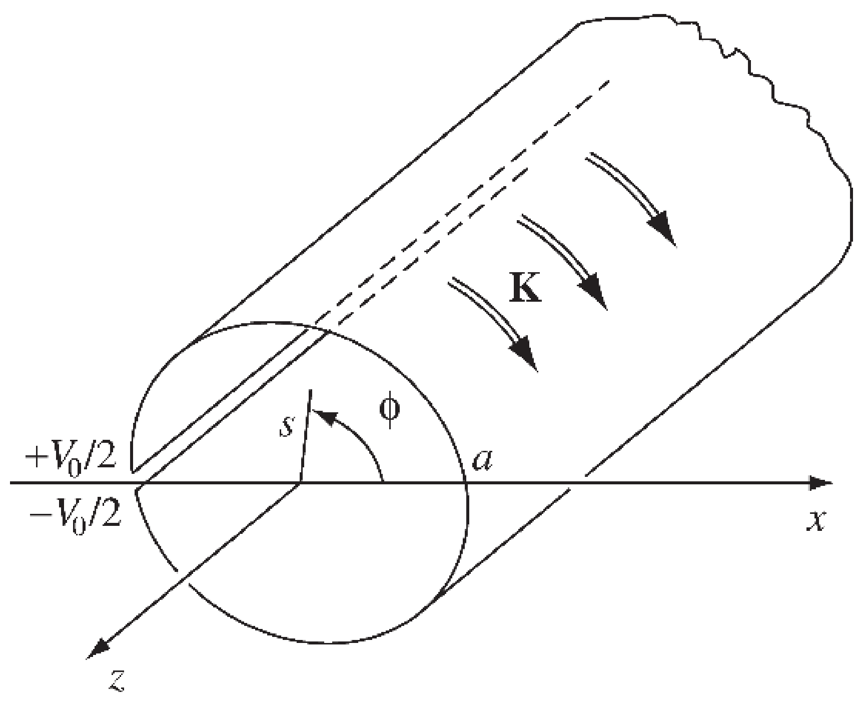


FIGURE 7.51

(a) Use separation of variables in cylindrical coordinates to determine $V(s,\phi)$ inside and outside the cylinder. [Answer: $(V_0/\pi)\tan^{-1}[(s \sin\phi)/(a+s\cos\phi)]$, $(s < a)$; $(V_0/\pi)\tan^{-1}[(a\sin\phi)/(s+a\cos\phi)]$, $(s < a)$]

(b) Find the surface charge density on the cylinder:[Answer: $(\varepsilon_0 V_0/\pi a)\tan[(\phi/2)]$]

Homework of Chap.7 (part III)

Problem 7.53 The current in a long solenoid is increasing linearly with time, so the flux is proportional to $t: \Phi = \alpha t$. Two voltmeters are connected to diametrically opposite points (A and B), together with resistors (R_1 and R_2), as shown in Fig. 7.55. What is the reading on each voltmeter? Assume that these are *ideal* voltmeters that draw negligible current (they have huge internal resistance), and that a voltmeter registers $\int_a^b \mathbf{E}\cdot d\mathbf{l}$ between the terminals and through the meter. [*Answer*: $V_1 = \alpha R_1/(R_1+R_2)$; $V_2 = -\alpha R_2/(R_1+R_2)$. Notice that $V_1 \neq V_2$, even though they are connected to the same points![32]]

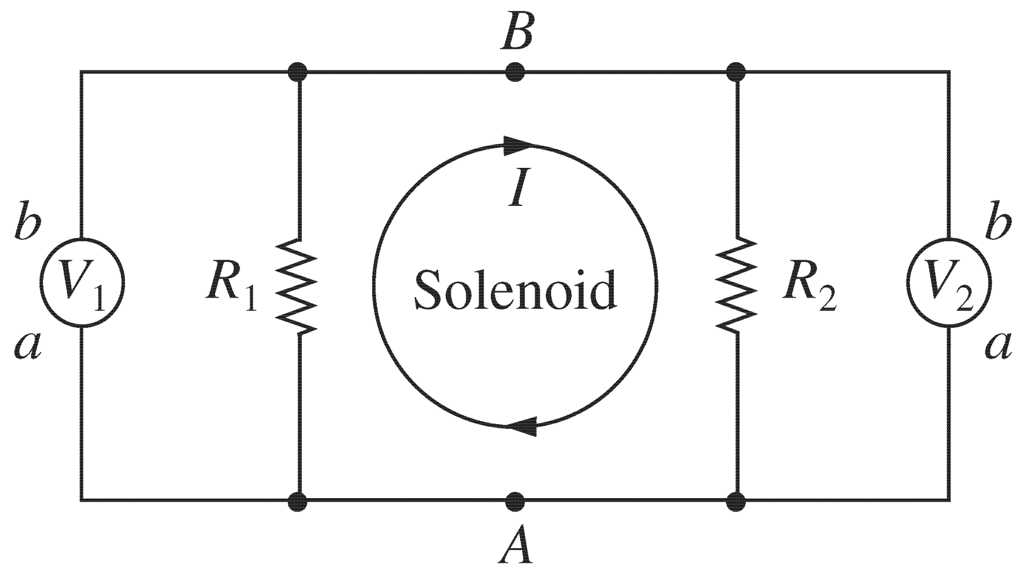


FIGURE 7.55

Problem 7.57 Two coils are wrapped around a cylindrical form in such a way that the *same flux passes through every turn of both coils*. (In practice this is achieved by inscrting an iron core through the cylinder; this has he effect of coneentrating the flux.) The primary coil has N_1 turns and the **secondary** the N_2 (Fig.7.57). If the current I in the primary is changing, show that the emf in the secondary is given by

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{N_2}{N_1}, \qquad (7.67)$$

where ε_1 is the (back) emf of the primary. [This is a primitive **transformer**-a device for raising or lowering the emf of an alternating current source. By chossing the appropriate number of turns, any desired secondary emf can be obtained. If you think this violates the conservation of energy, study Prob. 7.58]

Various Systems of Electromagnetic Units.

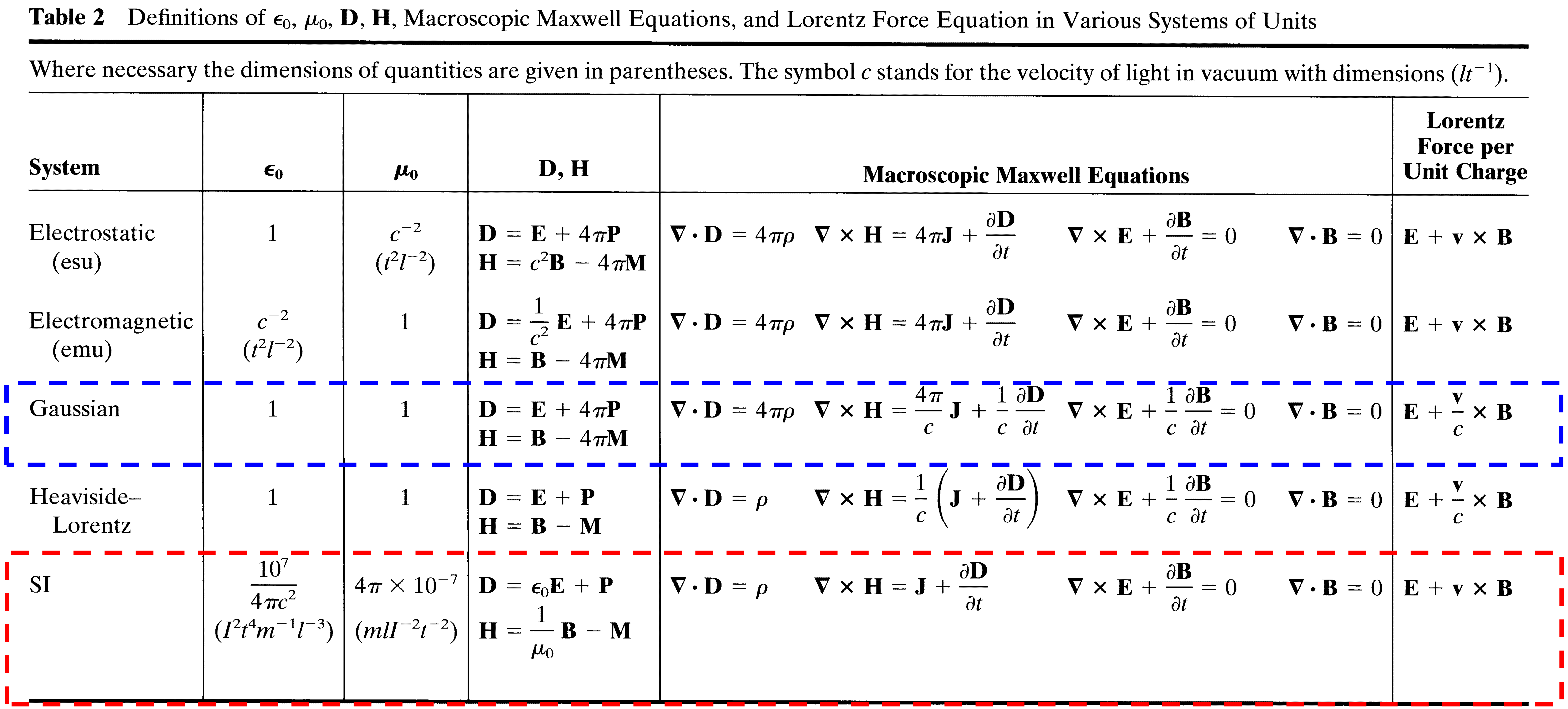


Table 2 Definitions of ϵ_0, μ_0, **D**, **H**, Macroscopic Maxwell Equations, and Lorentz Force Equation in Various Systems of Units

Where necessary the dimensions of quantities are given in parentheses. The symbol c stands for the velocity of light in vacuum with dimensions (lt^{-1}).

System	ϵ_0	μ_0	D, H	Macroscopic Maxwell Equations				Lorentz Force per Unit Charge
Electrostatic (esu)	1	c^{-2} (t^2l^{-2})	$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ $\mathbf{H} = c^2\mathbf{B} - 4\pi\mathbf{M}$	$\nabla\cdot\mathbf{D} = 4\pi\rho$	$\nabla\times\mathbf{H} = 4\pi\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\mathbf{E} + \mathbf{v}\times\mathbf{B}$
Electromagnetic (emu)	c^{-2} (t^2l^{-2})	1	$\mathbf{D} = \frac{1}{c^2}\mathbf{E} + 4\pi\mathbf{P}$ $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$	$\nabla\cdot\mathbf{D} = 4\pi\rho$	$\nabla\times\mathbf{H} = 4\pi\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\mathbf{E} + \mathbf{v}\times\mathbf{B}$
Gaussian	1	1	$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$	$\nabla\cdot\mathbf{D} = 4\pi\rho$	$\nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}$
Heaviside–Lorentz	1	1	$\mathbf{D} = \mathbf{E} + \mathbf{P}$ $\mathbf{H} = \mathbf{B} - \mathbf{M}$	$\nabla\cdot\mathbf{D} = \rho$	$\nabla\times\mathbf{H} = \frac{1}{c}\left(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right)$	$\nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}$
SI	$\frac{10^7}{4\pi c^2}$ $(I^2t^4m^{-1}l^{-3})$	$4\pi\times10^{-7}$ $(mlI^{-2}t^{-2})$	$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$	$\nabla\cdot\mathbf{D} = \rho$	$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\mathbf{E} + \mathbf{v}\times\mathbf{B}$

Jackson: Appendix on Units and Dimensions