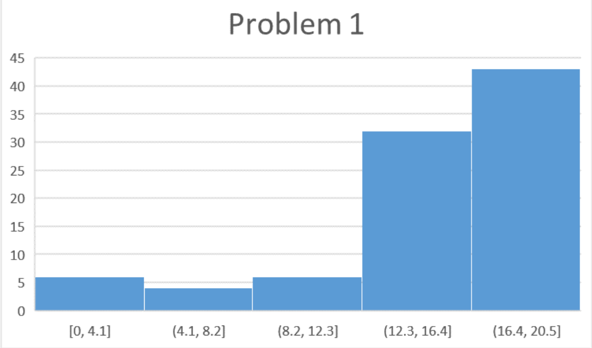
Midterm exam



1. (a) Find the divergence of the function $\mathbf{v} = (r\cos\theta)\,\hat{\mathbf{r}} + (r\sin\theta)\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\hat{\boldsymbol{\phi}}$. (7%)

   (b) Test the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius *R*, resting on the *xy* plane and center at the origin. (7%)

   (c) Find the curl of **v**. (6%)

Griffiths1.40+1.43 (exercise)

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\partial_r\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\partial_\phi\left(v_\phi\right)$$

$$= \frac{1}{r^2}\partial_r\left(r^3\cos\theta\right) + \frac{1}{r\sin\theta}\partial_\theta\left(r\sin^2\theta\right) + \frac{1}{r\sin\theta}\partial_\phi\left(r\sin\theta\cos\phi\right) = 3\cos\theta + 2\cos\theta - \sin\phi = 5\cos\theta - \sin\phi$$

$$\int\nabla\cdot\mathbf{v}\,d\tau = \int_{\frac{\pi}{2}}^{0}\int_0^{2\pi}\int_0^R\left(5\cos\theta - \sin\phi\right)r^2\,dr\,d\phi\,d\left(-\cos\theta\right) = \frac{1}{3}R^3\left(10\pi\right)\left(\frac{1}{2}\right) = \frac{5\pi R^3}{3}$$

$$\oint\mathbf{v}\cdot d\mathbf{a} = \int_0^{2\pi}\int_{\frac{\pi}{2}}^{0} r\cos\theta r^2\sin\theta\,d\theta\,d\phi\Big|_{r=R} + \int_0^{2\pi}\int_0^R r\sin\theta\,r\,dr\,d\phi\Big|_{\theta=\frac{\pi}{2}} = \pi R^3 + \frac{2\pi R^3}{3} = \frac{5\pi R^3}{3}$$

Avg. :15.9 Stdev. :4.2

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\partial_\theta\left(r\sin^2\theta\cos\phi\right)\hat{\mathbf{r}} - \frac{1}{r}\partial_r\left(r^2\sin\theta\cos\phi\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\partial_r\left(r^2\sin\theta\right) - \partial_\theta\left(r\cos\theta\right)\right]\hat{\boldsymbol{\phi}} = 2\cos\theta\cos\phi\hat{\mathbf{r}} - 2\sin\theta\cos\phi\hat{\boldsymbol{\theta}} + 3\sin\theta\hat{\boldsymbol{\phi}}$$

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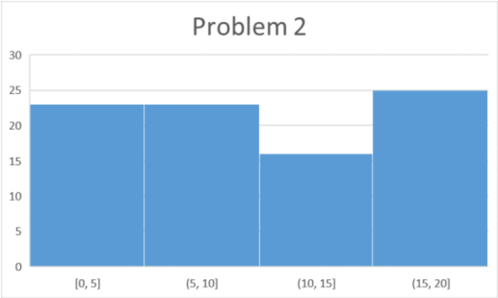
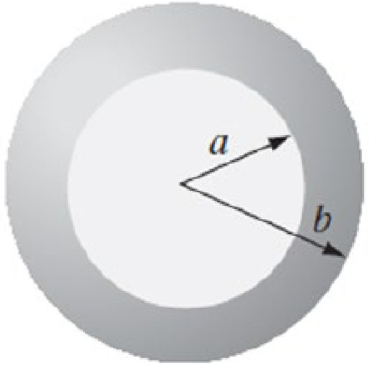


2. Consider a thick spherical shell with the charge density:

$$\rho = \begin{cases} \dfrac{\rho_0 r}{a}, & r < a \\ \dfrac{\rho_0 a^2}{r^2}, & a < r < b \end{cases}$$



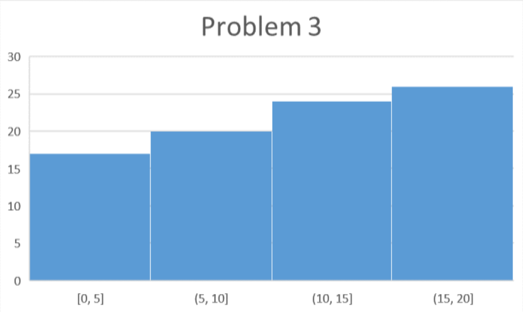
and a perfect conducting shell with surface charge density σ_b is placed at $r = b$, making $\mathbf{E} = 0$ for $r > b$. Find the electric field in the three regions:

$$\begin{cases} r < a \Rightarrow r^2 E_r \varepsilon_0 = \dfrac{\rho_0}{a}\displaystyle\int r'^3 dr' = \dfrac{r^4 \rho_0}{4a} \\ a < r < b \Rightarrow r^2 E_r \varepsilon_0 = \dfrac{a^3 \rho_0}{4} + \rho_0 a^2 \displaystyle\int dr' = \dfrac{a^3 \rho_0}{4} + \rho_0 a^2 (r - a) \end{cases} \Rightarrow E_r = \begin{cases} \dfrac{r^2 \rho_0}{4\varepsilon_0 a}, r < a \\ \dfrac{a^3 \rho_0}{4\varepsilon_0 r^2} + \dfrac{\rho_0 a^2 (r-a)}{\varepsilon_0 r^2}, b < r < a \end{cases}$$

, also, $E_r(r > b) = 0$ (Given)

$$\therefore \sigma_b = -\left[\frac{a^3 \rho_0}{4b^2} + \frac{\rho_0 a^2 (b-a)}{b^2}\right]$$

Griffiths2.15+Gauss Law (exercise)
Avg. :10.9 Stdev. :6.3



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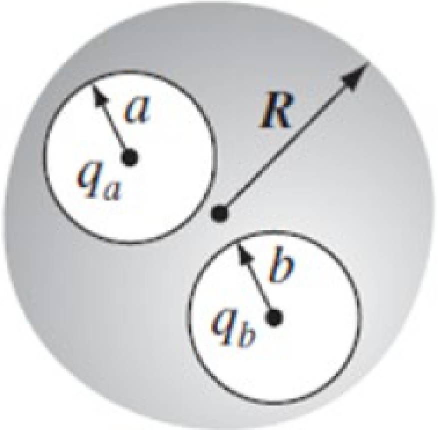
3. Two spherical cavities, of radii a and b, are hollowed out from the interior of a (neutral) conducting sphere of radius R (See the figure). At the center of each cavity a point charge is placed – call these charge q_a and q_b.
   (a) Find the surface charge densities σ_a, σ_b, and σ_R. (5%)
   (b) What is the field outside the conductor? (5%)
   (c) What is the field within each cavity? (5%)
   (d) What is the force on q_a and q_b? (5%)



(a) $\sigma_a = -\dfrac{q_a}{4\pi a^2}, \sigma_b = -\dfrac{q_b}{4\pi b^2}, \sigma_R = \dfrac{q_a + q_b}{4\pi R^2}$ (b) $\mathbf{E}_{\text{outside}} = \dfrac{q_a + q_b}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$

(c) $\mathbf{E}_a = \dfrac{q_b}{4\pi\varepsilon_0 r_a^2}\hat{\mathbf{r}}_a, \mathbf{E}_b = \dfrac{q_b}{4\pi\varepsilon_0 r_b^2}\hat{\mathbf{r}}_b$ (d) 0 (Shielded!)

Griffiths 2.39(Exercise)

Avg. :11.4 Stdev. :5.9

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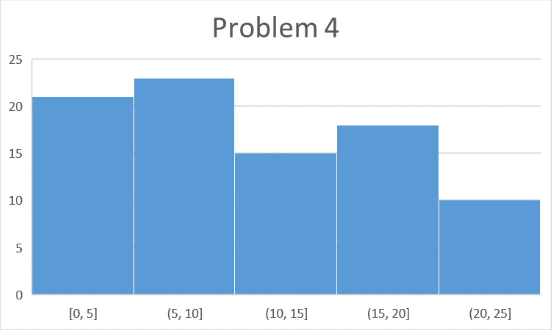
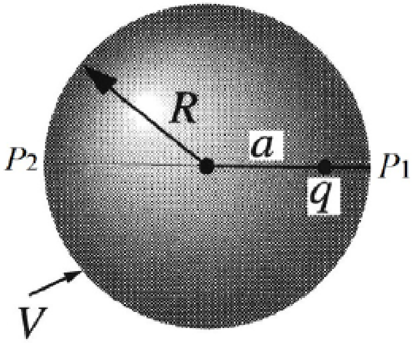
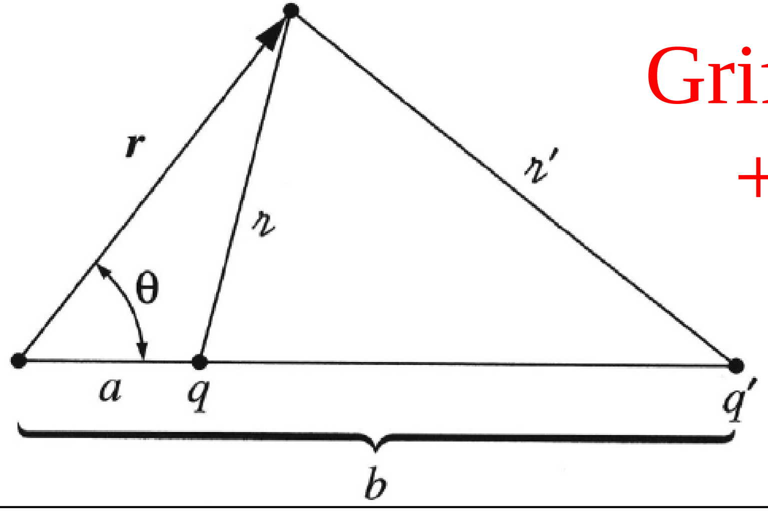


4. A point charge q is situated at distance a from the center of a conducting spherical *shell* of radius R. The spherical shell is grounded V=0.
   (a) Find the position and value of the image charge (outside the sphere). (10%)
   (b) Find the electric field on the (inner) surface of the metal sphere. (10%)
   (c) Find the (inner) surface charge density on the metal sphere. (5%)
   [Hint: 1. Use the notations shown below. 2. Assume q lays on the z-axis]





Griffiths Ex3.2
+Prob 3.8

Avg. :11.6
Stdev. :7.0

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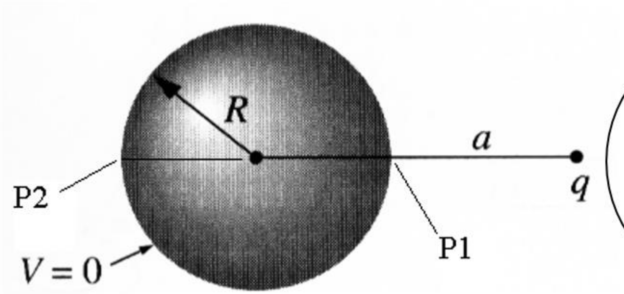
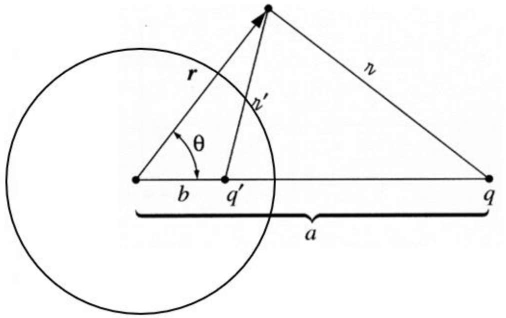
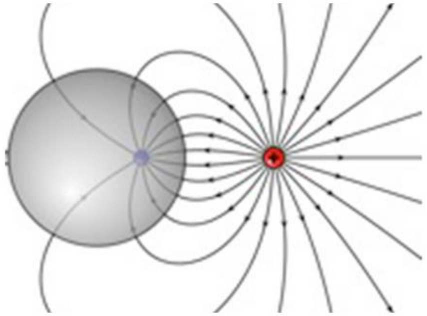
Example 3.2 A point charge is situated a distance a from the center of a grounded conducting sphere of radius R. Find the potential outside the sphere.







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Problem 3.8

(a) Using the law of cosines, show that Eq. 3.17 can be written as follows:

$$V(r,\theta)=\frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}}-\frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right], \tag{3.19}$$

where r and θ are the usual spherical polar coordinates, with the z axis along the line through q. In this form, it is obvious that $V=0$ on the sphere, $r=R$.

(b) Find the induced surface charge on the sphere, as a function of θ. Integrate this to get the total induced charge. (What *should* it be?)

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4. A point charge q is situated at distance a from the center of a conducting spherical *shell* of radius R. The spherical shell is grounded V=0.
   (a) Find the position and value of the image charge (outside the sphere). (10%)

(a) Dimensional Analysis: $\begin{cases} q' = -q \cdot f\left(R/a\right) \\ b = R \cdot f\left(R/a\right) \end{cases}.$

With the limits: $\begin{cases} r \to R \Rightarrow q' \to -q, b \to a\left(\text{or } R\right) \\ r \to \infty \Rightarrow q' \to 0, b \to 0 \end{cases} \Rightarrow \begin{cases} f\left(x \gg 1\right) \to 1 \\ f\left(x \to 1\right) \to 1 \end{cases}$

The answer is therefore reached to be $\begin{cases} q' = -q \cdot \dfrac{R}{a} \\ b = \dfrac{R^2}{a} \end{cases}.$

At $r = R$:

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R-a} + \frac{q'}{b-R}\right) = 0$$

At $r = -R$:

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R+a} + \frac{q'}{b+R}\right) = 0$$

$$\Rightarrow \frac{q}{R-a} = -\frac{q'}{b-R}, \quad \frac{q}{R+a} = -\frac{q'}{b+R}$$

$$\Rightarrow \frac{R+a}{R-a} = \frac{b+R}{b-R} \Rightarrow \frac{a}{R-a} = \frac{R}{b-R} \Rightarrow b = \frac{R^2}{a} \Rightarrow q' = -\frac{qR}{a}$$

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(b) Find the electric field on the (inner) surface of the metal sphere. (10%)

(c) Find the (inner) surface charge density on the metal sphere. (5%)

(b) Before going to the electric field, first we solve for the electric potential.

$$V(r)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}}+\frac{q'}{\mathscr{r}'}\right)\Rightarrow \mathbf{E}=-\nabla V=\frac{-1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}^2}\nabla\mathscr{r}+\frac{q'}{\mathscr{r}'^2}\nabla\mathscr{r}'\right)$$

$$\nabla\mathscr{r}=\nabla\left(a^2+r^2-2ar\cos\theta\right)^{1/2}=\frac{1}{\sqrt{a^2+r^2-2ar\cos\theta}}\left[(r-a\cos\theta)\hat{\mathbf{r}}+a\sin\theta\hat{\boldsymbol{\theta}}\right]$$

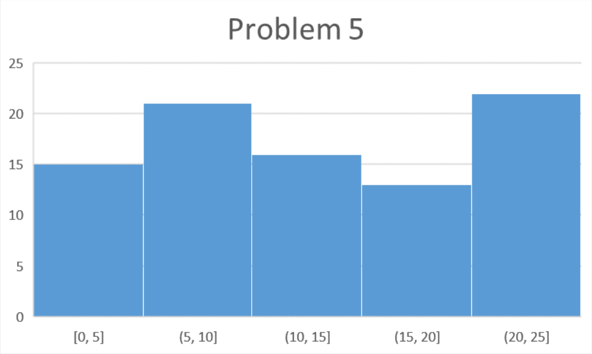
$$\nabla\mathscr{r}'=\nabla\left(b^2+r^2-2br\cos\theta\right)^{1/2}=\frac{1}{\sqrt{b^2+r^2-2br\cos\theta}}\left[(r-b\cos\theta)\hat{\mathbf{r}}+b\sin\theta\hat{\boldsymbol{\theta}}\right]$$

$$\therefore \mathbf{E}(\mathbf{r}=\mathbf{R})=\frac{-1}{4\pi\varepsilon_0}\left(\frac{q}{\left(a^2+R^2-2aR\cos\theta\right)^{3/2}}\left[(R-a\cos\theta)\hat{\mathbf{r}}+a\sin\theta\hat{\boldsymbol{\theta}}\right]+\frac{q'}{\left(b^2+R^2-2bR\cos\theta\right)^{3/2}}\left[(R-b\cos\theta)\hat{\mathbf{r}}+b\sin\theta\hat{\boldsymbol{\theta}}\right]\right)$$

$$=\frac{q}{4\pi\varepsilon_0 R}\frac{\left(R^2-a^2\right)}{\left(R^2+a^2-2aR\cos\theta\right)^{3/2}}\hat{\mathbf{r}}$$

(c) Induced Surface Charge $\sigma=+\varepsilon_0\left.\frac{\partial V}{\partial n}\right|_{r=R}$ $\left(\cancel{\mathbf{E}^{\perp}_{\text{above}}}-\mathbf{E}^{\perp}_{\text{below}}=\frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}\right)$

$$\left.\frac{\partial V}{\partial n}\right|_{r=R}=\frac{1}{4\pi\varepsilon_0}\frac{1}{2}\frac{q\left(\frac{2a^2}{R}-2R\right)}{\left(R^2+a^2-2aR\cos\theta\right)^{3/2}}\Rightarrow\sigma=\frac{-q}{4\pi R}\frac{\left(R^2-a^2\right)}{\left(R^2+a^2-2aR\cos\theta\right)^{3/2}}$$



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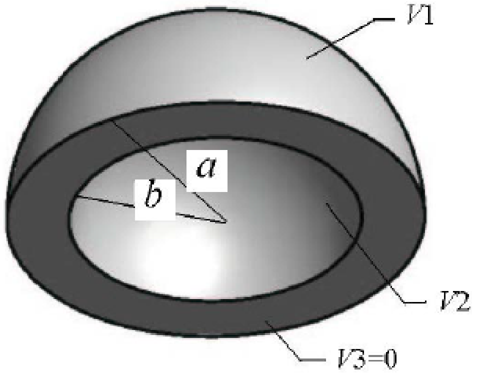
5. Suppose the potential on the surface of a hollow hemisphere is specified, as shown in the figure, where $V_1(a,\theta)=V_0\cos^3\theta$, $V_2(b,\theta)=0$, $V_3(r,\pi/2)=0$. V_0 is a constant.



(a) Show the general solution in the region $b \le r \le a$. (5%)

(b) Determine the potential in the region $b \le r \le a$, using the boundary conditions. (10%)

(c) Calculate the electric field on the surface of the inner shell $\mathbf{E}(r=b)$. (10%)

[Hint: $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=(3x^2-1)/2$, and $P_3(x)=(5x^3-3x)/2$.]

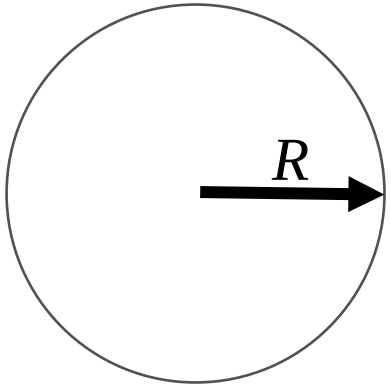
Griffiths 3.19, 3.23, 97midterm (Website)

Avg. :13.4
Stdev. :7.4

Midterm exam

Problem 3.19 The potential at the surface of a sphere (radius R) is given by

$$V_0 = k\cos 3\theta,$$



$V(R,\theta) = k\cos 3\theta$

Laplace equation: $\nabla^2 V = 0 \overset{\text{Azimuthal symmetry}}{\Rightarrow} V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$

Given: $V(R,\theta) = k\cos 3\theta = k\left(4\cos^3\theta - 3\cos\theta\right) = k\left[\alpha P_3(\cos\theta) + \beta P_1(\cos\theta)\right] \Leftarrow$ $\because$ Odd function!

$\left[\text{Recall: } P_0(x) = 0,\ P_1(x) = x,\ P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},\ P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x\right]$

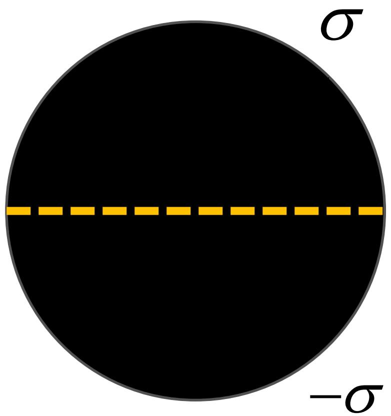
$\therefore V(R,\theta) = \frac{k}{5}\left[8P_3(\cos\theta) - 3P_1(\cos\theta)\right] \Rightarrow V(r,\theta) = \left(A_1 r + \frac{B_1}{r^2}\right)P_1(\cos\theta) + \left(A_3 r^3 + \frac{B_3}{r^4}\right)P_3(\cos\theta)$

$$\Rightarrow \begin{cases} V_{in} \text{ should be finite when } r \to 0 \Rightarrow V_{in} = A_1 r P_1(\cos\theta) + A_3 r^3 P_3(\cos\theta) \\ V_{out} \text{ should be finite when } r \to \infty \Rightarrow V_{out} = \frac{B_1}{r^2}P_1(\cos\theta) + \frac{B_3}{r^4}P_3(\cos\theta) \end{cases}$$

(Based on physics!)

Continuous constraint on the surface: $V_{in}\big|_{r=R} = V_{out}\big|_{r=R} \Rightarrow \begin{cases} A_1 R = \frac{B_1}{R^2} = \frac{-3}{5}k \\ A_3 R^3 = \frac{B_3}{R^4} = \frac{8}{5}k \end{cases} \Rightarrow \begin{cases} V_{in} = \frac{-3kr}{5R}P_1(\cos\theta) + \frac{8kr^3}{5R^3}P_3(\cos\theta) \\ V_{out} = \frac{-3kR}{5r^2}P_1(\cos\theta) + \frac{8kR^4}{5r^4}P_3(\cos\theta) \end{cases}$

Midterm exam



Problem 3.23 A spherical shell of radius R carries a uniform surface charge σ_0 on the "northern" hemisphere and a uniform surface charge $-\sigma_0$ on the "southern" hemisphere. Find the potential inside and outside the sphere, calculating the coefficients explicitly up to A_6 and B_6.

$$\nabla^2 V = 0 \Rightarrow \begin{cases} V_{in} = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \\ V_{out} = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta) \end{cases} \Leftarrow \text{Finite fact of physics}$$

B.C.1: $V_{in}\big|_{r=R} = V_{out}\big|_{r=R} \Rightarrow B_\ell = A_\ell R^{2\ell+1}$

B.C.2: $\left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} - \left.\frac{\partial V_{in}}{\partial r}\right|_{r=R} = -\frac{\sigma}{\varepsilon_0} \Rightarrow \sigma(\theta) = \sum_\ell \varepsilon_0 (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos\theta)$

Now, the coefficients can be determined by Fourier's trick:

$$\int_{-1}^{1} \sigma(\theta) P_{\ell'}(\cos\theta)\, d(\cos\theta) = \int_{-1}^{1} \sum_\ell \varepsilon_0 (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\, d(\cos\theta)$$

$$\Rightarrow A_\ell = \frac{1}{\varepsilon_0}\frac{1}{2\ell+1}\frac{1}{R^{\ell-1}}\frac{2\ell+1}{2}\int_0^\pi \sigma(\theta) P_\ell(\cos\theta)\sin\theta\, d\theta$$

Recall the given surface condition

$$A_\ell = \frac{\sigma_0}{2\varepsilon_0 R^{\ell-1}}\left(\int_0^{\pi/2} P_\ell(\cos\theta)\sin\theta\, d\theta - \int_{\pi/2}^{\pi} P_\ell(\cos\theta)\sin\theta\, d\theta\right) = \frac{\sigma_0}{2\varepsilon_0 R^{\ell-1}}\left(\int_0^1 P_\ell(x)\,dx - \int_{-1}^{0} P_\ell(x)\,dx\right)$$

$$\Rightarrow A_\ell = \begin{cases} 0, & \text{if } \ell \text{ is even} \\ \frac{\sigma_0}{\varepsilon_0 R^{\ell-1}}\int_0^1 P_\ell(x)\,dx, & \text{if } \ell \text{ is odd} \end{cases} \Rightarrow \begin{cases} A_0 = A_2 = A_4 = \ldots\ldots = 0 \\ A_1 = \frac{\sigma_0}{2\varepsilon_0},\ A_3 = \frac{-\sigma_0}{8\varepsilon_0 R^2},\ A_5 = \frac{\sigma_0}{16\varepsilon_0 R^4},\ \ldots\ldots \end{cases}$$

$$\because B_\ell = A_\ell R^{2\ell+1} \Rightarrow B_1 = \frac{\sigma_0 R^3}{2\varepsilon_0},\ B_3 = \frac{-\sigma_0 R^5}{8\varepsilon_0},\ B_5 = \frac{\sigma_0 R^7}{16\varepsilon_0}$$

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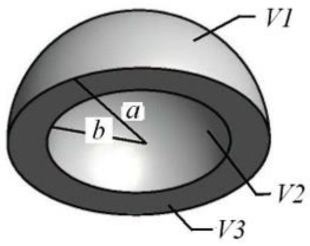
1. (6%, 7%, 7%) Suppose the potential at the surface of a hollow hemisphere is specified, as shown in the figure, where $V_1(a,\theta)=0$, $V_2(b,\theta)=V_0(2\cos\theta-5\cos\theta\sin^2\theta)$, $V_3(r,\pi/2)=0$. V_0 is a constant.
   (a) Show the general solution in the region $b\le r\le a$.
   (b) Determine the potential in the region $b\le r\le a$, using the boundary conditions.
   (c) Calculate the electric field in the region $b\le r\le a$.
   [Hint: $P_0(x)=1,\ P_1(x)=x,\ P_2(x)=(3x^2-1)/2,$ and $P_3(x)=(5x^3-3x)/2$.]



97midterm (Website)

1.

(a)

Boundary condition $\begin{cases}\text{(i) } V_1(a,\theta)=0\\ \text{(ii) } V_2(b,\theta)=V_0(2\cos\theta-5\cos\theta\sin^2\theta)=V_0(5\cos^3\theta-3\cos\theta)=2V_0P_3\\ \text{(ii) } V_3(r,\theta=\pi/2)=0\end{cases}$

General solution $V(r,\theta)=\sum_{\ell=0}^{\infty}(A_\ell r^\ell+B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)$

(b)

B.C. (i) $\rightarrow V(a,\theta)=\sum_{\ell=0}^{\infty}(A_\ell a^\ell+B_\ell a^{-(\ell+1)})P_\ell(\cos\theta)=0 \Rightarrow B_\ell=-A_\ell a^{2\ell+1}$

B.C. (ii) $\rightarrow V(b,\theta)=\sum_{\ell=0}^{\infty}(A_\ell b^\ell+B_\ell b^{-(\ell+1)})P_\ell(\cos\theta)=2V_0P_3(\cos\theta)$

Comparing the coefficiency $\Rightarrow A_3b^3+B_3b^{-4}=2V_0,\ A_\ell=B_\ell=0$ for $\ell=0,1,2,4,5,\ldots$

B.C. (iii) $\rightarrow V(r,\theta=\frac{\pi}{2})=(A_3r^3+B_3r^{-4})P_3(0)=0$

$\Rightarrow A_\ell=B_\ell=0$ except $\ell=3$,

$$A_3=\frac{2V_0b^4}{b^7-a^7}\ \text{and}\ B_3=-\frac{2V_0b^4a^7}{b^7-a^7}$$

$$\therefore V(r,\theta)=\left(\frac{2V_0}{b^7-a^7}b^4r^3-\frac{2V_0}{b^7-a^7}b^4a^7r^{-4}\right)\left(\frac{5\cos^3\theta-3\cos\theta}{2}\right)$$

(c)

$$V(r,\theta)=\left(\frac{2V_0}{b^7-a^7}b^4r^3-\frac{2V_0}{b^7-a^7}b^4a^7r^{-4}\right)\left(\frac{5\cos^3\theta-3\cos\theta}{2}\right)$$

$$\mathbf{E}=-\nabla V=-\frac{\partial V}{\partial r}\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}}$$

$$=-\left(\frac{6V_0}{b^7-a^7}b^4r^2-\frac{8V_0}{b^7-a^7}b^4a^7r^{-5}\right)\left(\frac{5\cos^3\theta-3\cos\theta}{2}\right)\hat{\mathbf{r}}$$

$$+\left(\frac{6V_0}{b^7-a^7}b^4r^2-\frac{8V_0}{b^7-a^7}b^4a^7r^{-5}\right)\left(\frac{15\cos^2\theta\sin\theta-3\sin\theta}{2}\right)\hat{\boldsymbol{\theta}}$$

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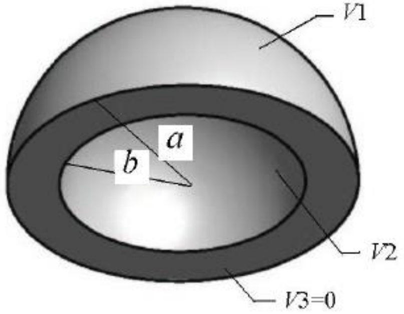


Laplace equation: $\nabla^2 V = 0 \xrightarrow{\text{Azimuthal symmetry}} V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$ (5pts)

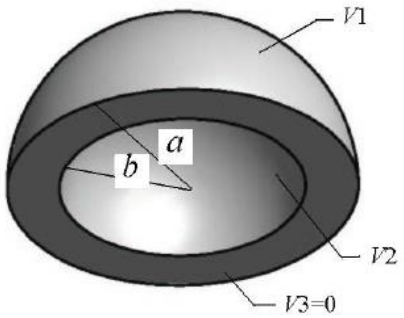
(i) $V = V_1(a,\theta) = V_0 \cos^3\theta = V_0 \frac{2P_3(\cos\theta) + 3P_1(\cos\theta)}{5} \Rightarrow \begin{cases} \left(A_1 a + \frac{B_1}{a^2} \right) P_1(\cos\theta) = V_0 \frac{3P_1(\cos\theta)}{5} \\ \left(A_3 a^3 + \frac{B_3}{a^4} \right) P_3(\cos\theta) = V_0 \frac{2P_3(\cos\theta)}{5} \\ \left(A_l a^l + \frac{B_l}{a^{l+1}} \right) P_3(\cos\theta) = 0 \quad , \forall l \neq 1,3 \end{cases} \Rightarrow \begin{cases} A_1 a + \frac{B_1}{a^2} = \frac{3}{5} V_0 \\ A_3 a^3 + \frac{B_3}{a^4} = \frac{2}{5} V_0 \\ A_l a^l + \frac{B_l}{a^{l+1}} = 0 \quad , \forall l \neq 1,3 \end{cases}$

(ii) $V = V_2(b,\theta) = 0 \Rightarrow A_l b^l + \frac{B_l}{b^{l+1}} = 0 \Rightarrow B_l = -A_l b^{2l+1}$

(iii) $V = V_3(r,\pi/2) = 0 \Rightarrow A_l r^l + \frac{B_l}{r^{l+1}} = 0 \quad , l$ even (Extra boundary, do no affect the solution)

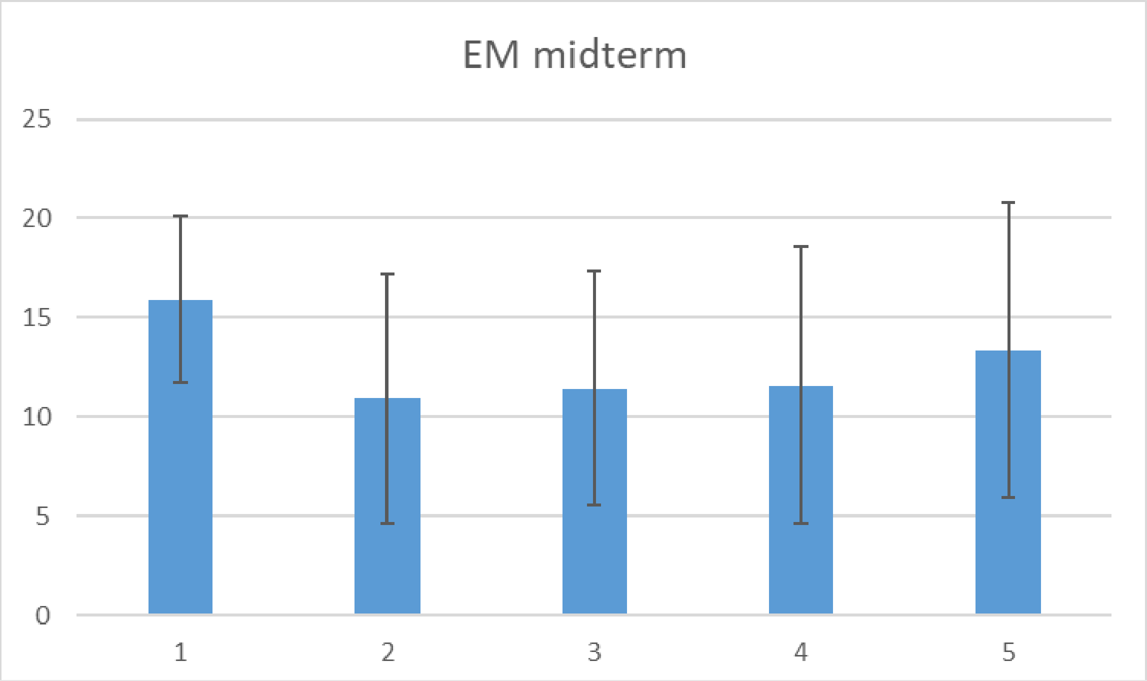
$$\therefore \begin{cases} A_1 = \frac{3}{5} \frac{a^2}{a^3 - b^3} V_0 \quad , B_1 = \frac{3}{5} \frac{-a^2 b^3}{a^3 - b^3} V_0 \\ A_3 = \frac{2}{5} \frac{a^4}{a^7 - b^7} V_0 \quad , B_1 = \frac{2}{5} \frac{-a^4 b^7}{a^7 - b^7} V_0 \\ A_l = B_l = 0 \quad , l \neq 1,3 \end{cases} \Rightarrow V(r,\theta) = V_0 \left[\frac{3}{5} \frac{a^2}{a^3 - b^3} \left(r - \frac{b^3}{r^2} \right) P_1(\cos\theta) + \frac{2}{5} \frac{a^4}{a^7 - b^7} \left(r^3 - \frac{b^7}{r^4} \right) P_3(\cos\theta) \right]$$ (10pts)

Midterm exam



$$\mathbf{E} = -\nabla V = -\left\{ \left[\frac{3}{5}\frac{a^2}{a^3-b^3}\left(1+2\frac{b^3}{r^3}\right)P_1(\cos\theta) + \frac{2}{5}\frac{a^4}{a^7-b^7}V_0\left(3r^2+4\frac{b^7}{r^5}\right)P_3(\cos\theta)\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{3}{5}\frac{a^2}{a^3-b^3}\left(r-\frac{b^3}{r^3}\right)(-\sin\theta) + \frac{2}{5}\frac{a^4}{a^7-b^7}V_0\left(r^3-\frac{b^7}{r^4}\right)\frac{-15\cos^2\theta\sin\theta+3\sin\theta}{2}\right]\hat{\boldsymbol{\theta}} \right\}$$

$$\Rightarrow \mathbf{E}(r=b) = -\left[3A_1P_1(\cos\theta)+7b^2A_3P_3(\cos\theta)\right]\hat{\mathbf{r}} \text{(10pts)}$$



Avg. :63.2

Stdev. :21.1

Chap.4 Supplement

For the EM Course Lectured by Prof. Tsun-Hsu Chang

Teaching Assistants: Hung-Chun Hsu, Yi-Wen Lin, and Tien-Fu Yang

2022 Fall at National Tsing Hua University

Homework Exercises

Griffiths:

2, 9, 10, 16, 21, 28, 33, 36, 39

6, 13, 14, 15, 18, 19, 22, 24, 26, 31, 32

In Chap.3, we said…

Problem 3.2 In one sentence, justify **Earnshaw's Theorem:** *A charged particle cannot be held in a stable equilibrium by electrostatic forces alone.* As an example, consider the cubical arrangement of fixed charges in Fig. 3.4. It *looks,* off hand, as though a positive charge at the center would be suspended in midair, since it is repelled away from each corner. Where is the leak in this "electrostatic bottle"? [To harness nuclear fusion as a practical energy source it is necessary to heat a plasma (soup of charged particles) to fantastic temperatures—so hot that contact would vaporize any ordinary pot. Earnshaw's theorem says that electrostatic containment is also out of the question. Fortunately, it *is* possible to confine a hot plasma magnetically.]

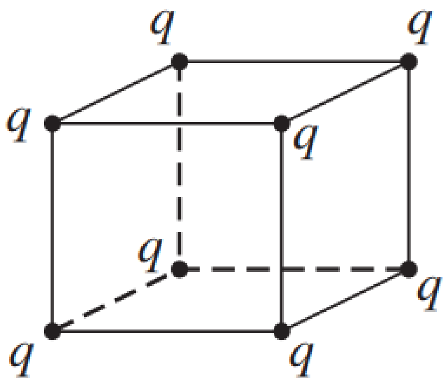
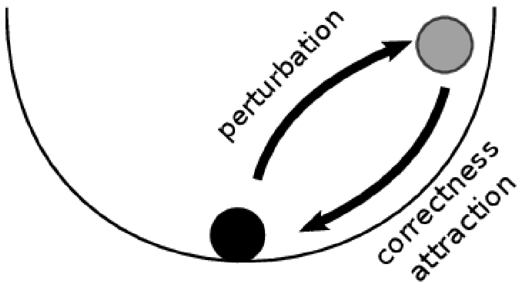
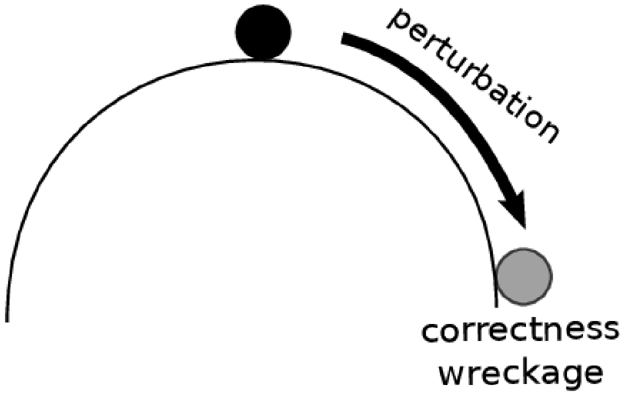


FIGURE 3.4

In Chap.3, we said…

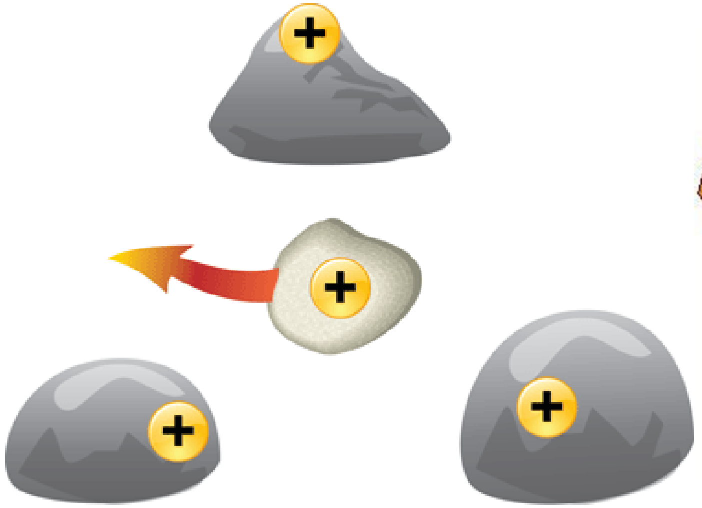


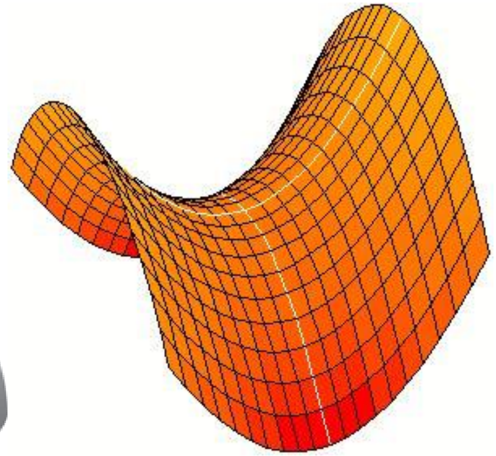
Stable equilibrium



Unstable equilibrium

Picture Credit: B. Danglot et al., *Empir. Softw. Eng.* **23**, 2086 (2018).





In Chap.3, we said…

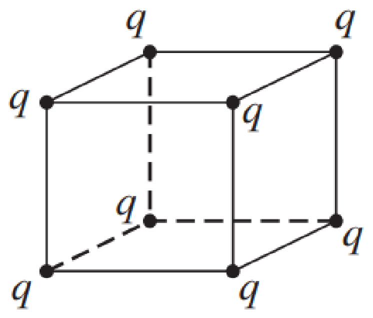
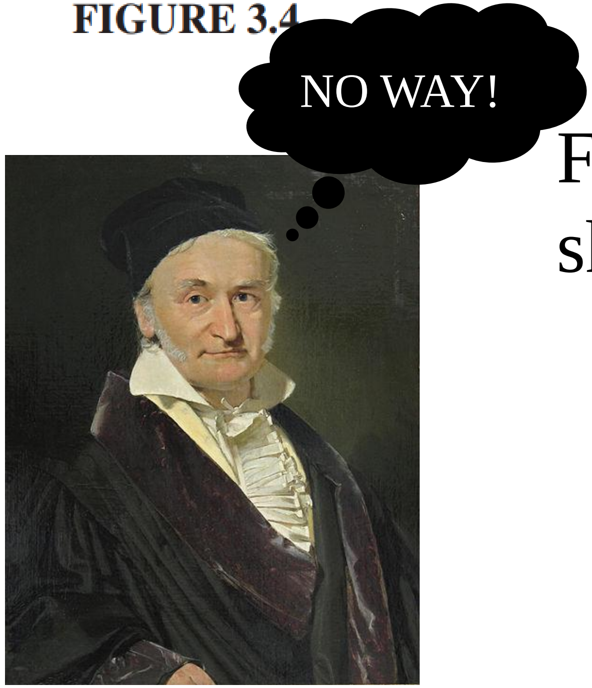


FIGURE 3.4



Stable equilibrium: small perturbations ("pushes") on the particle in any direction should not break the equilibrium.

⬇

Particle "falls back" to its previous position

⬇

Force field lines around the particle's equilibrium position should all point inward, toward that position.

⬇

Divergence of the field at that point must be negative.

Gauss's law says that the divergence of any possible electric force field is zero in free space.

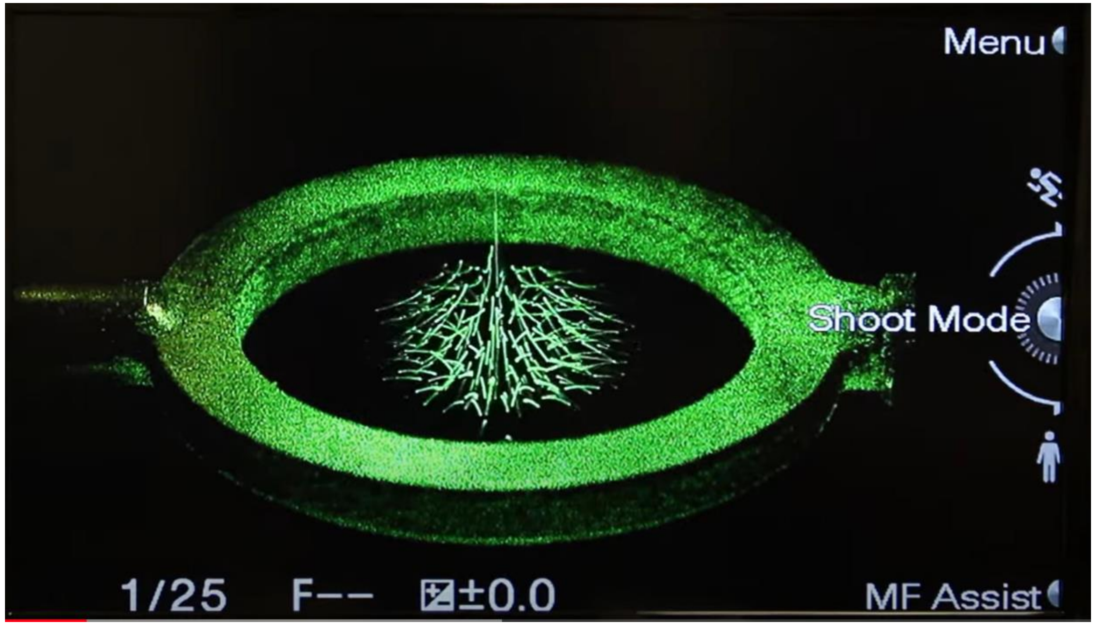
Prob 4.32 in Griffiths

Problem 4.32 Earnshaw's theorem (Prob. 3.2) says that you cannot trap a charged particle in an electrostatic field. *Question:* Could you trap a neutral (but polarizable) atom in an electrostatic field?

(a) Show that the force on the atom is $\mathbf{F} = \frac{1}{2}\alpha\nabla(E^2)$.

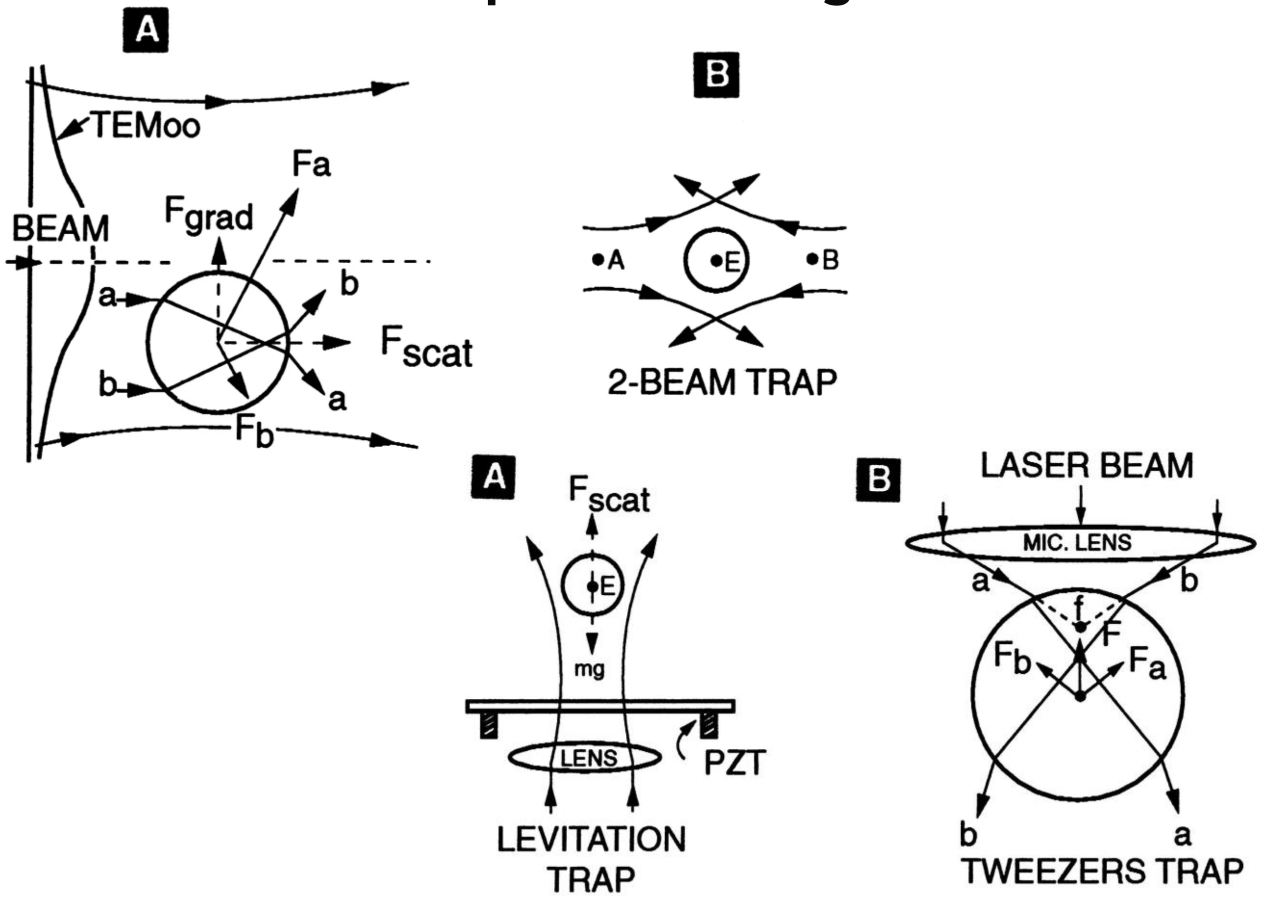
(b) The question becomes, therefore: Is it possible for E^2 to have a local maximum (in a charge-free region)? In that case the force would push the atom back to its equilibrium position. Show that the answer is *no*. [*Hint:* Use Prob. 3.4(a).][22]

Prob 4.32 in Griffiths



Source: https://www.youtube.com/watch?v=tT9e_QrrJzM

Optical trapping and manipulation of neutral particles using lasers

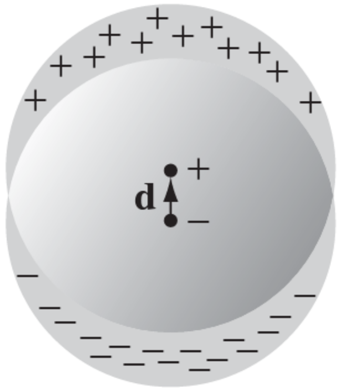


https://doi.org/10.1073/pnas.94.10.4853

Problem 4.2 According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a},$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom. [*Hint*: First calculate the electric field of the electron cloud, $E_e(r)$; then expand the exponential, assuming $r \ll a$.[1]



$\boxed{\mathbf{p} = \alpha \mathbf{E}_e}$ Rule of Thumb! Not a fundamental Law!

$$\oint \mathbf{E}_e \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} Q_{enc} \Rightarrow E_e(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \int_0^r \rho\, d\tau = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \frac{q}{\pi a^3} \int_0^r e^{-\frac{2r'}{a}} 4\pi r'^2 dr' = \frac{1}{\pi\varepsilon_0} \frac{q}{r^2} \frac{1}{a^3} \int_0^r e^{-\frac{2}{a}r'} r'^2 dr'$$

Feynman's Trick of Integration: Introduct another parameter λ to simply the calculation

Consider: $I = \int_0^r e^{-\lambda y} dy = \frac{1}{\lambda}\left(1 - e^{-\lambda r}\right)$

1. $\partial_\lambda I = \partial_\lambda \int_0^r e^{-\lambda y} dy = \int_0^r \partial_\lambda \left(e^{-\lambda y}\right) dy = \int_0^r (-y) e^{-\lambda y} dy = \frac{-1}{\lambda^2}\left(1 - e^{-\lambda r}\right) + \frac{1}{\lambda}\left(1 + \lambda e^{-\lambda r}\right)$

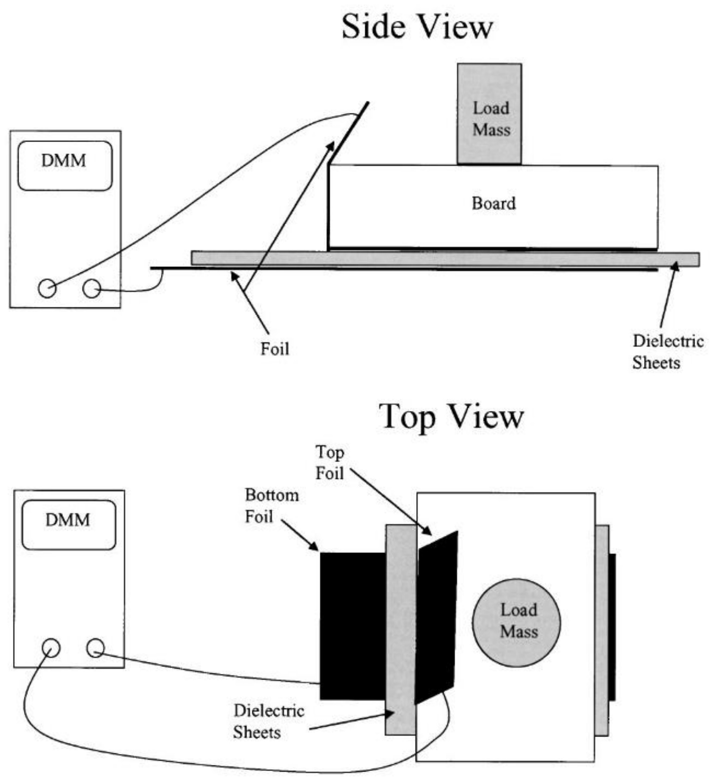
2. $\partial_\lambda^2 I = \partial_\lambda^2 \int_0^r e^{-\lambda y} dy = \int_0^r \partial_\lambda^2 \left(e^{-\lambda y}\right) dy = \int_0^r (-y)^2 e^{-\lambda y} dy = \frac{2}{\lambda^3}\left[1 - e^{-\lambda r}\left(1 + \lambda r + \frac{(\lambda r)^2}{2}\right)\right] \Leftarrow$ Same form as required integration!

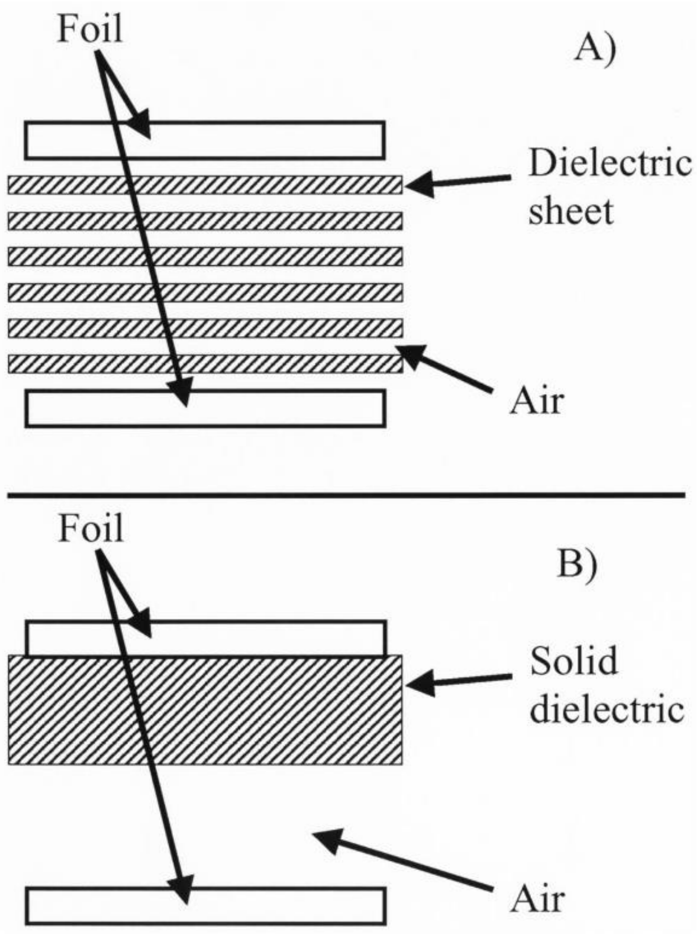
$$\therefore E_e(\mathbf{r}) = \frac{1}{\pi\varepsilon_0} \frac{q}{r^2} \frac{1}{a^3} \frac{2}{(2/a)^3}\left[1 - e^{-2r/a}\left(1 + (2/a) r + \frac{(2r/a)^2}{2}\right)\right] \sim \frac{qd}{3\pi\varepsilon_0 a^3} \sim \frac{p}{\alpha}$$

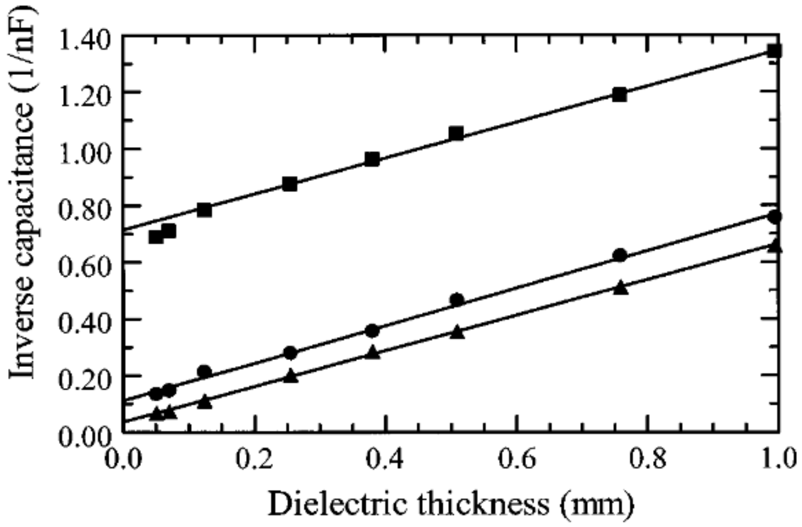
Experiment
Hydrogen atom ~ 0.667e-30

$$\begin{cases} \dfrac{\alpha_{qm}}{4\pi\varepsilon_0} \sim \dfrac{3}{4} a^3 \sim 0.09 \times 10^{-30} m^3 \\ \dfrac{\alpha_{cl}}{4\pi\varepsilon_0} \sim a^3 \sim 0.12 \times 10^{-30} m^3 \end{cases}$$

Parallel Plate Method







$$C = \frac{\varepsilon_r \varepsilon_0 A}{d}$$

$$\Rightarrow \frac{1}{C_{measured}} = \frac{D_{teflon}}{\varepsilon_r \varepsilon_0 A} + \frac{d_{air}}{\varepsilon_0 A}$$

$$\Rightarrow \frac{1}{C_{measured}} = \frac{ND_{teflon}}{\varepsilon_r \varepsilon_0 A} + \frac{2d_{a,f} + (N-1) d_{d,f}}{\varepsilon_0 A}$$

$$\Rightarrow \frac{1}{C_{measured}} = \frac{ND_{teflon}}{\varepsilon_{r,eff} \varepsilon_0 A} + \frac{2d_{a,f} - d_{d,f}}{\varepsilon_0 A}$$

Ref: *AJP* **73**, 52 (2005)

Polarizability and susceptibility

$$\mathbf{E}_{macro} = \mathbf{E}_{self} + \mathbf{E}_{ext}$$

In a linear dielectric, the polarization is said to be proportional to the filed $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$ ← Macroscopic

If the material consists of atoms (or nonpolar molecules), the induced dipole moment $\mathbf{p} = \alpha \mathbf{E}$ ← Microscopic

$$\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E} \overset{?}{\Rightarrow} \chi_e = \frac{N\alpha}{\varepsilon_0}$$

If the density of atoms is low, it's not far off. However, the fields used are from different viewpoints!

$$\mathbf{E}_{self} = \frac{-\mathbf{p}}{4\pi\varepsilon_0 R^3} \Rightarrow \mathbf{E}_{macro} = \frac{-\alpha}{4\pi\varepsilon_0 R^3}\mathbf{E}_{ext} + \mathbf{E}_{ext} = \left(1 - \frac{N\alpha}{3\varepsilon_0}\right)\mathbf{E}_{ext} = \frac{\mathbf{P}}{\varepsilon_0 \chi_e} = \frac{N\alpha}{\varepsilon_0 \chi_e}\mathbf{E}_{ext}$$

$$\therefore \chi_e = \frac{N\alpha/\varepsilon_0}{1 - N\alpha/3\varepsilon_0} \Rightarrow \alpha = \frac{3\varepsilon_0}{N}\frac{\chi_e}{3+\chi_e} = \frac{3\varepsilon_0}{N}\frac{\varepsilon_r - 1}{\varepsilon_r + 2} \approx \frac{3\varepsilon_0}{N}\frac{n^2 - 1}{n^2 + 2}$$

What about polar substance?

Clausius-Mossotti formula

Lorentz-Lorenz relation

Polarizability and susceptibility

Energy of a dipole in an external field $u = -\mathbf{p}\cdot\mathbf{E} = -pE\cos\theta$

Statistical mechanics says that for a material in equilibrium at absolute temperature, the probability of a given molecule having energy is proportional to the Boltzmann factor

$$\exp(-u/kT)$$

The average energy of the dipoles is therefore

$$\langle u\rangle = \frac{\int u e^{-u/kT}\,d\Omega}{\int e^{-u/kT}\,d\Omega} = \frac{\int_{-pE}^{pE} u e^{-u/kT}\,du}{\int_{-pE}^{pE} e^{-u/kT}\,du} = kT - pE\left[\frac{e^{pE/kT}+e^{-pE/kT}}{e^{pE/kT}-e^{-pE/kT}}\right] = kT - pE\coth(pE/kT)$$

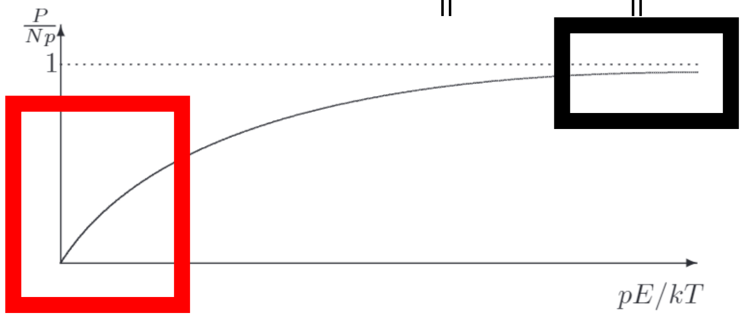
Linear region

$$\therefore \|\mathbf{P}\| = N\|\langle\mathbf{p}\rangle\| = N\left\|\langle\mathbf{p}\cdot\mathbf{E}\rangle\frac{\hat{\mathbf{E}}}{E}\right\| = -Np\frac{\langle u\rangle}{pE} = Np\left\{\coth\left(\frac{pE}{kT}\right) - \frac{kT}{pE}\right\}$$

Langevin equation

$$P \approx \frac{Np^2}{3kT}E = \varepsilon_0\chi_e E$$

$$\Rightarrow \chi_e = \frac{Np^2}{3\varepsilon_0 kT}$$



Comment: For large fields/low temperatures, all the molecules are lined up, and the material is nonlinear.