Quiz 3 (Chap. 4)

For the EM Course Lectured by Prof. Tsun-Hsu Chang

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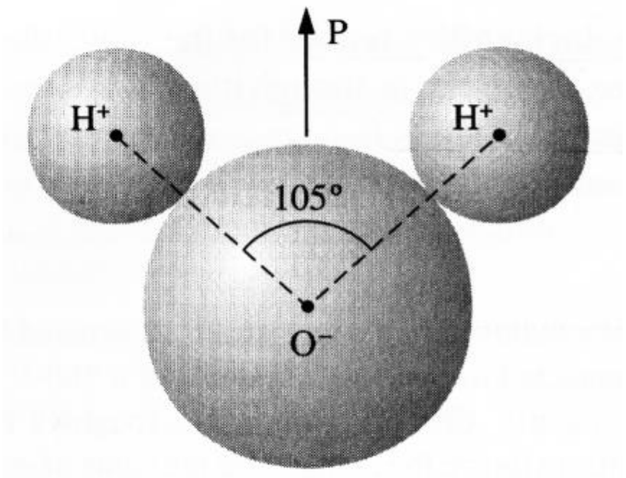
2022 Fall at National Tsing Hua University

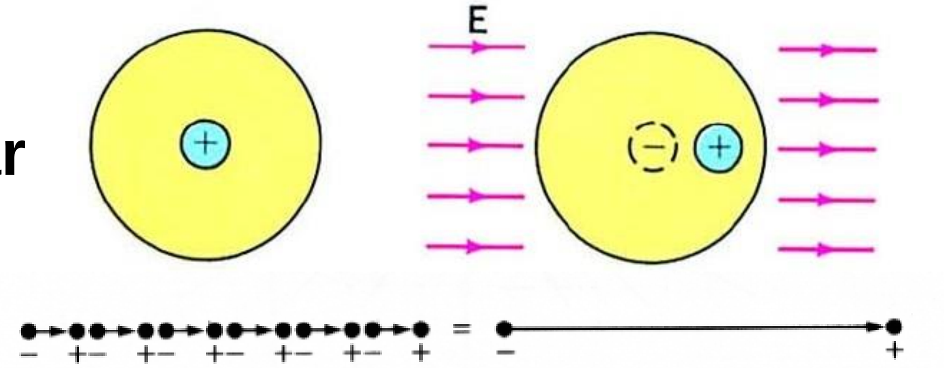
What did we learn in Chap. 4?

Conductors vs. **dielectrics** (or **insulators**)

Polar



Non-Polar



The atom or molecule now has a tiny dipole moment $\mathbf{p}$, which points in the same direction as $\mathbf{E}$ and is proportional to the field.

$$\mathbf{p} = \alpha_{ij}\mathbf{E}, \quad \alpha = \text{atomic polarizability}$$

Rule of Thumb! Not a fundamental Law!

α_{ij}: polarizability tensor for the molecule

Always possible to choose "principal" axes such that the off-diagonal terms vanish, leaving just three nonzero polarizabilities.

What did we learn in Chap. 4?

Example 4.1 A primitive model for an atom consists of a point nucleus (+q) surrounded by a uniformly charged spherical cloud (–q) of radius a.

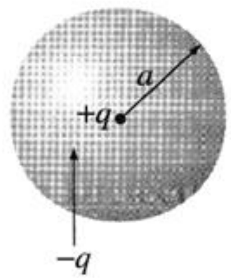


Figure 4.1

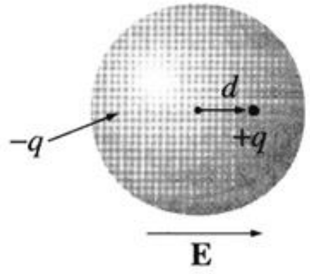


Figure 4.2

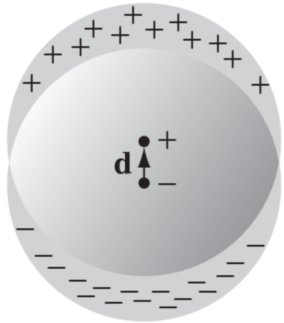
$$E_e = \frac{1}{4\pi\varepsilon_0}\frac{qd}{a^3} \qquad p = qd = (4\pi\varepsilon_0 a^3)E = \alpha E$$

$\alpha = 4\pi\varepsilon_0 a^3$ the atomic polarizability

Prob.4.2 According to **quantum mechanics**, the electron cloud for a hydrogen atom in ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a},$$

$$E_{e,QM}(\mathbf{r}) \sim \frac{qd}{3\pi\varepsilon_0 a^3} \sim \frac{p}{\alpha}$$

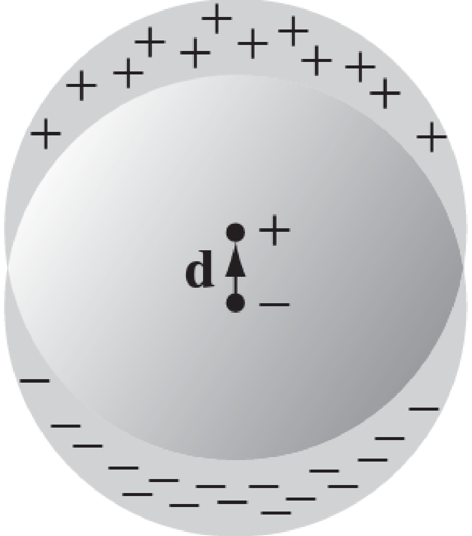


$$\begin{cases} \dfrac{\alpha_{qm}}{4\pi\varepsilon_0} \sim \dfrac{3}{4}a^3 \sim 0.09\times 10^{-30}\, m^3 \\ \dfrac{\alpha_{cl}}{4\pi\varepsilon_0} \sim a^3 \sim 0.12\times 10^{-30}\, m^3 \end{cases}$$

Experiment
Hydrogen atom ~ 0.667e-30

What did we learn in Chap. 4?

Example 4.3 If we have two spheres of charge: a positive sphere and a negative sphere. When the material is uniformly polarized, all the plus charges move slightly upward (the z-direction), all the minus charges move slightly downward. The two sphere no longer overlap perfectly. Find the polarizability.



Sol. The electric field inside a uniform charged sphere of radius a

$$\mathbf{E}_e(r) = \frac{1}{4\pi r^2}\frac{\frac{4}{3}\pi r^3\rho}{\varepsilon_0}\hat{\mathbf{r}} = \frac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{a^3}\hat{\mathbf{r}}, \text{ where } q = \tfrac{4}{3}\pi a^3\rho$$

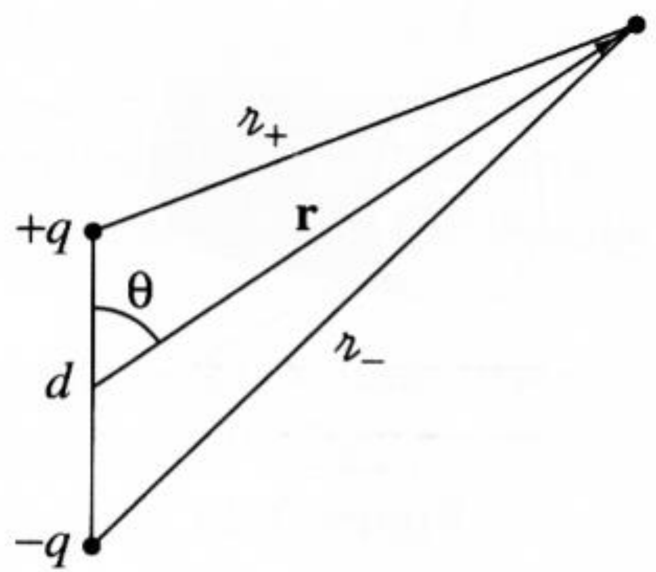
$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathscr{r}}\cdot\mathbf{p}}{\mathscr{r}^2}$$

Two uniformly charged spheres separated by **d** produce the electric field:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{q+}(\mathbf{r}_+) + \mathbf{E}_{q-}(\mathbf{r}_-) = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^3}(\mathbf{r}_+ - \mathbf{r}_-) = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^3}((\mathbf{r} - \frac{1}{2}\mathbf{d}) - (\mathbf{r} + \frac{1}{2}\mathbf{d})) = -\frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{d}}{a^3} = \frac{-1}{4\pi\varepsilon_0 a^3}\mathbf{p}$$

What did we learn in Chap. 4?

Example 3.10 An electric dipole consists of two equal and opposite charges separated by a distance d. Find the approximate potential V at points far from the dipole.



$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2} = \frac{p}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2}$$

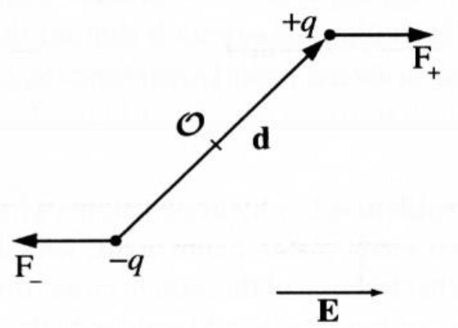
where $\mathbf{p} = q\mathbf{d}$ pointing from negative charge to positive charge.

What did we learn in Chap. 4?

In a uniform field, the force on the positive end, $\mathbf{F} = q\mathbf{E}$, exactly cancels the force on the negative end. However, there will be a torque:

$$\mathbf{N} = (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) = q\mathbf{d} \times \mathbf{E} = \mathbf{p} \times \mathbf{E}$$

in such a direction as to line $\mathbf{p}$ up parallel to $\mathbf{E}$



The force on a dipole in a nonuniform field

$$\mathbf{F} = q(\mathbf{E}_+ - \mathbf{E}_-) \cong q((\mathbf{d} \cdot \nabla)\mathbf{E}) \cong (\mathbf{p} \cdot \nabla)\mathbf{E}$$

What happens to a piece of dielectric material when it is placed in an electric field?

A lot of little dipoles point along the direction of the field and the material becomes polarized.

A convenient measure of this effect is $\mathbf{P} \equiv$ dipole moment per unit volume, which is called the Polarization.

What did we learn in Chap. 4?

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\hat{\mathscr{r}}\cdot \mathrm{d}\mathbf{p}}{\mathscr{r}^2} = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\hat{\mathscr{r}}\cdot \mathbf{P}(\mathbf{r}')}{\mathscr{r}^2}\mathrm{d}\tau' = \frac{1}{4\pi\varepsilon_0}\int_v \mathbf{P}\cdot\nabla'(\frac{1}{\mathscr{r}})\,\mathrm{d}\tau'$$

$$V = \frac{1}{4\pi\varepsilon_0}[\int_v \nabla'\cdot(\frac{\mathbf{P}}{\mathscr{r}})\,\mathrm{d}\tau' - \int_v \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\,\mathrm{d}\tau'\,]$$

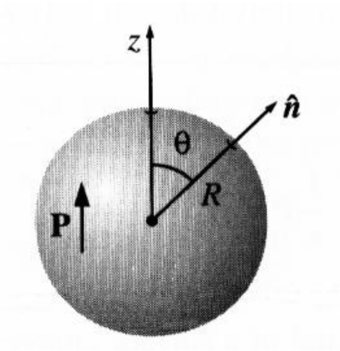
$$= \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\mathbf{P}}{\mathscr{r}}\cdot d\mathbf{a}' + \frac{1}{4\pi\varepsilon_0}\int_v \frac{1}{\mathscr{r}}(-\nabla'\cdot\mathbf{P})\,\mathrm{d}\tau'$$

$$\begin{cases}\sigma_b = \mathbf{P}\cdot\hat{\mathrm{n}} \\ \rho_b = -\nabla'\cdot\mathbf{P}\end{cases} \qquad V = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\sigma_b}{\mathscr{r}}\mathrm{d}a' + \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho_b}{\mathscr{r}}\mathrm{d}\tau'$$

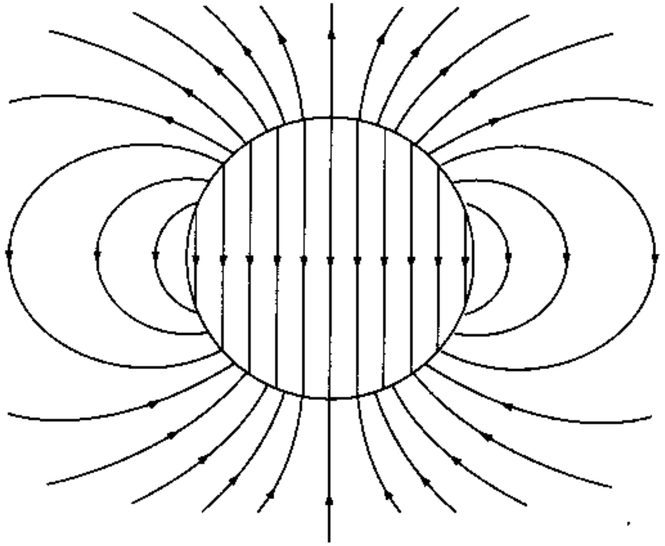
The potential of a polarized object
＝produced by a surface charge density
＋ a volume charge density

What did we learn in Chap. 4?

Ex. 4.2 Find the electric field produced by a uniformly polarized sphere of radius R.



$$\begin{cases} \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = P\cos\theta' \\ \rho_b = -\nabla'\cdot\mathbf{P} = 0 \end{cases}$$



$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_0^{\pi}\frac{P\cos\theta'}{\mathscr{r}}2\pi R^2\sin\theta' d\theta'$$

$$V(r,\,\theta,\,0) = \begin{cases} \dfrac{1}{3\varepsilon_0}\dfrac{PR^3}{r^2}\cos\theta & (r \geq R) \\ \dfrac{P}{3\varepsilon_0}r\cos\theta & (r \leq R) \end{cases}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n P_n(\cos\theta') \qquad r \geq R$$

$$= \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\theta') \qquad r \leq R$$

$\because$ orthogonality $\quad\therefore$ only $n=1$ survive

$$\mathbf{E} = -\nabla V = -\frac{P}{3\varepsilon_0}\hat{\mathbf{z}} \quad \text{uniformly}$$

What did we learn in Chap. 4?

The electric field inside matter

Microscopic level →Too Complicated to calculate

Macroscopic field → Defined as the average field over regions large enough

For many substances, the polarization is proportional to the field, provided $\mathbf{E}$ is not too strong.

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

χ_e : the electric susceptibility

The total field $\mathbf{E}$ may be due in part to free charges and in part to the polarization itself.

Materials that obey above equation are called linear dielectrics.

We cannot compute $\boldsymbol{P}$ *directly from this equation.*

$$\mathbf{E}_0 \to \mathbf{P}_0$$

$$\mathbf{P}_0 \to \mathbf{E}_0 + \Delta\mathbf{E}'_P$$

$$\mathbf{E}_0 + \Delta\mathbf{E}'_P \to \mathbf{P}_0 + \Delta\mathbf{P}'_0$$

What did we learn in Chap. 4?

Now we are going to treat the field caused by both bound charge and free charge.

$$\rho = \rho_f + \rho_b$$
$$= \rho_f - \nabla \cdot \mathbf{P} = \varepsilon_0 \nabla \cdot \mathbf{E}$$

where $\mathbf{E}$ is now the total field, not just that portion generated by polarization.

$$\varepsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{P} = \rho_f$$
$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) \equiv \nabla \cdot \mathbf{D} = \rho_f$$

Electric displacement

$$\nabla \cdot \mathbf{D} = \rho_f \Rightarrow \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f\,\text{enc}}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E}$$

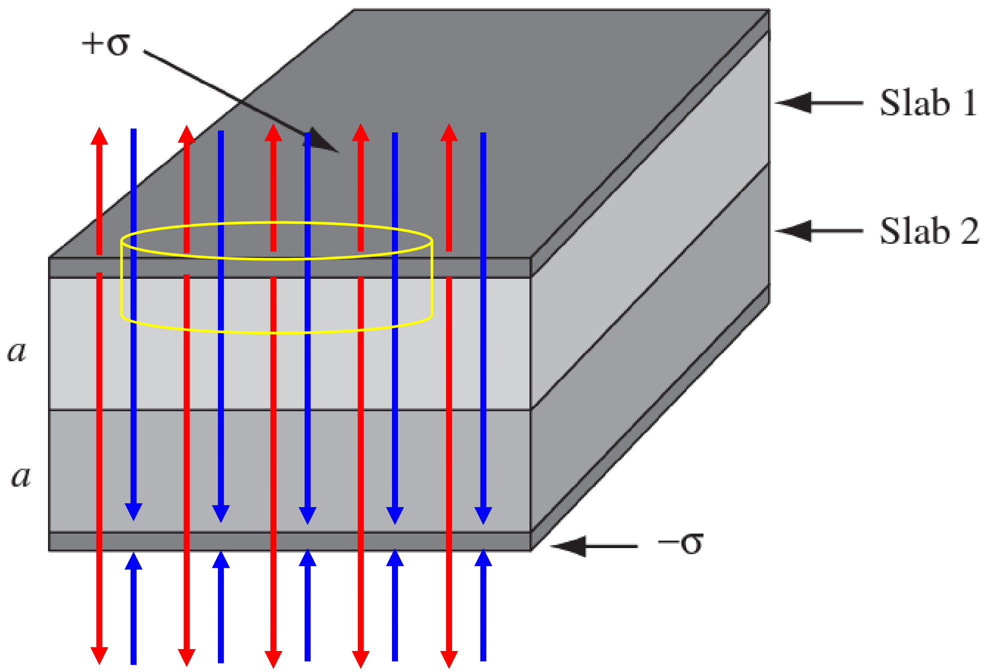
$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e$$ Relative permittivity

$$\mathbf{E} = \frac{1}{\varepsilon} \mathbf{D} = \frac{1}{\varepsilon_r} \mathbf{E}_{vac}$$

Space filled with a homogenous linear dielectric → field reduced by a factor of one over the dielectric constant
→ Polarization partially shields the charge

What did we learn in Chap. 4?

Problem 4.18 The space between the plates of a parallel-plate capacitor (Fig. 4.24) is filled with two slabs of linear dielectric material. Each slab has thickness a, so the total distance between the plates is $2a$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate $-\sigma$.



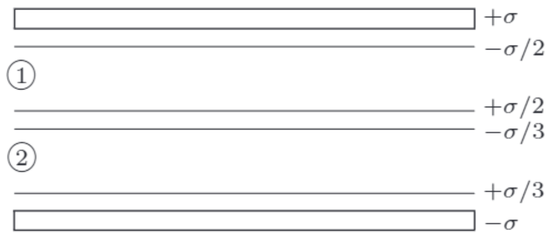
$$\oint_S \mathbf{D}\cdot d\mathbf{a} = Q_{f_{enc}} \Rightarrow 2DA = \sigma A \Rightarrow \mathbf{D} = \frac{\sigma}{2}\hat{\mathbf{n}}$$

$$\therefore \mathbf{D} = \begin{cases} \mathbf{0}, \text{ outside the plates} \\ \dfrac{-\sigma}{2}\hat{\mathbf{z}} + \dfrac{-\sigma}{2}\hat{\mathbf{z}} = -\sigma\hat{\mathbf{z}} \end{cases}$$

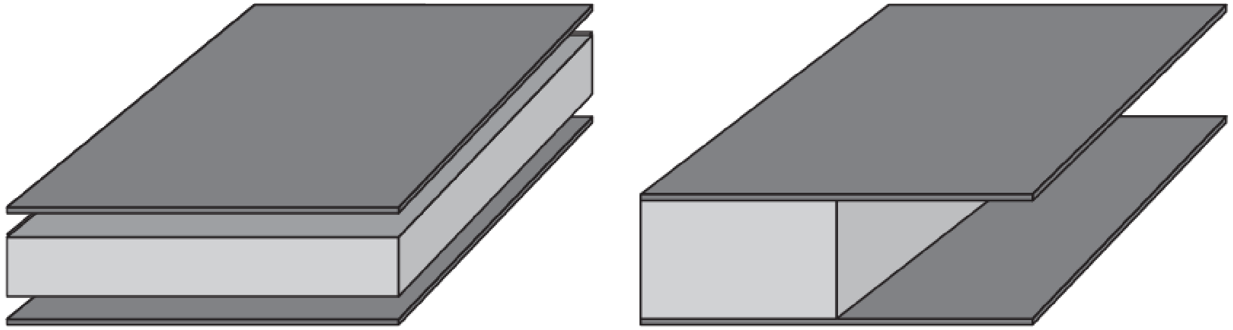
$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} \Rightarrow \begin{cases} \mathbf{E}_1 = -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}, \text{ for slab 1} \\ \mathbf{E}_2 = -\dfrac{2\sigma}{3\varepsilon_0}\hat{\mathbf{z}}, \text{ for slab 2} \end{cases} \Rightarrow \Delta V = -\int_-^+ \mathbf{E}\cdot d\vec{\ell} = \frac{7\sigma a}{6\varepsilon_0}$$

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} = \varepsilon_0(\varepsilon_r - 1)\mathbf{E} = \varepsilon_0(\varepsilon_r - 1)\frac{-\sigma}{\varepsilon_0\varepsilon_r}\hat{\mathbf{z}} = -\sigma\left(1 - \frac{1}{\varepsilon_r}\right)\hat{\mathbf{z}}$$

$$\therefore \mathbf{P}_1 = -\frac{\sigma}{2}\hat{\mathbf{z}},\ \ \mathbf{P}_2 = -\frac{\sigma}{3}\hat{\mathbf{z}} \Rightarrow \nabla\cdot\mathbf{P} = 0 \Rightarrow \rho_b = 0 \text{ everywhere}$$



What did we learn in Chap. 4?



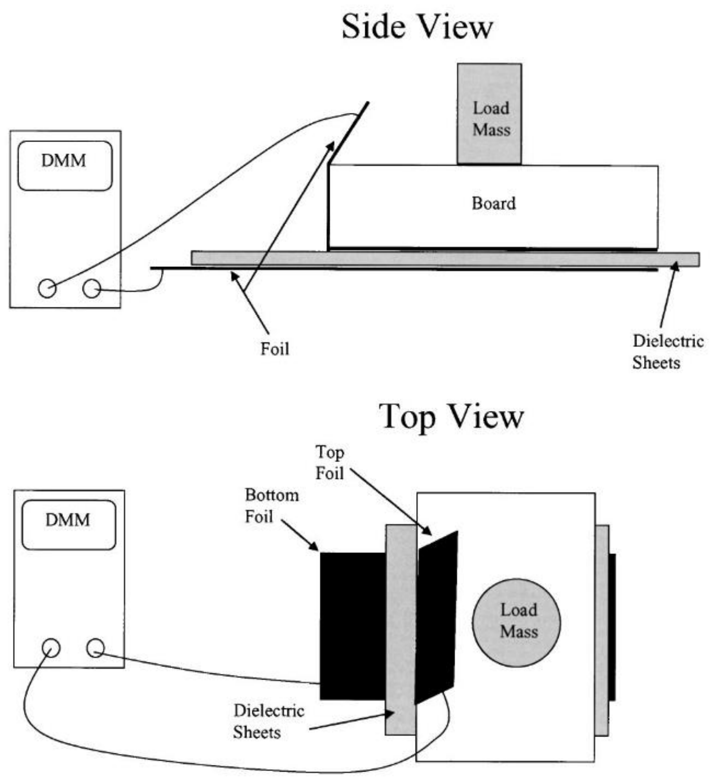
(a) (b)

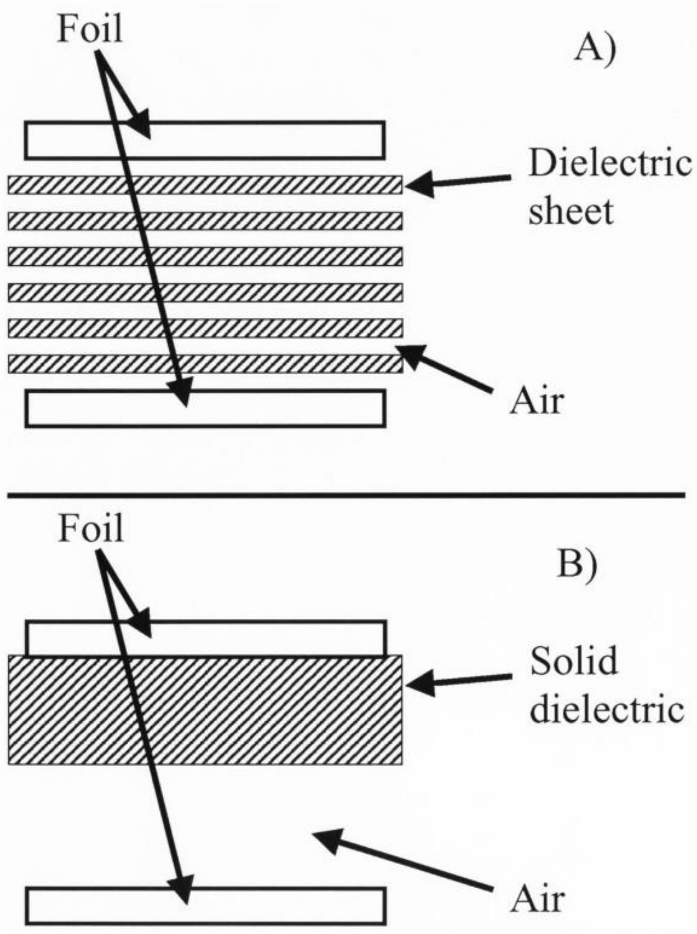
	E	D	P
(a) air	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	0
(a) dielectric	$\frac{2}{(\epsilon_r+1)}\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$
(b) air	$\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	0
(b) dielectric	$\frac{V}{d}\,\hat{\mathbf{x}}$	$\epsilon_r\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$(\epsilon_r-1)\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$

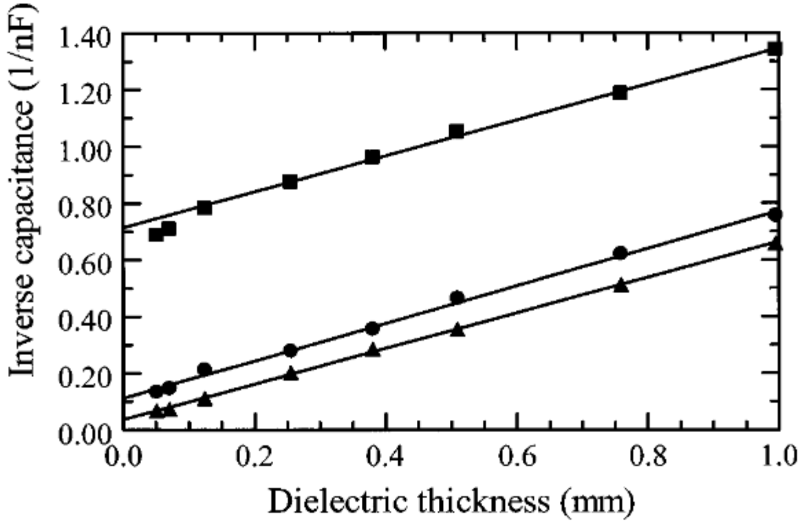
	σ_b (top surface)	σ_f (top plate)
(a)	$-\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}$
(b)	$-(\epsilon_r-1)\frac{\epsilon_0 V}{d}$	$\epsilon_r\frac{\epsilon_0 V}{d}$ (left); $\frac{\epsilon_0 V}{d}$ (right)

$$\left(\frac{C_b}{C_0}=\frac{1+\varepsilon_r}{2}\right)>\left(\frac{C_a}{C_0}=\frac{2}{1+\varepsilon_r^{-1}}\right)$$

What did we learn in Chap. 4?







Ref: *AJP* **73**, 52 (2005)

What did we learn in Chap. 4?

$$\rho_b = -\nabla\cdot\mathbf{P} = -\nabla\cdot\left(\varepsilon_0\chi_e\frac{\mathbf{D}}{\varepsilon}\right) = -\frac{\chi_e}{1+\chi_e}\rho_f \leftarrow \text{in a homogenous linear dielectric}$$

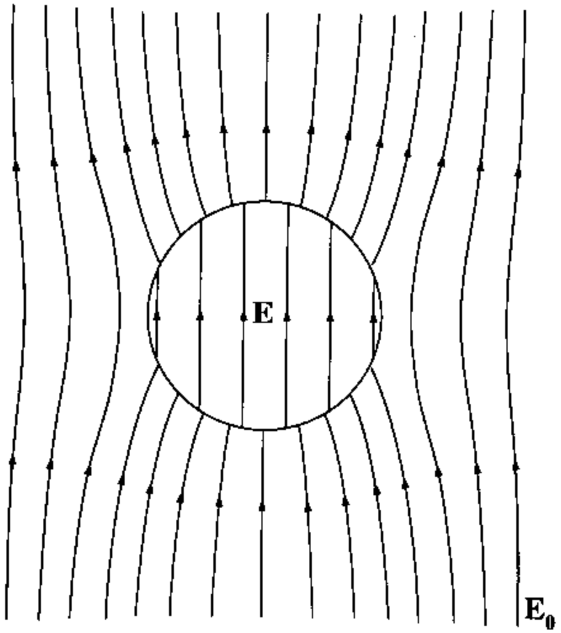
$$\varepsilon = \varepsilon_0\left(1+\chi_e\right)$$

shielding effect

$$D_{above}^{\perp} - D_{below}^{\perp} = \sigma_f \quad \Rightarrow \quad \varepsilon_{above}E_{above}^{\perp} - \varepsilon_{below}E_{below}^{\perp} = \sigma_f$$

$$\left(\varepsilon_{\text{above}}\nabla V_{\text{above}} - \varepsilon_{\text{below}}\nabla V_{\text{below}}\right) = -\sigma_f\hat{\mathbf{n}}$$

What did we learn in Chap. 4?



$$\left.\begin{array}{lll} \text{(i)} & V_{\text{in}} = V_{\text{out}} & \text{at } r = R \\ \text{(ii)} & \varepsilon \dfrac{\partial V_{\text{in}}}{\partial r} = \varepsilon_0 \dfrac{\partial V_{\text{out}}}{\partial r} & \text{at } r = R \\ \text{(iii)} & V_{\text{out}} \to -E_0 r\cos\theta & \text{for } r \gg R \end{array}\right\}$$

$$\begin{cases} V_{in}(r,\theta) = -\dfrac{3E_0}{\varepsilon_r + 2} r\cos\theta \\ V_{out}(r,\theta) = -E_0 r\cos\theta + \left(\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) R^3 E_0 r^{-2}\cos\theta \end{cases}$$

$$\mathbf{E}_{in} = -\nabla V_{in} = \frac{3E_0}{\varepsilon_r + 2}\hat{\mathbf{z}} \quad \leftarrow \text{uniform}$$

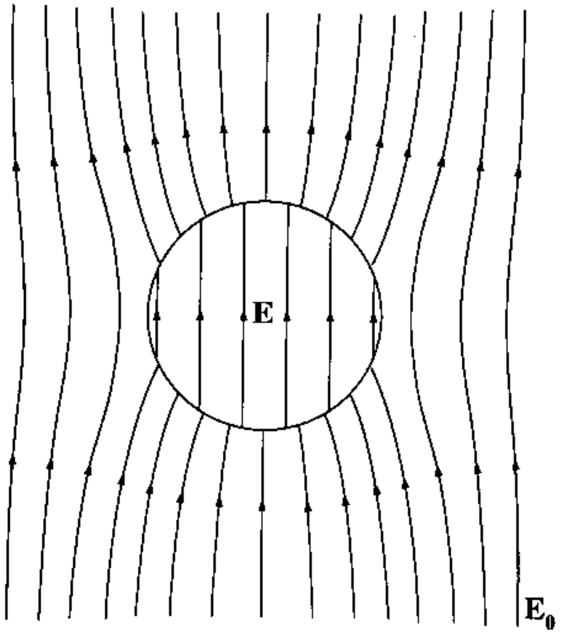
$$V(r,\theta) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos\theta)$$

$$\begin{cases} V_{in}(r,\theta) = \displaystyle\sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & r \le R \\ V_{out}(r,\theta) = -E_0 r\cos\theta + \displaystyle\sum_{\ell=0}^{\infty} B_\ell r^{-(\ell+1)} P_\ell(\cos\theta) & r \ge R \end{cases}$$

How to connect this result with Polarization?

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

What did we learn in Chap. 4?



$$\mathbf{E}_0 \to \mathbf{P}_0 = \varepsilon_0 \chi_e \mathbf{E}_0 \to \mathbf{E}_1 = \frac{-\mathbf{P}_0}{3\varepsilon_0} = \frac{-\chi_e}{3}\mathbf{E}_0$$

$$\mathbf{E}_1 \to \mathbf{P}_1 = \varepsilon_0 \chi_e \mathbf{E}_1 = \frac{-\varepsilon_0 \chi_e^2}{3}\mathbf{E}_0 \to \mathbf{E}_2 = \frac{-\mathbf{P}_1}{3\varepsilon_0} = \frac{\chi_e^2}{9}\mathbf{E}_0$$

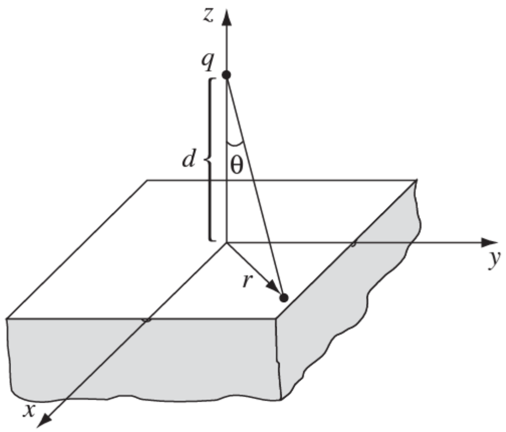
$$\vdots$$

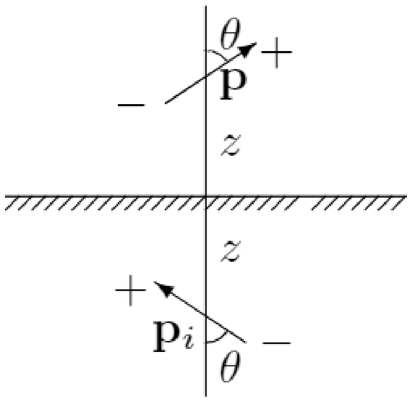
$$\mathbf{E}_n = \left(\frac{-\chi_e}{3}\right)^n \mathbf{E}_0$$

$$\therefore \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \ldots = \sum_{n=0}^{\infty}\left(\frac{-\chi_e}{3}\right)^n \mathbf{E}_0 = \frac{\mathbf{E}_0}{1+\left(\chi_e/3\right)} = \frac{3\mathbf{E}_0}{\varepsilon_r + 2}$$

Remark:1. Consistent with the previous result
2. This method formally requires $\chi_e < 3$

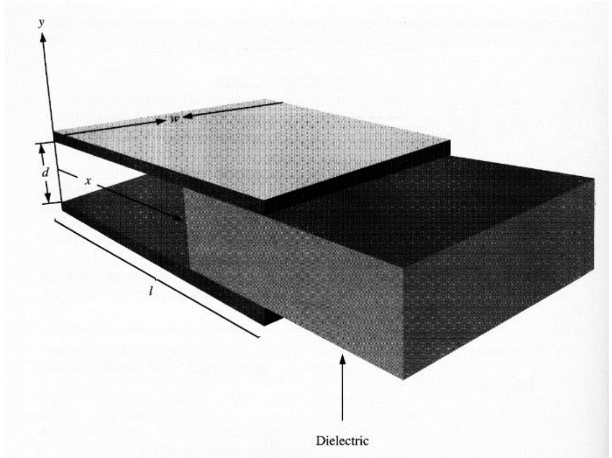
What did we learn in Chap. 4?





$$W = \frac{1}{2}\int (\varepsilon_0 \mathbf{E}\cdot\mathbf{E})d\tau$$

$$W = \frac{1}{2}\int (\mathbf{E}\cdot\mathbf{D})d\tau$$



Partial Image Charge

Energy in Dielectric systems

Force on Dielectric
Fringing Field Effect

Further Reading: *Optik* **124**, 16 (2013)

What did we learn in Chap. 4?

$$\mathbf{E}_{macro} = \mathbf{E}_{self} + \mathbf{E}_{else}$$

In a linear dielectric, the polarization is said to be proportional to the filed $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}_{macro}$ ← Macroscopic

If the material consists of atoms (or nonpolar molecules), the induced dipole moment $\mathbf{p} = \alpha \mathbf{E}_{else}$ ← Microscopic

$$\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E} \overset{?}{\Rightarrow} \chi_e = \frac{N\alpha}{\varepsilon_0}$$

If the density of atoms is low, it's not far off. However, the fields used are from different viewpoints!

$$\mathbf{E}_{self} = \frac{-\mathbf{p}}{4\pi\varepsilon_0 R^3} \Rightarrow \mathbf{E}_{macro} = \frac{-\alpha}{4\pi\varepsilon_0 R^3}\mathbf{E}_{else} + \mathbf{E}_{else} = \left(1 - \frac{N\alpha}{3\varepsilon_0}\right)\mathbf{E}_{else} = \frac{\mathbf{P}}{\varepsilon_0\chi_e} = \frac{N\alpha}{\varepsilon_0\chi_e}\mathbf{E}_{else}$$

$$\therefore \chi_e = \frac{N\alpha/\varepsilon_0}{1 - N\alpha/3\varepsilon_0} \Rightarrow \alpha = \frac{3\varepsilon_0}{N}\frac{\chi_e}{3+\chi_e} = \frac{3\varepsilon_0}{N}\frac{\varepsilon_r - 1}{\varepsilon_r + 2} \approx \frac{3\varepsilon_0}{N}\frac{n^2 - 1}{n^2 + 2}$$

Clausius-Mossotti formula   Lorentz-Lorenz relation

What about polar substance?

Further Reading: *Optik* **124**, 16 (2013)

What did we learn in Chap. 4?

Energy of a dipole in an external field $u = -\mathbf{p}\cdot\mathbf{E} = -pE\cos\theta$

Statistical mechanics says that for a material in equilibrium at absolute temperature, the probability of a given molecule having energy is proportional to the Boltzmann factor

$$\exp(-u/kT)$$

The average energy of the dipoles is therefore

$$\langle u\rangle = \frac{\int u e^{-u/kT}\,d\Omega}{\int e^{-u/kT}\,d\Omega} = \frac{\int_{-pE}^{pE} u e^{-u/kT}\,du}{\int_{-pE}^{pE} e^{-u/kT}\,du} = kT - pE\left[\frac{e^{pE/kT}+e^{-pE/kT}}{e^{pE/kT}-e^{-pE/kT}}\right] = kT - pE\coth(pE/kT)$$

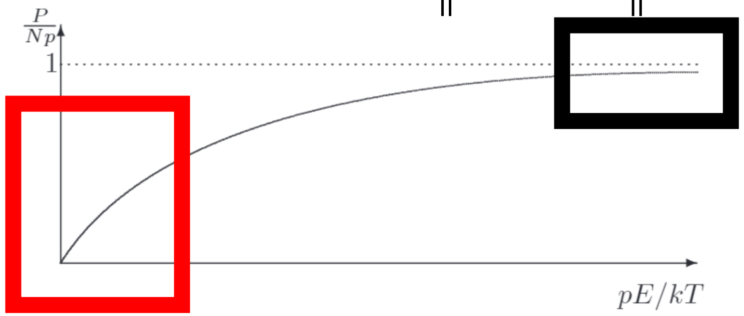
Linear region

$$\therefore \|\mathbf{P}\| = N\|\langle\mathbf{p}\rangle\| = N\left\|\langle\mathbf{p}\cdot\mathbf{E}\rangle\frac{\hat{\mathbf{E}}}{E}\right\| = -Np\frac{\langle u\rangle}{pE} = Np\left\{\coth\left(\frac{pE}{kT}\right) - \frac{kT}{pE}\right\}$$

Langevin equation

$$P \approx \frac{Np^2}{3kT}E = \varepsilon_0\chi_e E$$

$$\Rightarrow \chi_e = \frac{Np^2}{3\varepsilon_0 kT}$$



Comment: For large fields/low temperatures, all the molecules are lined up, and the material is nonlinear.

Remarks on Dielectrics

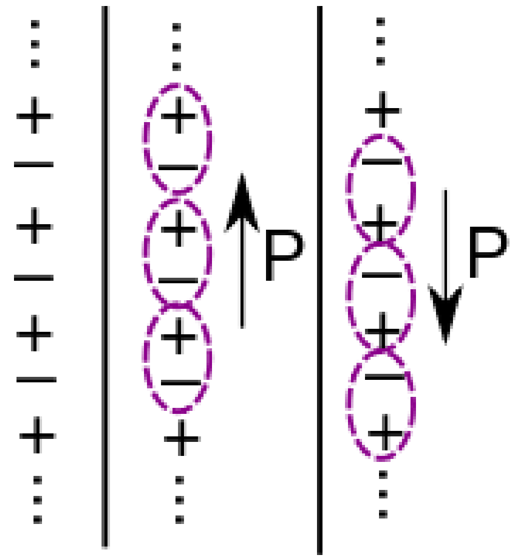
Anisotropic Materials:
i-th component of the polarization is related to the j-th component of the electric field.

$$P_i = \sum_j \varepsilon_0 \chi_{ij} E_j$$

Polarize in the x direction by applying a field in the z direction → Crystal Optics.

How to choose "unit volume" in reality?
e.g., Plasma in microscopic scale is regarded as a gas of free charges (P = 0). However, in macroscopic scale, it serves as a continuous medium, exhibiting non-zero permittivity.

Polarization Ambiguity



Non-uniqueness of P is not problematic, because every measurable consequence of P is in fact a consequence of a continuous change in P.