Chapter 4 Electric Fields in Matter
4.1 Polarization: 4.1.1 Dielectrics

Most everyday objects belong to one of two large classes: **conductors** and **insulators** (or **dielectrics**)

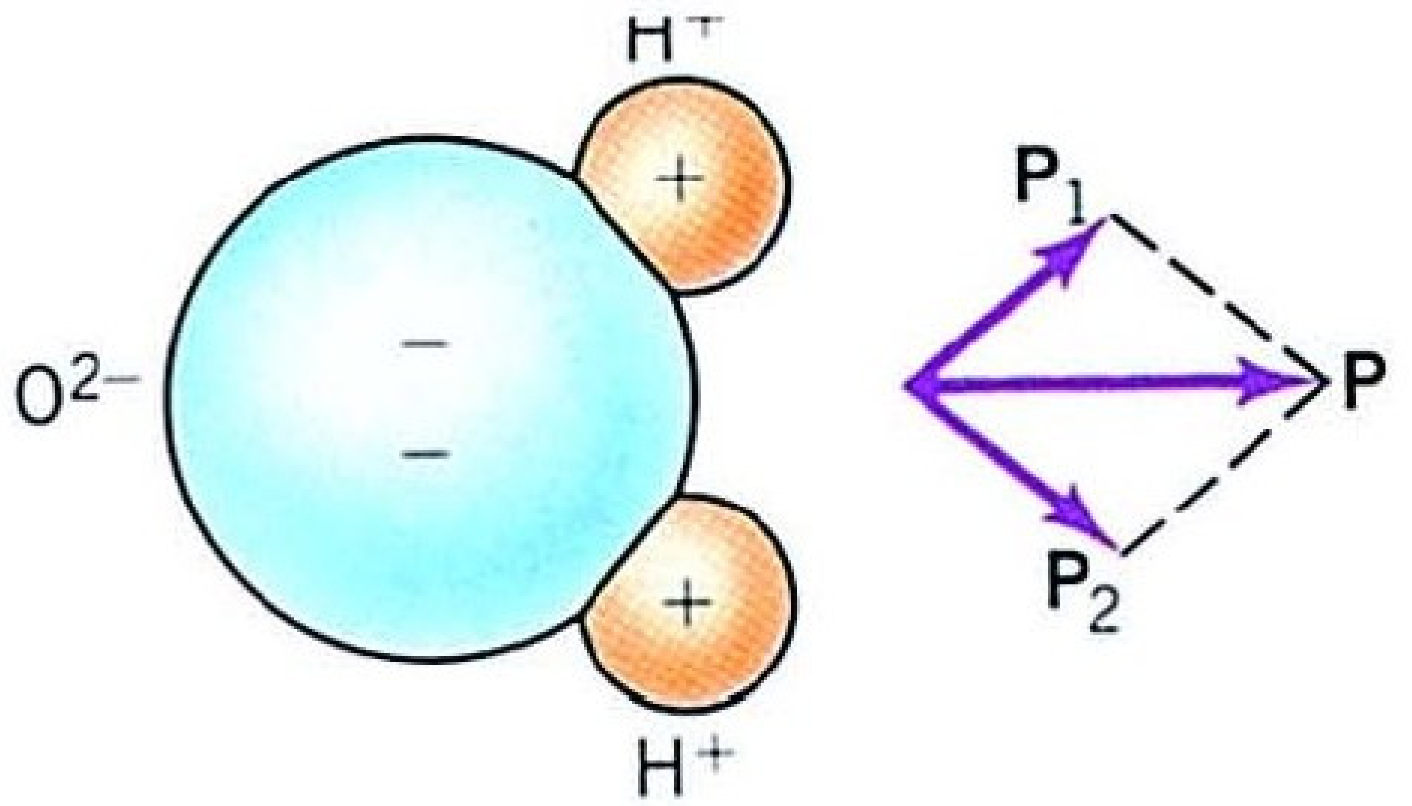
Conductors: Substances contain an "unlimited" supply of charges that are free to move about through the material.

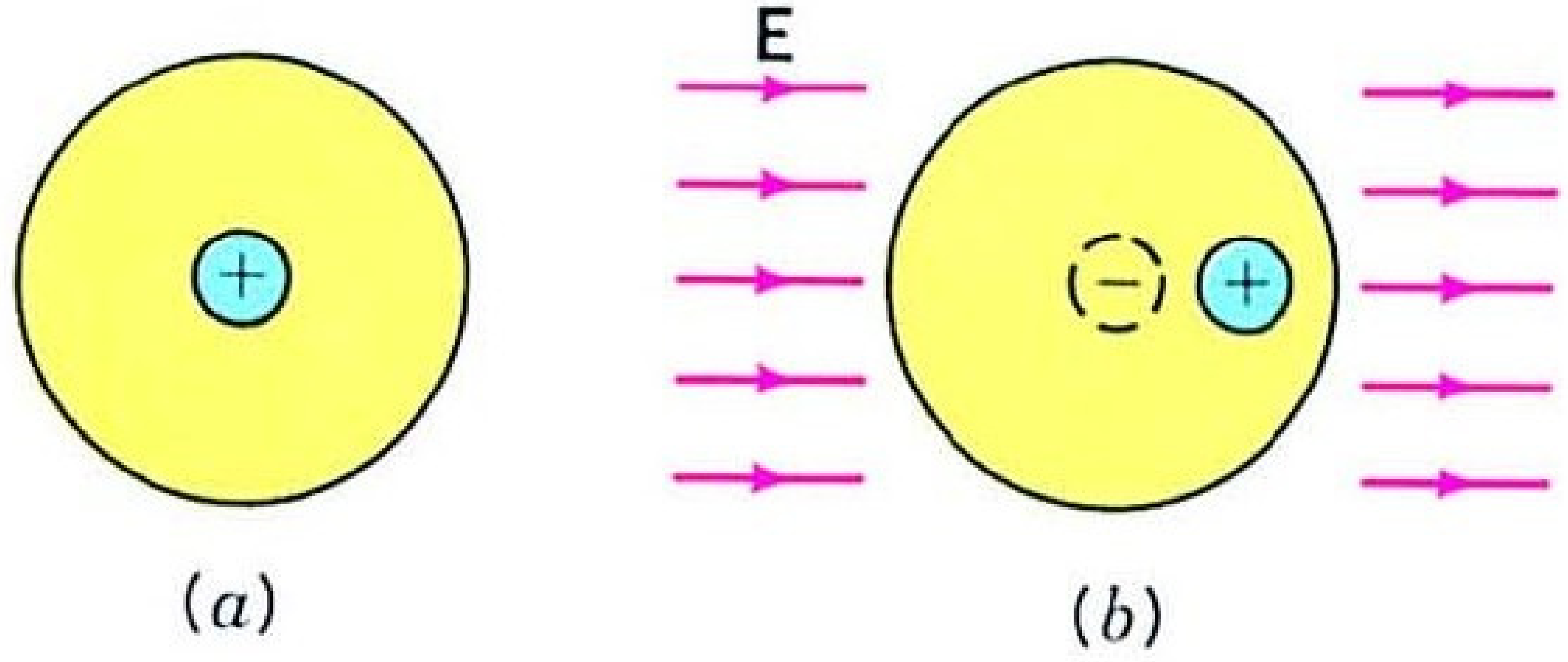
Dielectrics: All charges are attached to specific atoms or molecules. All they can do is move a bit within the atom or molecule.

Dielectrics

Dielectrics: Microscopic displacements are not as dramatics as the wholesale rearrangement of charge in conductor, but their **cumulative effects** account for the characteristic behavior of dielectric materials.

There are actually two principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule: **rotating** and **stretching**.





4.1.2 Induced Dipoles

What happens to a neutral atom when it is placed in an electric field $\mathbf{E}$?

Although the atom as a whole is electrically neutral, there is a *positively* charged core (the nucleus) and a *negatively* charged electron cloud surrounding it.

Thus, the nucleus is pushed in the direction of the field, and the electron the opposite way.

The electric fields *pull* the electrons and the nucleus *apart*, their mutual attraction drawing them together - reach balance, leaving the atom polarized.

4.1.2 Induced Dipoles

The atom or molecule now has a tiny dipole moment $\mathbf{p}$, which points in the same direction as $\mathbf{E}$ and is proportional to the field.

$$\mathbf{p} = \alpha\mathbf{E}, \quad \alpha = \text{atomic polarizability}$$

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.67	0.396	24.1	1.64	43.4	59.4

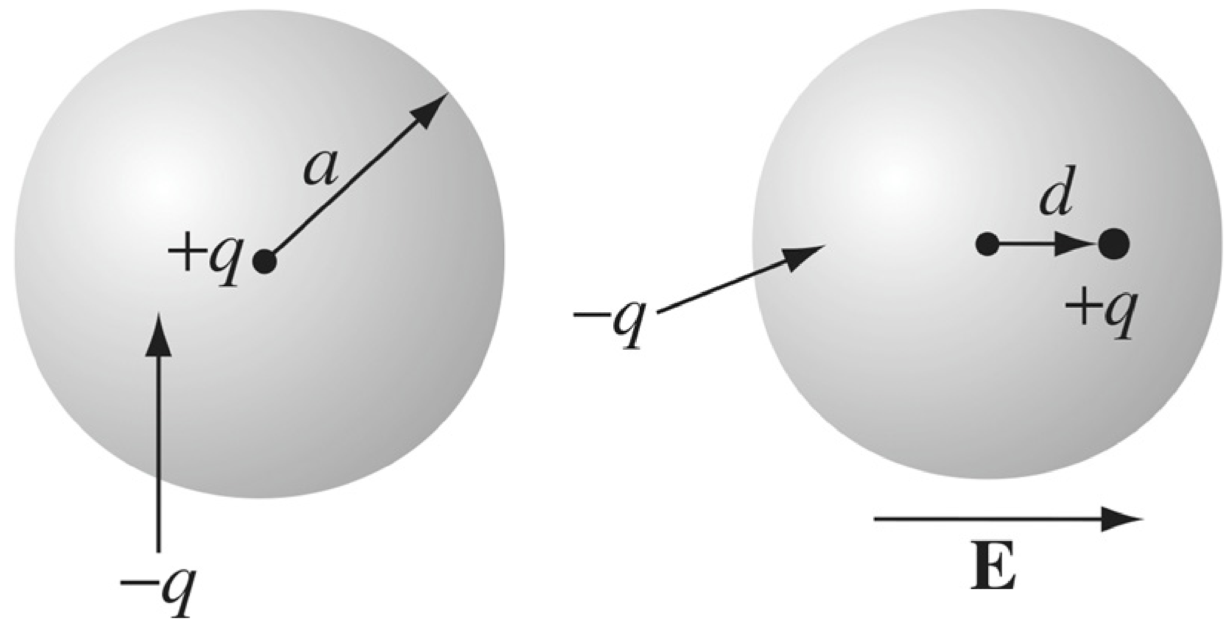
TABLE 4.1 Atomic Polarizabilities ($\alpha/4\pi\epsilon_0$, in units of 10^{-30} m^3). *Data from: Handbook of Chemistry and Physics*, 91st ed. (Boca Raton: CRC Press, 2010).

Example 4.1 A primitive model for an atom consists of a point nucleus ($+q$) surrounded by a uniformly charged spherical cloud ($-q$) of radius a. Calculate the atomic polarizability of such an atom.

Sol. The actual displacements involved are extremely small. It is reasonable to assume that the electron cloud retains its spherical shape.

The equilibrium occurs when the nucleus is displaced a distance d from the center of the sphere.

The external field pushing the nucleus to the right exactly balances the internal field pulling it to the left.



$$E_e = \frac{1}{4\pi\varepsilon_0}\frac{qd}{a^3} \qquad p = qd = (4\pi\varepsilon_0 a^3)E = \alpha E$$

$$\alpha = 4\pi\varepsilon_0 a^3 \quad \text{the atomic polarizability}$$

Example 4.3 If we have two spheres of charge: a positive sphere and a negative sphere. When the material is uniformly polarized, all the plus charges move slightly upward (the z-direction), all the minus charges move slightly downward. The two sphere no longer overlap perfectly. Find the polarizability.

Sol. The electric field inside a uniform charged sphere of radius a

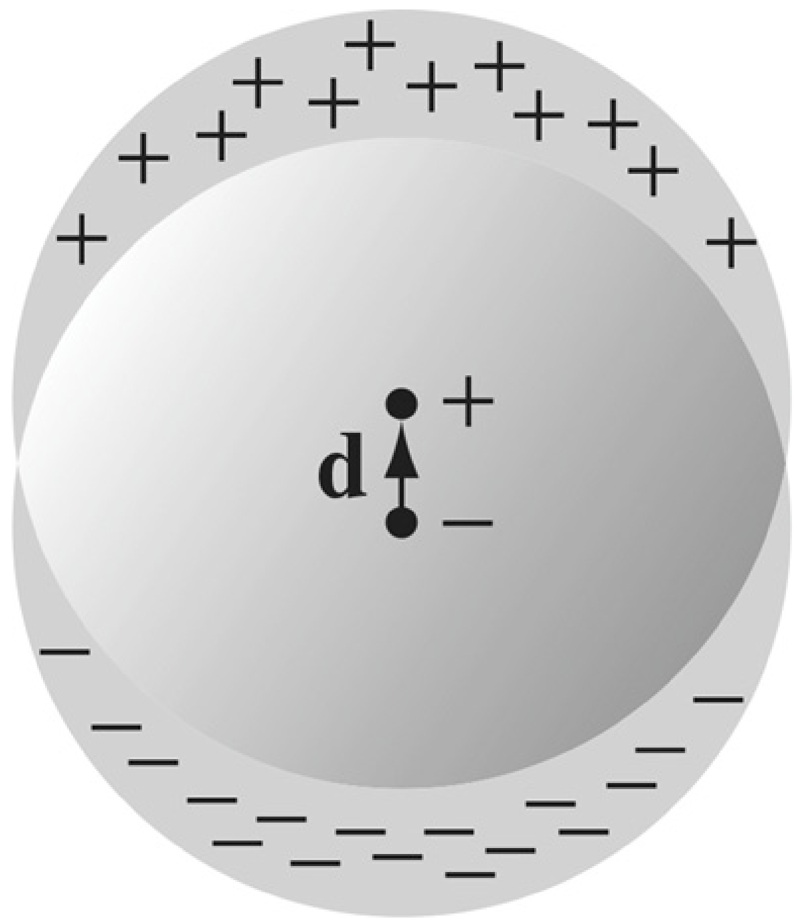
$$\mathbf{E}_e(r) = \frac{1}{4\pi r^2}\frac{\frac{4}{3}\pi r^3\rho}{\varepsilon_0}\hat{\mathbf{r}} = \frac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{a^3}\hat{\mathbf{r}}, \text{ where } q = \tfrac{4}{3}\pi a^3\rho$$

Two uniformly charged spheres separated by **d** produce the electric field:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{q+}(\mathbf{r}_+) + \mathbf{E}_{q-}(\mathbf{r}_-) = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^3}(\mathbf{r}_+ - \mathbf{r}_-)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{a^3}((\mathbf{r} - \frac{1}{2}\mathbf{d}) - (\mathbf{r} + \frac{1}{2}\mathbf{d})) = -\frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{d}}{a^3}$$

$$= -\frac{1}{4\pi\varepsilon_0 a^3}\mathbf{p} \qquad \therefore \quad \alpha = 4\pi\varepsilon_0 a^3$$



Prob.4.2 According to **quantum mechanics**, the electron cloud for a hydrogen atom in ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a},$$

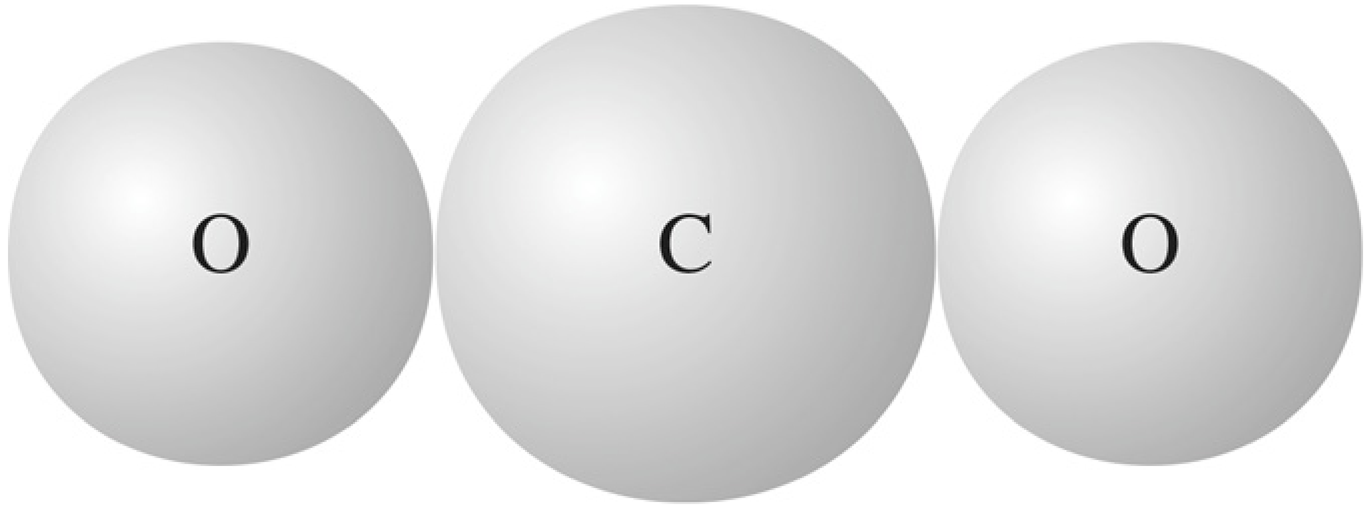
where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom. [Hint: First calculate the electric field of the electron cloud, $E_e(r)$; then expand the exponential, assume $r \ll a$.

Sol. For a more sophisticated approach, see W. A. Bowers, Am. J. Phys. **54**, 347 (1986).

Polarizability of Molecules

For molecules the situation is not quite so simple, because frequently they polarize more readily in some directions than others.

For instance, carbon dioxide CO_2



When the field is at some angle to the axis, you must resolve it into parallel and perpendicular components, and multiply each by the pertinent polarizability:

$$\mathbf{p} = \alpha_\perp \mathbf{E}_\perp + \alpha_{||} \mathbf{E}_{||}$$

In this case the induced dipole moment may not even be in the same direction as $\mathbf{E}$.

Polarizability Tensor

CO_2 is relatively simple, as molecules go, since the atoms at least arrange themselves in a straight line.

For a complete asymmetrical molecule, a more general linear relation between $\mathbf{E}$ and $\mathbf{p}$.

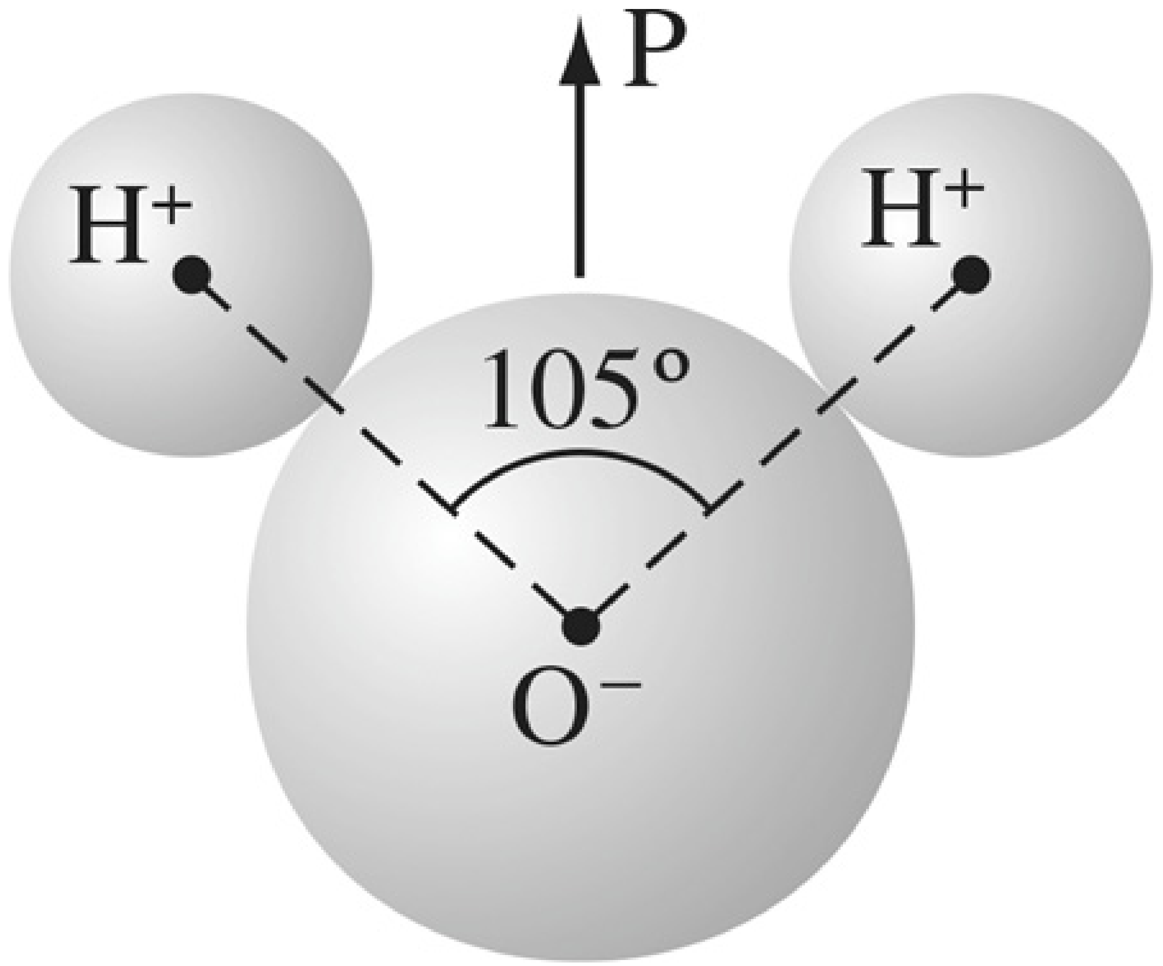
$$p_x = \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z$$
$$p_y = \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z$$
$$p_z = \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z$$

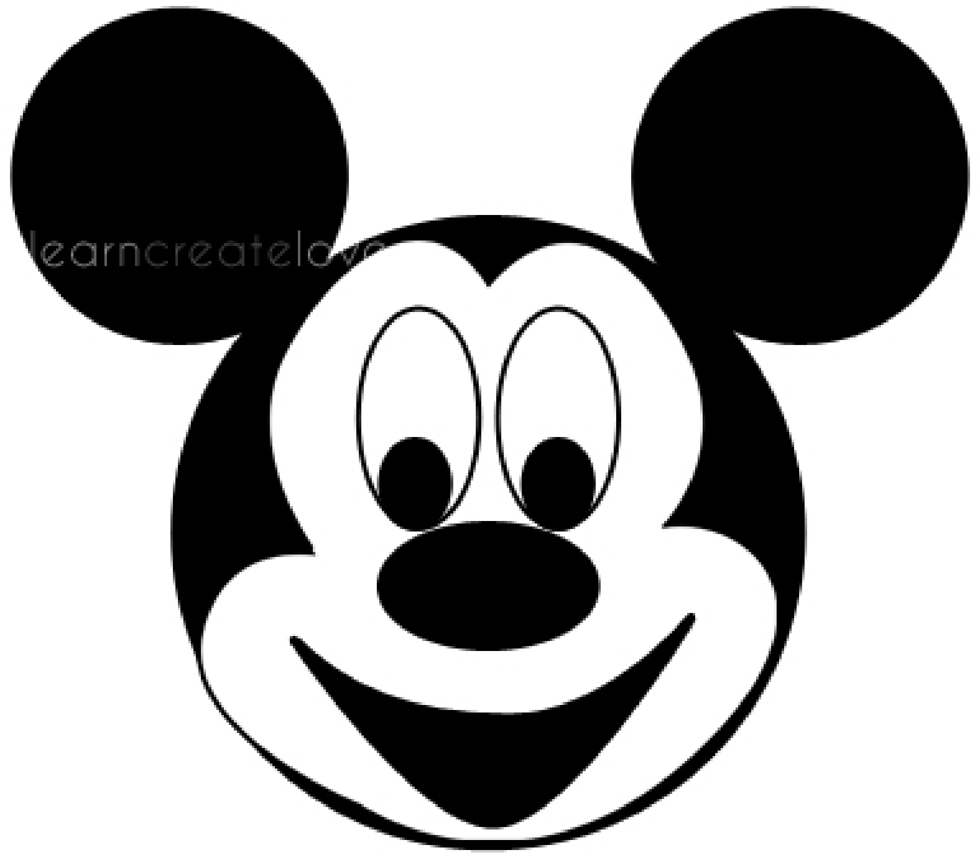
The set of nine constants α_{ij} constitute the polarizability tensor for the molecule.

It is always possible to choose "principal" axes such that the off-diagonal terms vanish, leaving just three nonzero polarizabilities.

4.1.3. Alignment of Polar Molecules

The neutral atom has no dipole moment to start with---$\mathbf{p}$ was induced by the applied field $\mathbf{E}$. However, some molecules have *built-in*, *permanent* dipole moment.



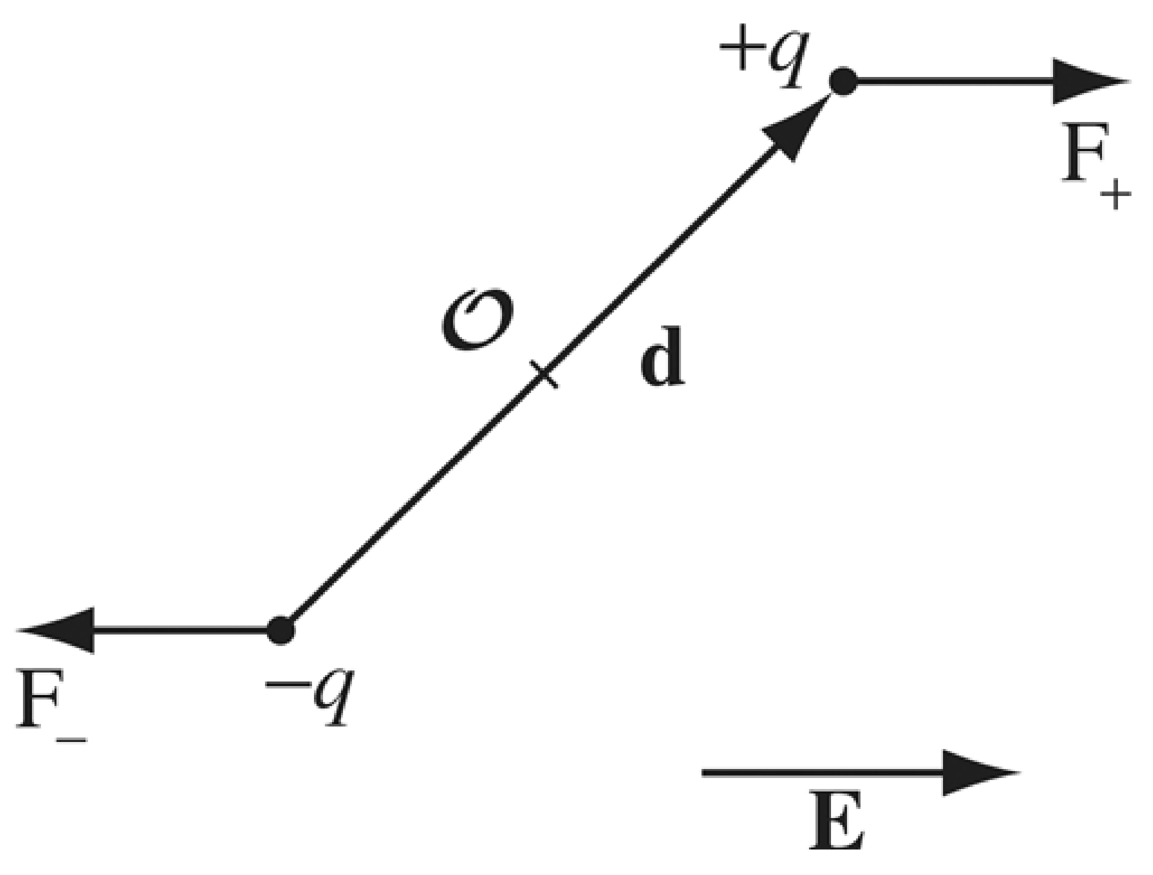


The dipole moment of water is usually large : $6.1*10^{-30}$ C·m, which accounts for its effectiveness as solvent.

What happens when polar molecules are placed in an electric field? Rotating

Torque for a Permanent Dipole in Uniform Field

In a uniform field, the force on the positive end, $\mathbf{F} = q\mathbf{E}$, exactly cancels the force on the negative end. However, there will be a torque:



$$\begin{aligned}\mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) \\ &= [(\mathbf{d}/2) \times (q\mathbf{E}) + (-\mathbf{d}/2) \times (-q\mathbf{E})] \\ &= q\mathbf{d} \times \mathbf{E}\end{aligned}$$

This dipole $\mathbf{p} = q\mathbf{d}$ in a uniform field experiences a torque $\mathbf{N} = \mathbf{p} \times \mathbf{E}$

$\mathbf{N}$ is in such a direction as to line $\mathbf{p}$ up parallel to $\mathbf{E}$.

A polar molecule that is free to rotate will swing around until it points in the direction of the applied field.

Net Force due to Field Nonuniformity

If the field is nonuniform, so that $\mathbf{F}_+$ does not exactly balance $\mathbf{F}_-$; There will be a net force on the dipole.

Of course, $\mathbf{E}$ must change rather abruptly for there to be significant in the space of one molecule, so this is not ordinarily a major consideration in discussing the behavior of dielectrics.

The formula for the force on a dipole in a nonuniform field is of some interest

Evaluated at different positions

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q(\mathbf{E}_+ - \mathbf{E}_-) = q(\Delta\mathbf{E}) \cong q((\mathbf{d} \cdot \nabla)\mathbf{E})$$

$$\mathbf{F} \cong (\mathbf{p} \cdot \nabla)\mathbf{E}$$

4.1.4. Polarization

What happens to a piece of dielectric material when it is placed in an electric field?

•Neutral atoms: Inducing tiny dipole moment, pointing in the same direction as the field (**stretching**).

•Polar molecules: experiencing a torque, tending to line it up along the field direction (**rotating**).

Results: A lot of little dipoles point along the direction of the field and the material becomes polarized.

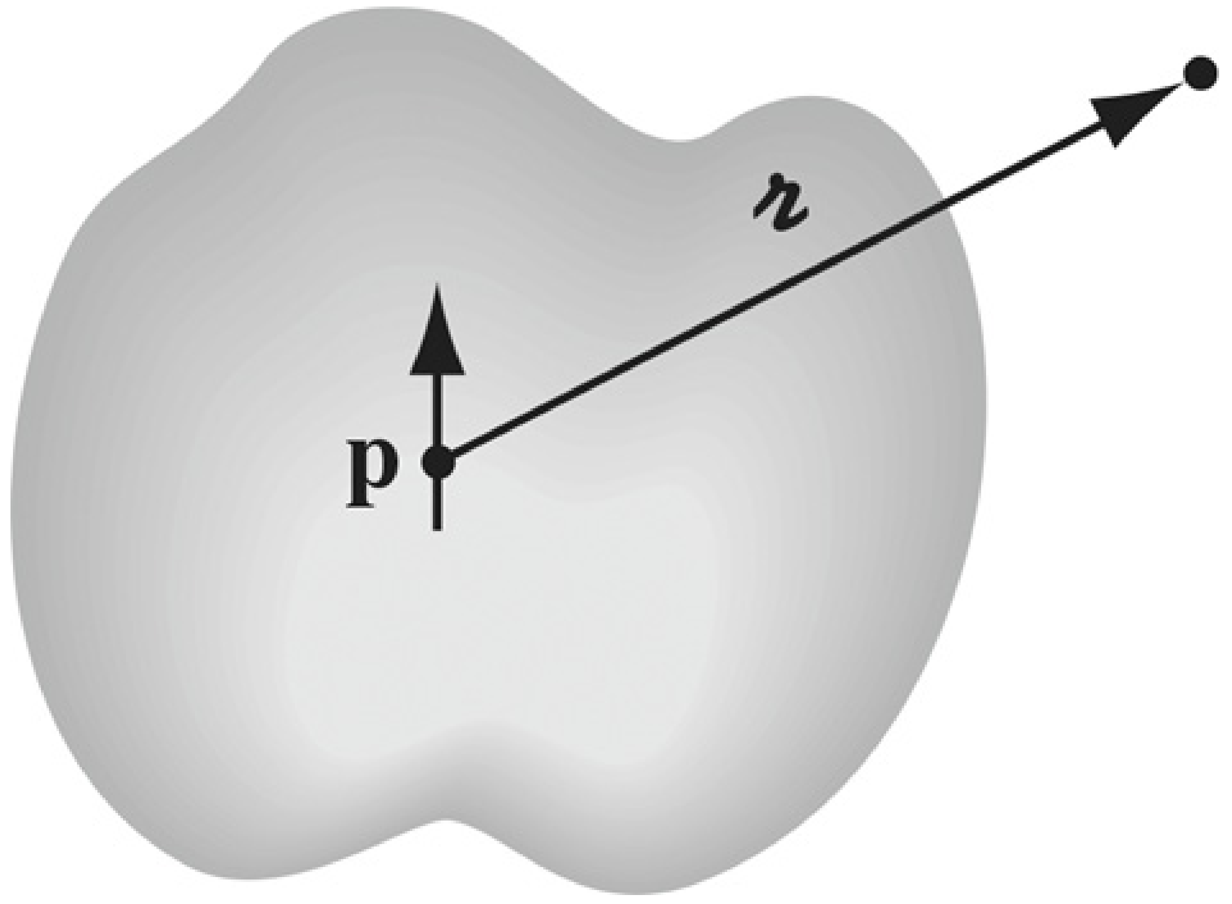
A convenient measure of this effect is

$\mathbf{P} \equiv$ dipole moment per unit volume, which is called the polarization.

4.2 The Field of a Polarized Object
4.2.1 Bound Charges

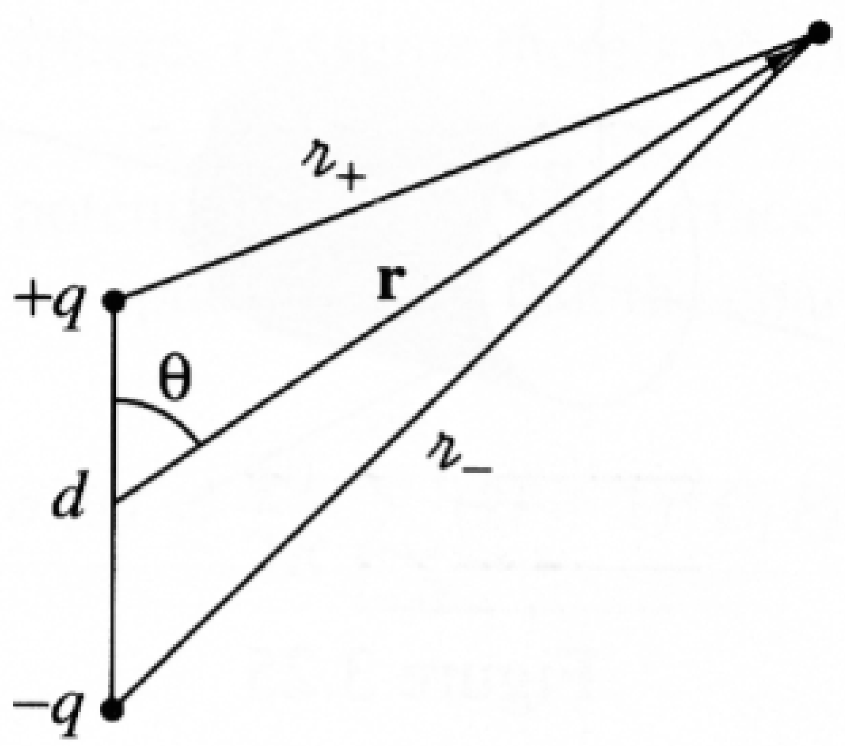
Suppose we have a piece of polarized material with polarization $\mathbf{P}$. What is the field produced by this object? (It is easier to work with potential.)

For a single dipole $\mathbf{p}$, the potential is $$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{p}}{\mathscr{r}^2}$$
where $\vec{\mathscr{r}}$ is the vector from the dipole to the point at which we are evaluating the potential.



Ref. Sec. 3.4

Example 3.10 An electric dipole consists of two equal and opposite charges separated by a distance d. Find the approximate potential V at points far from the dipole.



Sol: $V(\mathbf{r}) = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{\left|\mathbf{r} - \frac{d}{2}\hat{\mathbf{z}}\right|} - \dfrac{1}{\left|\mathbf{r} + \frac{d}{2}\hat{\mathbf{z}}\right|}\right) = \dfrac{q}{4\pi\varepsilon_0 r}\left((1+\varepsilon)^{-1/2} - (1-\varepsilon)^{-1/2}\right)$

where $\varepsilon = \dfrac{r'}{r}\left(\dfrac{r'}{r} - 2\cos\theta\right) \cong -\dfrac{d}{r}\cos\theta$ (since $\dfrac{r'}{r} \ll 1$ and $r' = \dfrac{d}{2}$)

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r}\left((1+\varepsilon)^{-1/2} - (1-\varepsilon)^{-1/2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left(\frac{d}{r}\cos\theta\right) = \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2} = \frac{p}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2}$$

where $\mathbf{p} = q\mathbf{d}$ pointing from negative charge to positive charge.

4.2.1 Bound Charges

For an infinitesimal dipole moment $d\mathbf{p} = \mathbf{P}d\tau$, the total potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\hat{\mathscr{r}}\cdot \mathrm{d}\mathbf{p}}{\mathscr{r}^2} = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\hat{\mathscr{r}}\cdot \mathbf{P}(\mathbf{r}')}{\mathscr{r}^2}\mathrm{d}\tau'$$

Note that $\nabla'(\frac{1}{\mathscr{r}}) = \frac{\vec{\mathscr{r}}}{\mathscr{r}^2}$ with respect to the source coordinate.

$$\boxed{\vec{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'}$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_v \mathbf{P}\cdot\nabla'(\frac{1}{\mathscr{r}})\,\mathrm{d}\tau'$$

Integrating by parts, using product rule, gives

$$\boxed{\nabla\cdot(\frac{\mathbf{A}}{g}) = \frac{1}{g}(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot\nabla(\frac{1}{g})}$$

Ref. Sec. 1.2.6

Bound Charges

$$V = \frac{1}{4\pi\varepsilon_0}\int_v \mathbf{P}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)\,\mathrm{d}\tau'$$

$$\boxed{\mathbf{P}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right) = \nabla'\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right) - \frac{1}{\mathscr{r}}\nabla'\cdot\mathbf{P}}$$

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_v \nabla'\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right)\,\mathrm{d}\tau' - \int_v \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\,\mathrm{d}\tau'\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\mathbf{P}}{\mathscr{r}}\cdot d\mathbf{a}' + \frac{1}{4\pi\varepsilon_0}\int_v \frac{1}{\mathscr{r}}(-\nabla'\cdot\mathbf{P})\,\mathrm{d}\tau'$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathrm{n}}$$

surface charge

$$\rho_b = -\nabla'\cdot\mathbf{P}$$

volume charge

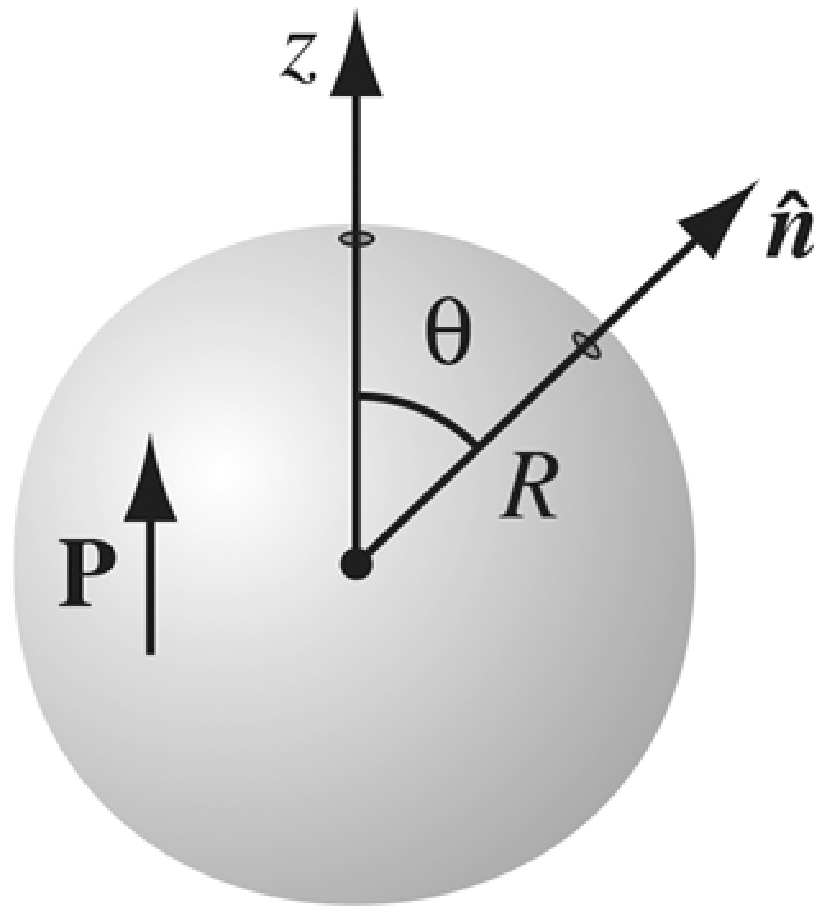
Bound Surface and Volume Charges

$$\begin{cases} \sigma_b = \mathbf{P} \cdot \hat{\mathrm{n}} \\ \rho_b = -\nabla' \cdot \mathbf{P} \end{cases} \qquad V = \frac{1}{4\pi\varepsilon_0} \oint_S \frac{\sigma_b}{\mathscr{r}} \mathrm{d}a' + \frac{1}{4\pi\varepsilon_0} \int_v \frac{\rho_b}{\mathscr{r}} \mathrm{d}\tau'$$

This means that the potential of a polarized object is the same as that produced by a surface charge density plus a volume charge density.

Ex. 4.2 Find the electric field produced by a uniformly polarized sphere of radius R.

Sol: See the next three pages.



$$\begin{cases} \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = P\cos\theta' \\ \rho_b = -\nabla'\cdot\mathbf{P} = 0 \end{cases}$$

$$\boxed{\begin{aligned} da' &= (Rd\theta')(R\sin\theta' d\phi') \\ &= R^2\sin\theta' d\theta' d\phi' \end{aligned}}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_0^\pi \frac{P\cos\theta'}{\mathscr{r}} 2\pi R^2 \sin\theta' d\theta'$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n P_n(\cos\vartheta) \qquad r \ge R$$

$$= \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\vartheta) \qquad r \le R$$

ϑ: angle between $\mathbf{r}$ and $\mathbf{r}' \Rightarrow \cos\vartheta = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'$

It will be easier if we let $\mathbf{r}$ lie on the z axis, so that the angle between them changes from ϑ to θ'

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n P_n(\cos\theta') \qquad r \ge R$$

$$= \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\theta') \qquad r \le R$$

Ref. Sec. 3.4

$$V(r,0,0) = \frac{1}{4\pi\varepsilon_0}\int_0^\pi \frac{1}{r}\sum_{n=0}^{\infty}(\frac{R}{r})^n\, P_n(\cos\theta')P\cos\theta' 2\pi R^2 \sin\theta' d\theta' \qquad r \ge R$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-1}^1 \frac{1}{r}\sum_{n=0}^{\infty}(\frac{R}{r})^n\, P_n(\cos\theta')P\cos\theta' 2\pi R^2 d\cos\theta'$$

$$= \frac{1}{2\varepsilon_0}\int_{-1}^1 \frac{PR^3}{r^2}\cos^2\theta' d\cos\theta' = \frac{1}{3\varepsilon_0}\frac{PR^3}{r^2} \qquad \begin{pmatrix}\text{orthogonality} \\ \text{only } n=1 \text{ survive}\end{pmatrix}$$

$$V(r,0,0) = \frac{1}{4\pi\varepsilon_0}\int_0^\pi \frac{1}{R}\sum_{n=0}^{\infty}(\frac{r}{R})^n\, P_n(\cos\theta')P\cos\theta' 2\pi R^2 \sin\theta' d\theta' \qquad r \le R$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-1}^1 \frac{1}{R}(\frac{r}{R})\, P_1(\cos\theta')\, P\cos\theta' 2\pi R^2 d\cos\theta'$$

$$= \frac{P}{3\varepsilon_0} r \qquad \begin{pmatrix}\text{orthogonality} \\ \text{only } n=1 \text{ survive}\end{pmatrix}$$

Allow **r** a θ-dependence.

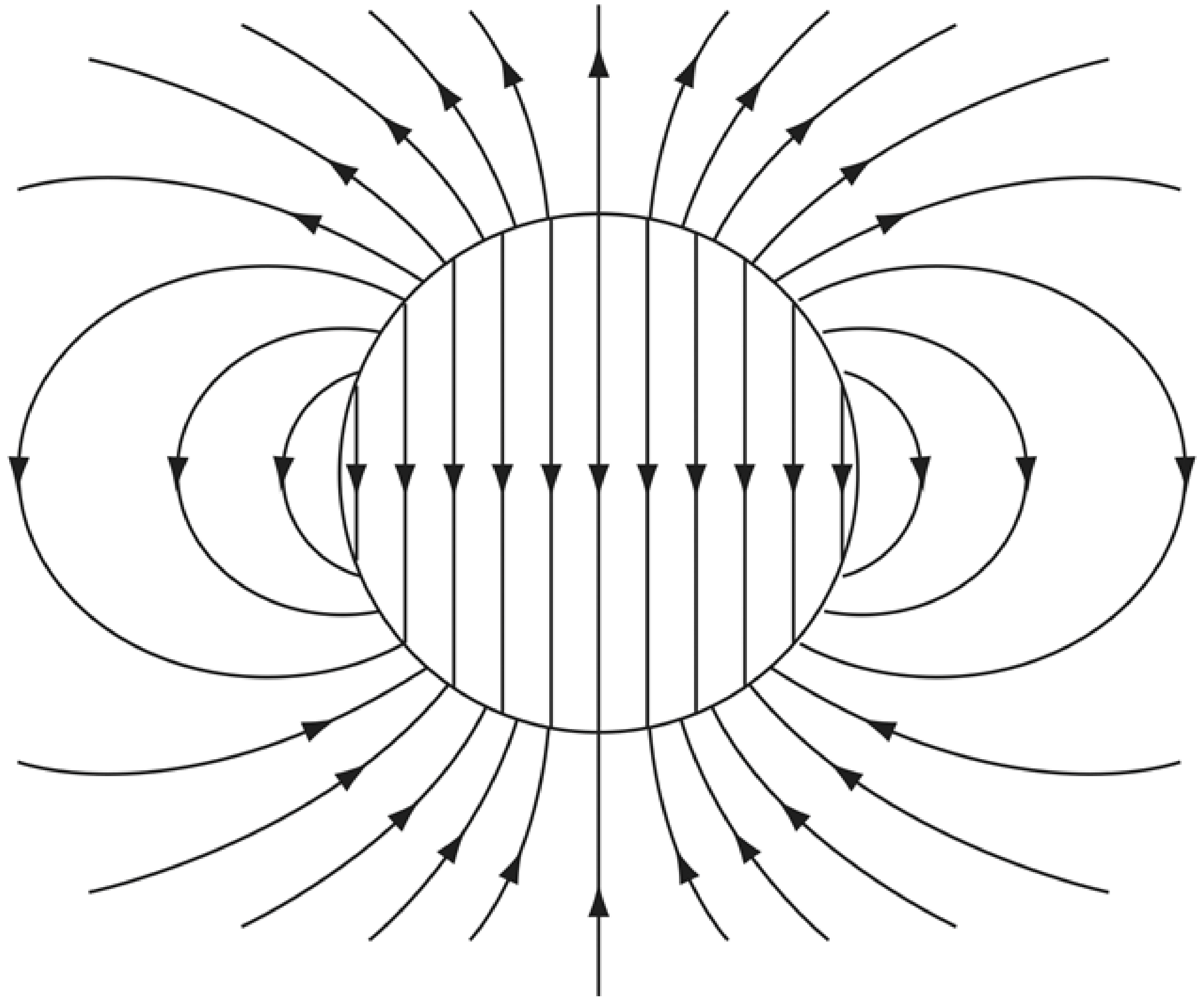
$$V(r,\theta,0) = \begin{cases} \dfrac{1}{3\varepsilon_0}\dfrac{PR^3}{r^2}\cos\theta & (r \ge R) \\ \dfrac{P}{3\varepsilon_0} r\cos\theta & (r \le R) \end{cases}$$

Electric Field of a Uniformly Polarized Sphere

$$V(r) = \frac{P}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta \quad \text{(outside)}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{(\frac{4}{3}\pi R^3 P)}{r^2}\cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2} \quad \text{where } p = \frac{4}{3}\pi R^3 P$$



$$V(r) = \frac{P}{3\varepsilon_0} r\cos\theta = \frac{P}{3\varepsilon_0} z \quad \text{(inside)}$$

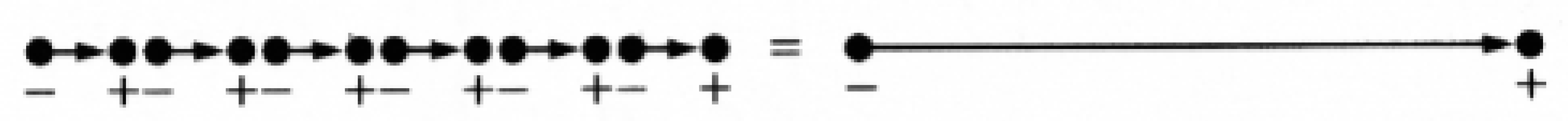
Why are the field lines not continuous?

$$\mathbf{E} = -\nabla V = -\frac{P}{3\varepsilon_0}\hat{\mathbf{z}} \quad \text{uniformly}$$

4.2.2 Physical Interpretation of Bound Charges

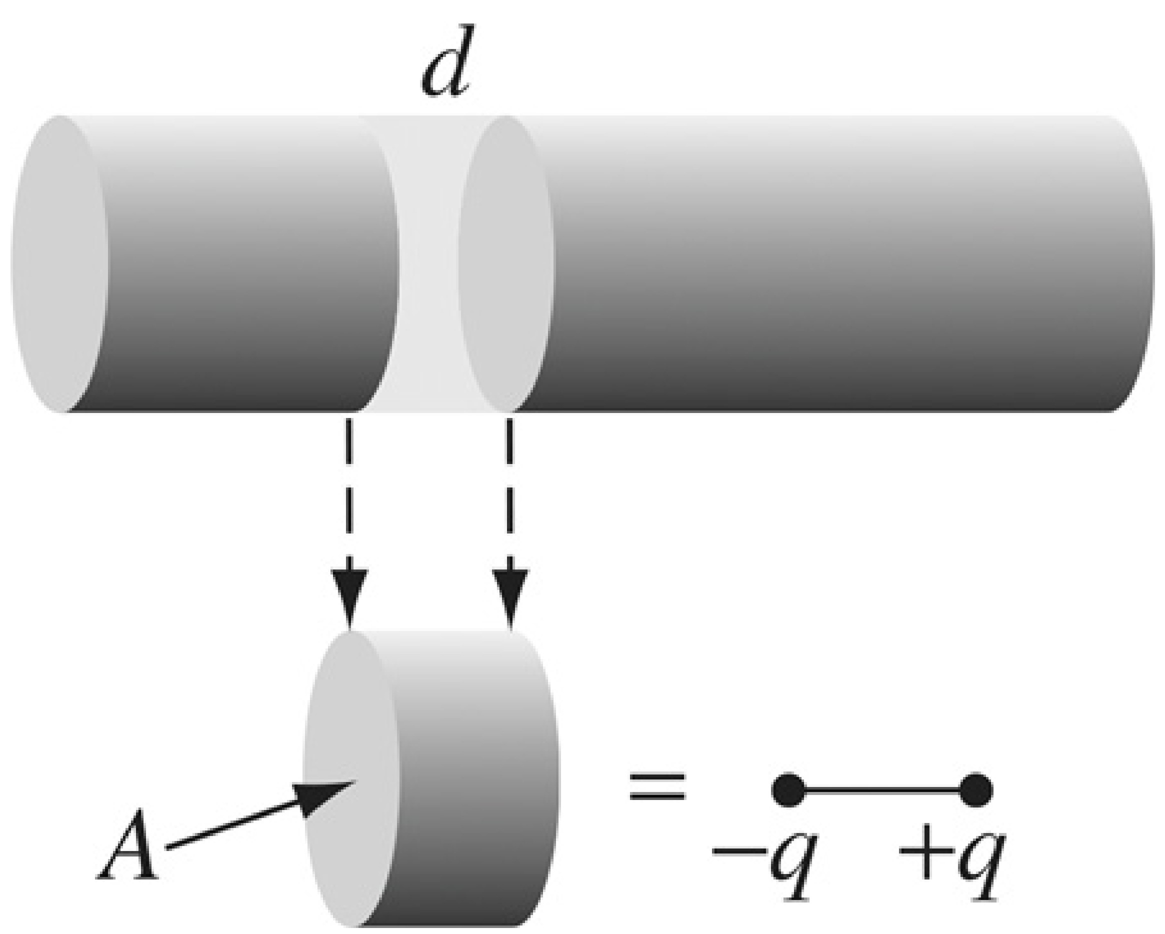
What is the physical meaning of the bound charge?

Consider a long string of dipoles.



The net charge at the ends is called the bound charge. The bound charge is no different from any other kind.

Consider a "tube" of dielectric with a given polarization $\mathbf{P}$.



$$\sigma_b = \frac{q}{A_{end}} = P\cos\theta = \mathbf{P}\cdot\hat{\mathbf{n}}$$

Nonuniform Polarization
→ The Bound Volume Charge

If the polarization is nonuniform, we get accumulations of bound charge within the material as well as on the surface.

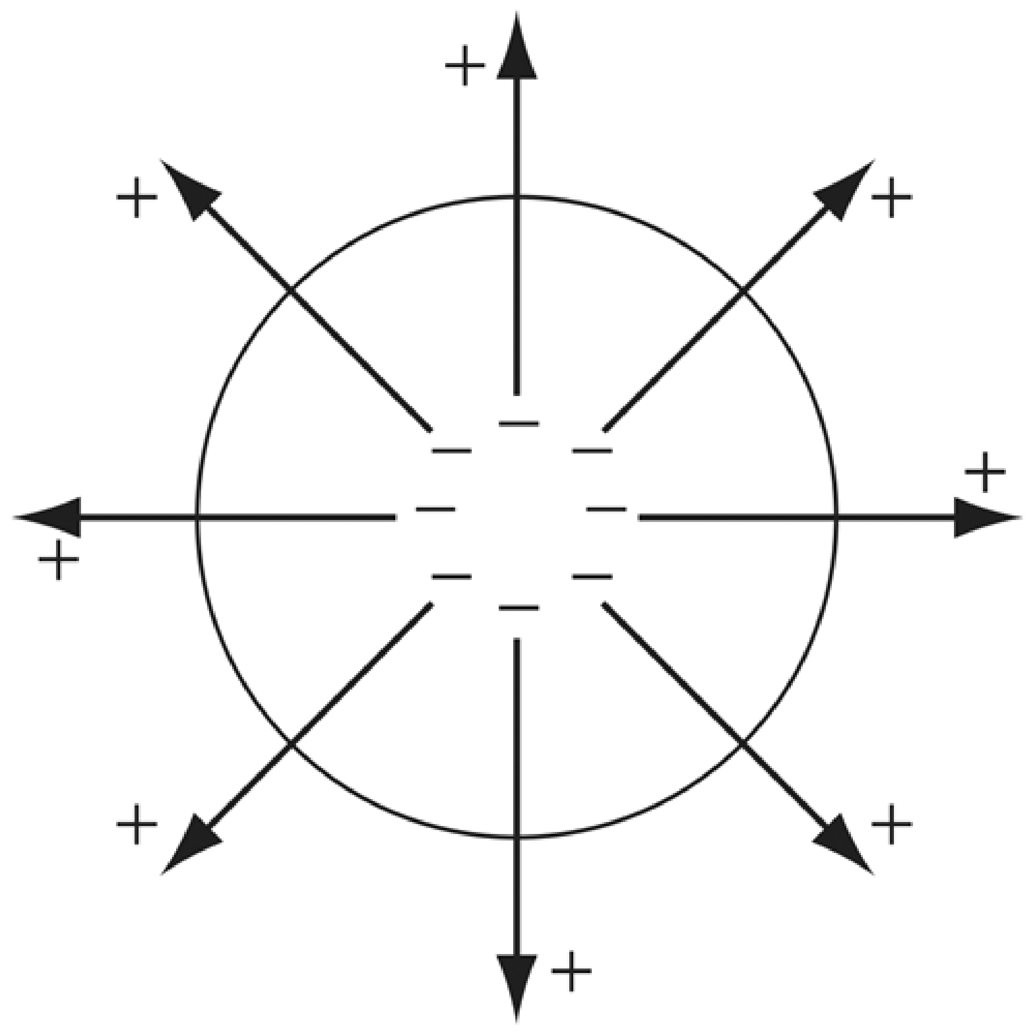
The net bound charge in a given volume is equal and opposite to the amount that has been pushed out through the surface.

$$\int_v \rho_b d\tau = -\oint_S \mathbf{P} \cdot d\mathbf{a} = \int_v (-\nabla \cdot \mathbf{P})\, d\tau$$

Gauss's laws

This is true for any volume bound charge.

$$\rho_b = -\nabla \cdot \mathbf{P}$$

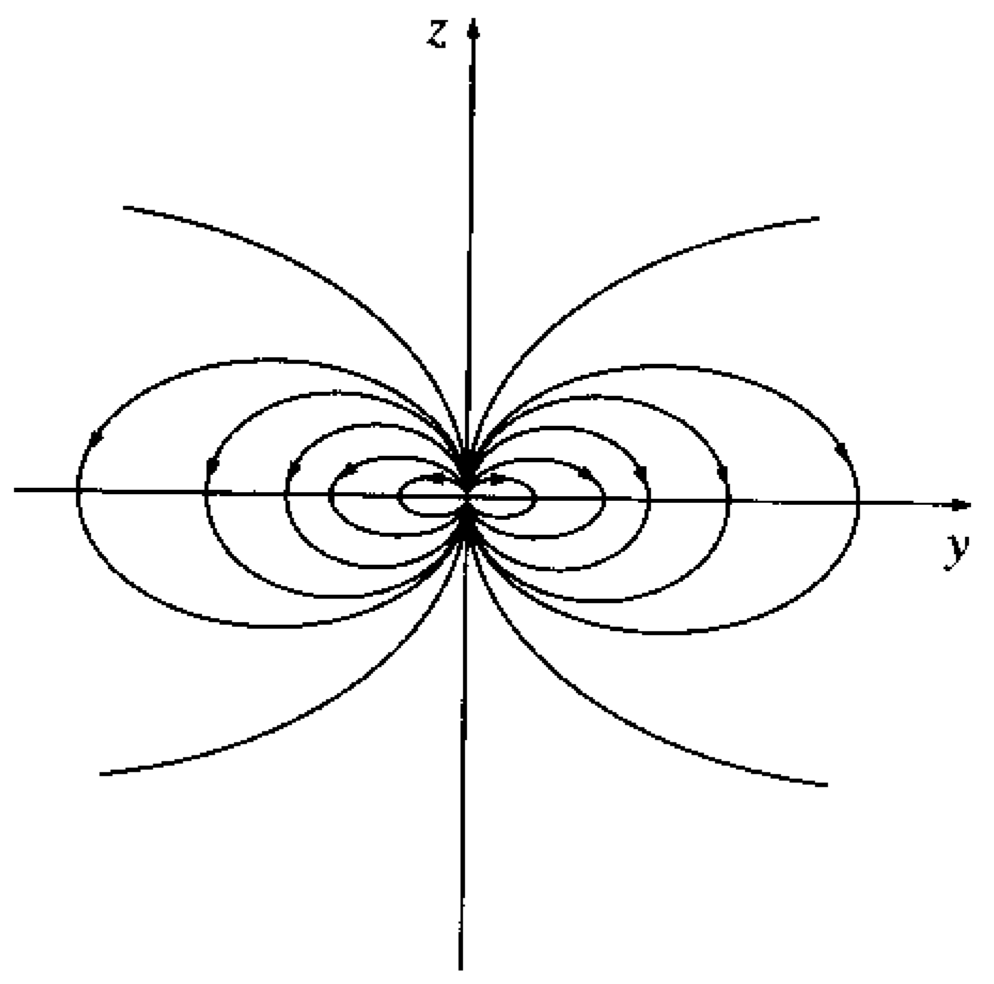


4.2.3 The Field Inside a Dielectric

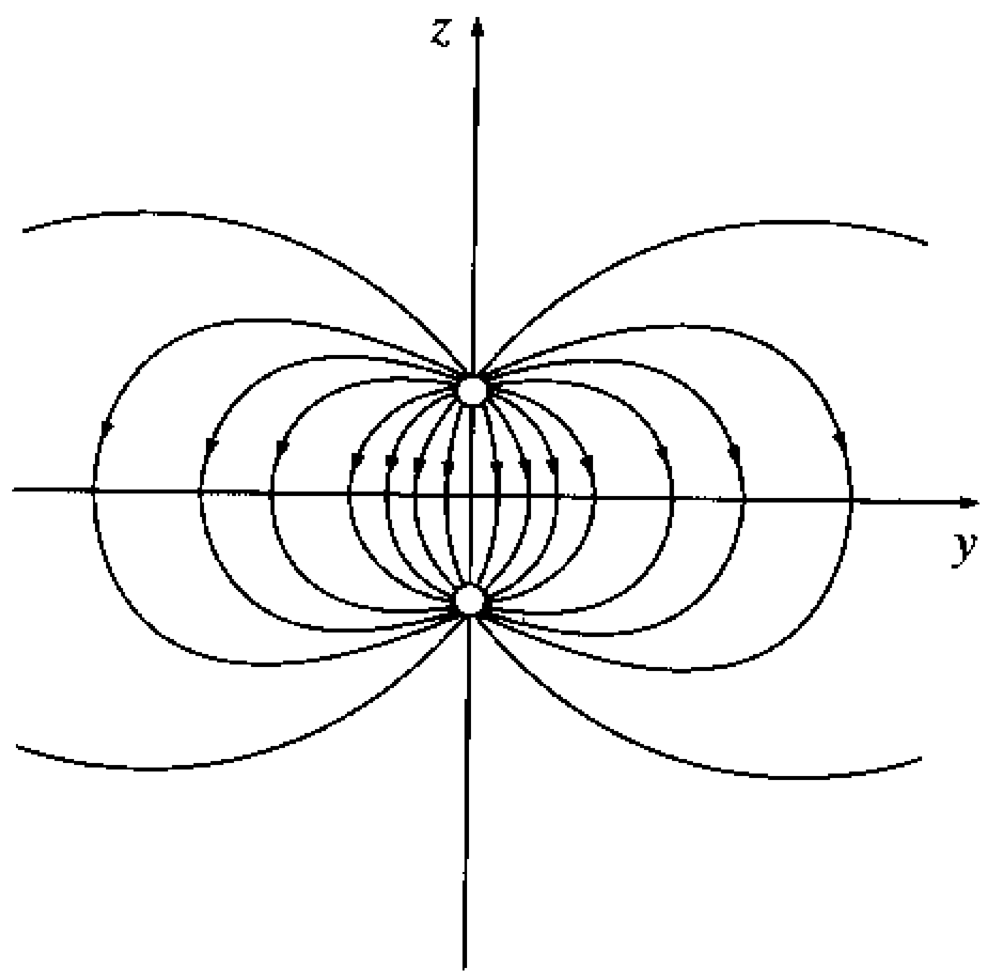
What kind of dipole is we actually dealing with, “pure” dipole or “physical” dipole?

Outside the dielectric there is no real problem, since we are far away from the molecules.

Inside the dielectric, however, we can hardly pretend to be far from all the dipoles.



(a) Field of a "pure" dipole



(a) Field of a "physical" dipole

4.2.3 The Field Inside a Dielectric

The electric field inside matter must be very complicated, on the ***microscopic*** level, which would be utterly impossible to calculate, nor would it be of much interest.

The ***macroscopic*** field is defined as the average field over regions large enough to contain many thousands of atoms.

The macroscopic field smoothes over the uninteresting microscopic fluctuation and is *what people mean* when they speak of “the field inside matter”.

Homework of Chap. 4 (part I)

Problem 4.2 According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a},$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom. [*Hint*: First calculate the electric field of the electron cloud, $E_e(r)$; then expand the exponential, assuming $r \ll a$.[1]

Problem 4.9 A dipole $\mathbf{p}$ is a distance r from a point charge q, and oriented so that $\mathbf{p}$ makes an angle θ with the vector $\mathbf{r}$ from q to $\mathbf{p}$.

(a) What is the force on $\mathbf{p}$?

(b) What is the force on q?

Problem 4.10 A sphere of radius R carries a polarization

$$\mathbf{P}(\mathbf{r}) = k\mathbf{r},$$

where k is a constant and $\mathbf{r}$ is the vector from the center.

(a) Calculate the bound charges σ_b and ρ_b.

(b) Find the field inside and outside the sphere.

Homework of Chap. 4 (part I)

Problem 4.16 Suppose the field inside a large piece of dielectric is $\mathbf{E}_0$, so that the electric displacement is $\mathbf{D}_0 = \varepsilon_0 \mathbf{E}_0 + \mathbf{P}$.

(a) Now a small spherical cavity (Fig. 4.19a) is hollowed out of the material. Find the field at the center of the cavity in terms of $\mathbf{E}_0$ and $\mathbf{P}$. Also find the displacement at the center of the cavity in terms of $\mathbf{D}_0$ and $\mathbf{P}$. Assume the polarization is "frozen in," so it doesn't change when the cavity is excavated.

(b) Do the same for a long needle-shaped cavity running parallel to $\mathbf{P}$ (Fig. 4.19b).

(c) Do the same for a thin wafer-shaped cavity perpendicular to $\mathbf{P}$ (Fig. 4.19c).

Assume the cavities are small enough that $\mathbf{P}$, $\mathbf{E}_0$, and $\mathbf{D}_0$ are essentially uniform. [*Hint*: Carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.]

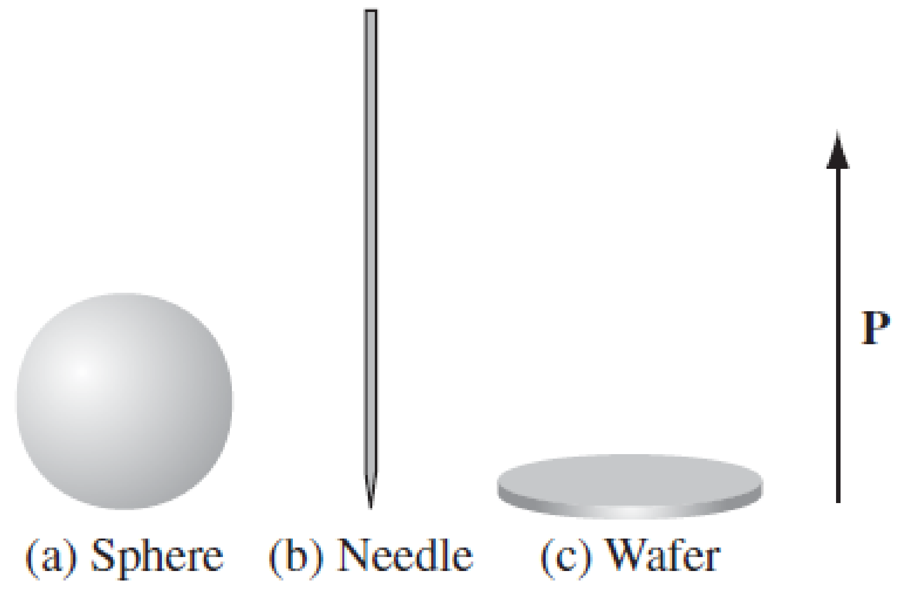


FIGURE 4.19

Problem 4.33 A dielectric cube of side a, centered at the origin, carries a "frozenin" polarization $\mathbf{P} = k\mathbf{r}$, where k is a constant. Find all the bound charges, and check that they add up to zero.

4.3 The Electric Displacement
4.3.1 Gauss's Law in the Presence of Dielectric

EM
Tsun-Hsu Chang

The effect of polarization is to produce accumulations of bound charge, $\rho_b = -\nabla \cdot \mathbf{P}$ within the dielectric and $\sigma_b = \mathbf{P} \cdot \hat{\mathrm{n}}$ on the surface.

Now we are going to treat the field caused by both bound charge and free charge. $\rho = \rho_f + \rho_b$

$$= \rho_f - \nabla \cdot \mathbf{P} = \varepsilon_0 \nabla \cdot \mathbf{E}$$

where $\mathbf{E}$ is now the total field, not just that portion generated by polarization . $\varepsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{P} = \rho_f$

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

Let $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ the electric displacement
Gauss's law reads $\nabla \cdot \mathbf{D} = \rho_f$

Gauss's Law in the Presence of Dielectric

$$\nabla \cdot \mathbf{D} = \rho_f \Rightarrow \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f\,\text{enc}}$$

The total free charge enclosed in the volume

In a typical problem, we know ρ_f, but not ρ_b. So this equation allows us to deal with the information at hand.

What is the contribution of the bound surface charge?

The bound surface charge σ_b can be considered as ρ_b varies rapidly but smoothly within the "skin". So Gauss's law can be applied elsewhere.

4.3.2 A Deceptive Parallel

"To solve problems involving dielectrics, you just forget all about the bound charge ─ calculate the field as you ordinarily would, only call the answer $\mathbf{D}$ instead of $\mathbf{E}$"

↑ This conclusion is false.

The divergence alone is insufficient to determine a vector field; you need to know the curl as well.

$$\nabla\times\mathbf{D}=\varepsilon_0\left(\nabla\times\mathbf{E}\right)+\nabla\times\mathbf{P}=\nabla\times\mathbf{P} \leftarrow \text{not always zero}$$

Since the curl of $\mathbf{D}$ is not always zero, $\mathbf{D}$ cannot be expressed as the gradient of a scalar.

Advice: If the problem exhibits *spherical*, *cylindrical*, or *plane* symmetry, then you can get $\mathbf{D}$ directly from the generalized Gauss's law i.e., $\nabla\cdot\mathbf{D}=\rho_f$

4.3.3 Boundary Conditions

The electrostatic boundary condition in terms of $\mathbf{E}$

$$E_{above}^{\perp} - E_{below}^{\perp} = \frac{\sigma}{\varepsilon_0} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\mathbf{E}_{above}^{||} - \mathbf{E}_{below}^{||} = 0 \qquad \nabla \times \mathbf{E} = 0$$

The electrostatic boundary condition in terms of $\mathbf{D}$

$$D_{above}^{\perp} - D_{below}^{\perp} = \sigma_f \qquad \nabla \cdot \mathbf{D} = \rho_f$$

$$\mathbf{D}_{above}^{||} - \mathbf{D}_{below}^{||} = \mathbf{P}_{above}^{||} - \mathbf{P}_{below}^{||} \qquad \nabla \times \mathbf{D} = \nabla \times \mathbf{P}$$

4.4 Linear Dielectric
4.4.1 Susceptibility and Permittivity

For many substances, the polarization is proportional to the field, provided $\mathbf{E}$ is not too strong.

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

χ_e : the electric susceptibility of the medium

dimensionless

Materials that obey above equation are called linear dielectrics.

The total field $\mathbf{E}$ may be due in part to free charges and in part to the polarization itself.

Permittivity and Dielectric Constant

We cannot compute $\mathbf{P}$ directly from this equation:

the external field will polarize the material

$$\mathbf{E}_0 \rightarrow \mathbf{P}_0$$

this polarization will produce its own field and contribute to the total field.

$$\mathbf{P}_0 \rightarrow \mathbf{E}_0 + \Delta\mathbf{E}'_P$$

The new total field will polarize the material.

$$\mathbf{E}_0 + \Delta\mathbf{E}'_P \rightarrow \mathbf{P}_0 + \Delta\mathbf{P}'_0$$

Will this series converge? Depends.

Linear Media & Dielectric Constant

In linear media ,

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = \varepsilon_0 \left(1 + \chi_e\right) \mathbf{E} = \varepsilon \mathbf{E}$$

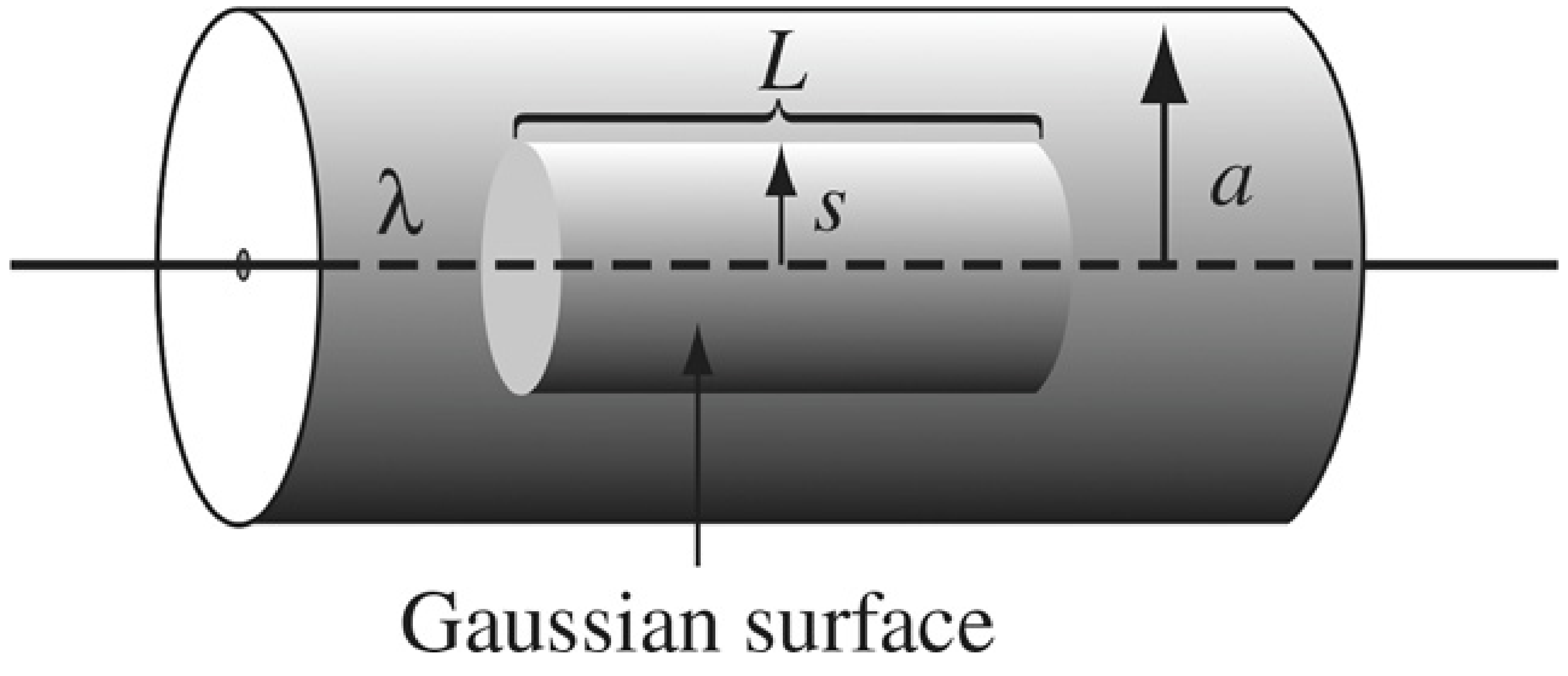
where $\underset{\uparrow}{\varepsilon} = \varepsilon_0 \left(1 + \chi_e\right)$ $\quad\quad$ $\varepsilon_r = \dfrac{\varepsilon}{\varepsilon_0} = 1 + \chi_e$

Permittivity of the material

Relative permittivity
or dielectric constant

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7-5.9
Neon	1.00013	Salt	5.9
Hydrogen (H_2)	1.000254	Silicon	11.7
Argon	1.000517	Methanol	33.0
Air (dry)	1.000536	Water	80.1
Nitrogen (N_2)	1.000548	Ice (-30° C)	104
Water vapor (100° C)	1.00589	$KTaNbO_3$ (0° C)	34,000

Example 4.4 A thin long straight wire, carrying uniform line charge density λ, is surrounded by rubber insulation out to a radius a. Find the electric displacement.



Sol: Drawing a cylindrical Gaussian surface, of radius s and length L, and applying the new Gauss's law, we find

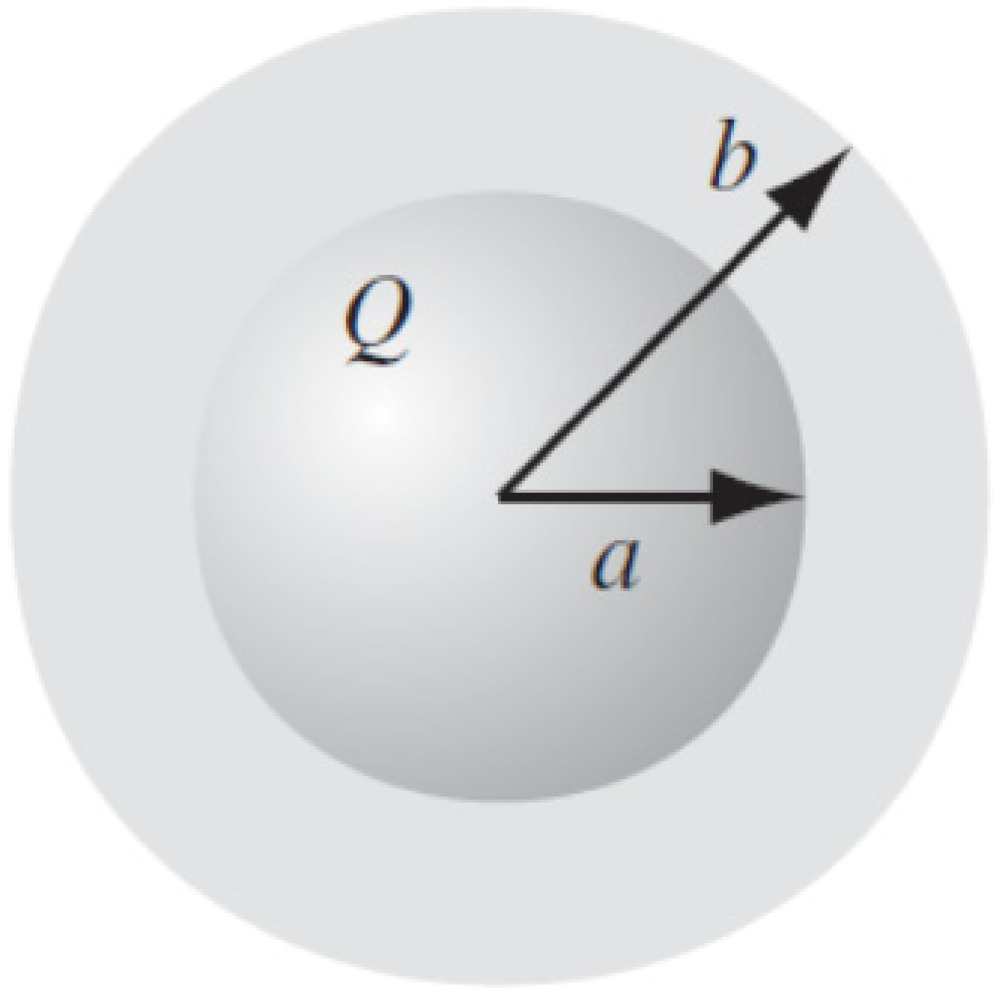
$$\nabla \cdot \mathbf{D} = \rho_f \Rightarrow \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f\,\text{enc}}$$

$$\text{Inside } (s \le a) \quad D(2\pi s L) = \lambda L \quad \Rightarrow \quad \mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}} \quad \therefore \mathbf{E} = \frac{\lambda}{2\pi s \varepsilon_r \varepsilon_0}\hat{\mathbf{s}}$$

$$\text{Outside } (s \ge a) \quad D(2\pi s L) = \lambda L \Rightarrow \mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}} \quad \therefore \mathbf{E} = \frac{\lambda}{2\pi s \varepsilon_0}\hat{\mathbf{s}}$$

Example 4.5 A metal sphere of radius a carries a charge Q. It is surrounded, out to radius b , by linear dielectric material of permittivity ε. Find the potential at the center (relative to infinity).

Sol: Use the generalized Gauss's law



$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}} \qquad \text{for all points} \quad r > a$$

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} & \text{for} \quad a < r < b \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} & \text{for} \quad r > b \end{cases}$$

The metal sphere is equalpotential

$$V = -\int_\infty^a \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \frac{Q}{4\pi\varepsilon_0 r^2}dr - \int_b^a \frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} + \frac{1}{\varepsilon_r a} - \frac{1}{\varepsilon_r b}\right)$$

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} = \frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} = \frac{Q}{4\pi r^2}\left(\frac{\chi_e}{1+\chi_e}\right)\hat{\mathbf{r}} \quad \text{for} \quad a < r < b$$

volume bound charge $\rho_b = -\nabla \cdot \mathbf{P} = -\frac{Q}{4\pi}\frac{\chi_e}{1+\chi_e}(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2}) = -\frac{Q\chi_e}{1+\chi_e}\delta^3(\mathbf{r})$

surface bound charge

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} \dfrac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon b^2} & \text{at the outer surface} \\[2ex] \dfrac{-\varepsilon_0 \chi_e Q}{4\pi\varepsilon a^2} & \text{at the inner surface} \end{cases}$$

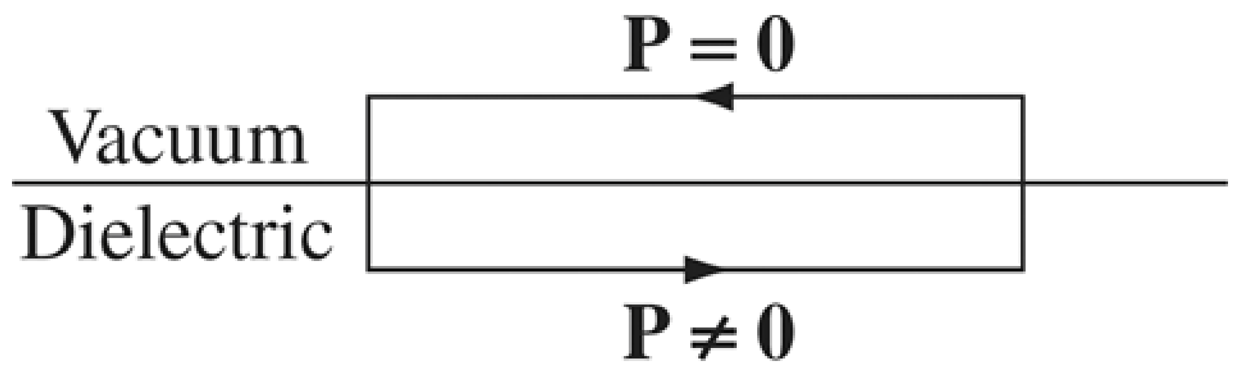
Note that $\hat{\mathbf{n}}$ always points outward with respect to the dielectric , which is $+\hat{\mathbf{r}}$ at b but $-\hat{\mathbf{r}}$ at a .

The surface bound charge at inner surface is negative. It is this layer of negative charge that reduces the field, within the dielectric by a factor of ε_r .

In this respect a dielectric is rather like an imperfect conductor.

Stokes' Theorem for the Polarization

In general, linear dielectrics cannot escape the defect that $\nabla \times \mathbf{P} \neq 0$



However, if the space is entirely filled with a homogenous linear dielectric, then this objection is void.

$$\nabla \cdot \mathbf{D} = \rho_f$$

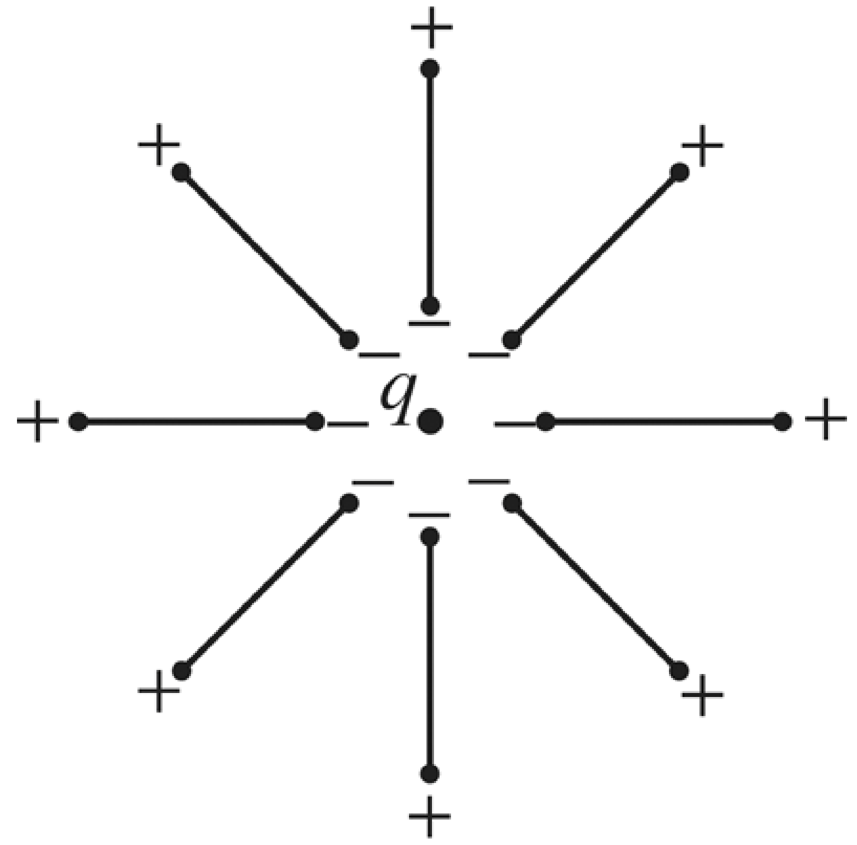
$$\nabla \times \mathbf{D} = 0$$

$$\mathbf{E} = \frac{1}{\varepsilon}\mathbf{D} = \frac{1}{\varepsilon_r}\mathbf{E}_{vac}$$

Remark : When all the space is filled with a homogenous linear dielectric, the field everywhere is simply reduced by a factor of one over the dielectric constant .

Shielding Effect & Susceptibility Tensor

The polarization of the medium partially "shields" the charge, by surrounding it with bound charge of the opposite sign.



For some material, it is generally easier to polarize in some directions than in others .

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

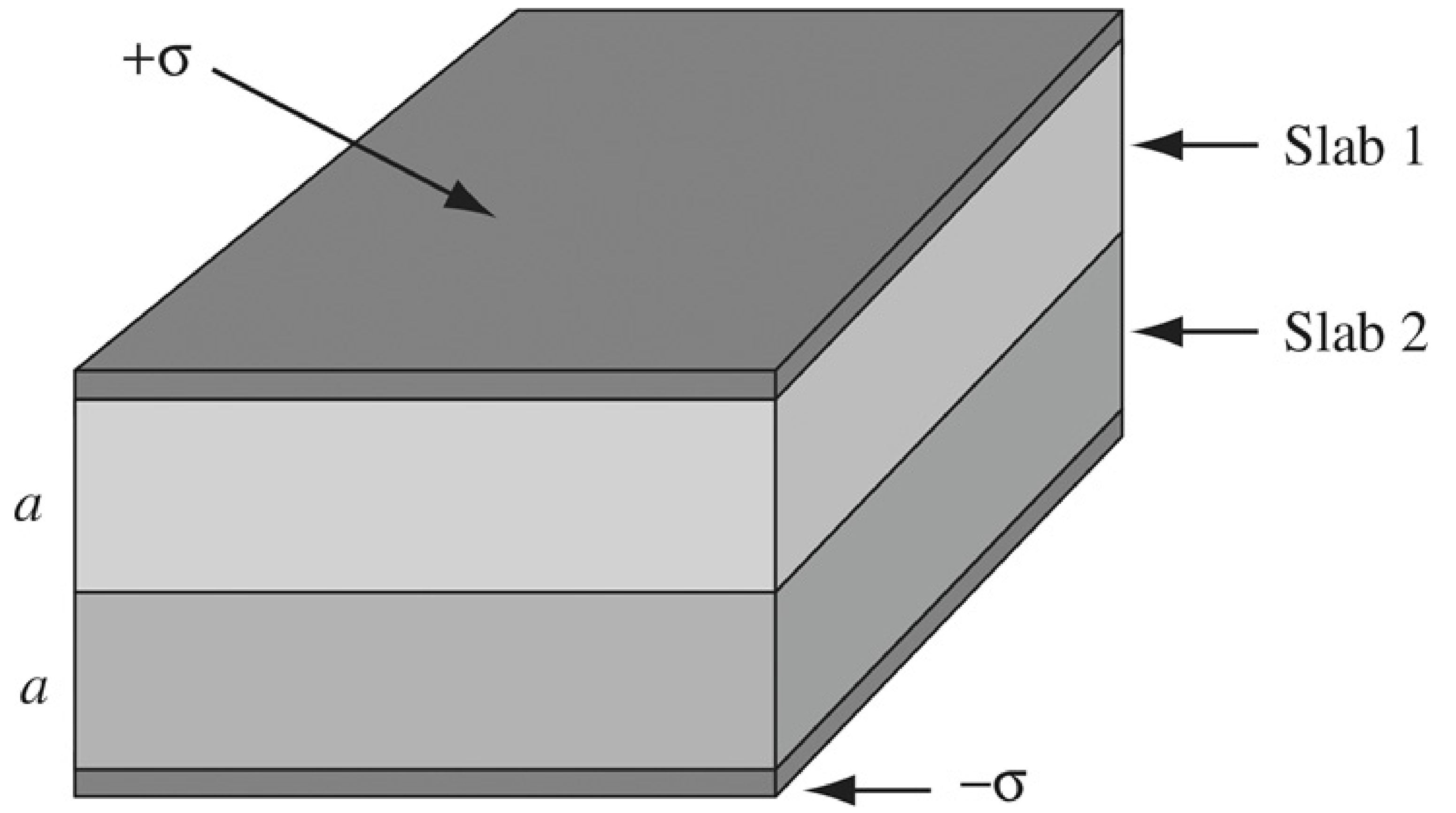
linear dielectric

$$P_x = \varepsilon_0 (\chi_{exx} E_x + \chi_{exy} E_y + \chi_{exz} E_z)$$
$$P_y = \varepsilon_0 (\chi_{eyx} E_x + \chi_{eyy} E_y + \chi_{eyz} E_z)$$
$$P_z = \varepsilon_0 (\chi_{ezx} E_x + \chi_{ezy} E_y + \chi_{ezz} E_z)$$

general case
the susceptibility tensor

Prob. 4.18 The space between the planes of a parallel-plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness a, so the total distance between the plates is $2a$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5 the free charge density on the top plate is σ and on the bottom plate $-\sigma$.

(a) Find the electric displacement $\mathbf{D}$ in each slab.
(b) Find the electric field $\mathbf{E}$ in each slab.
(c) Find the polarization $\mathbf{P}$ in each slab.
(d) Find the potential difference between the plates.
(e) Find the location and amount of all bound charge.
(f) Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to (d).



4.4.2 Boundary Value Problems with Linear Dielectrics

Relation between bound charge and free charge

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\varepsilon_0 \chi_e \frac{\mathbf{D}}{\varepsilon} \right) = -\frac{\chi_e}{1+\chi_e} \rho_f \leftarrow$$ in a homogenous linear dielectric

shielding effect

The boundary conditions that make reference only to the free charge .

$$D_{above}^{\perp} - D_{below}^{\perp} = \sigma_f \quad \Rightarrow \quad \varepsilon_{above} E_{above}^{\perp} - \varepsilon_{below} E_{below}^{\perp} = \sigma_f$$

$$(\varepsilon_{\text{above}} \nabla V_{\text{above}} - \varepsilon_{\text{below}} \nabla V_{\text{below}}) = -\sigma_f \hat{\mathbf{n}}$$

$$\text{or} \quad (\varepsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \varepsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n}) = -\sigma_f$$

$$\text{where} \quad \frac{\partial V_{\text{above}}}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}.$$

Homogeneous Linear Dielectric Sphere

Example 4.7 A sphere of homogeneous linear dielectric material is placed in a uniform electric field $\mathbf{E}$. Find the resultant electric field.

Sol: Look at Ex. 3.8 an uncharged conducting sphere. In that case the field of the induced charge ***completely*** canceled $\mathbf{E}$ within the sphere; However, in a dielectric the cancellation is only ***partial***.

The boundary conditions

$$\left.\begin{aligned}
&\text{(i)} \quad V_{\text{in}} = V_{\text{out}} && \text{at } r = R \\
&\text{(ii)} \quad \varepsilon \frac{\partial V_{\text{in}}}{\partial r} = \varepsilon_0 \frac{\partial V_{\text{out}}}{\partial r} && \text{at } r = R \\
&\text{(iii)} \, V_{\text{out}} \to -E_0 r \cos\theta && \text{for } r \gg R
\end{aligned}\right\}$$

no free charge at the surface

$$V(r,\theta)=\sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)$$

$$\begin{cases} V_{in}(r,\theta)=\sum_{\ell=0}^{\infty}A_\ell r^\ell P_\ell(\cos\theta) & r\le R \\ V_{out}(r,\theta)=-E_0 r\cos\theta+\sum_{\ell=0}^{\infty}B_\ell r^{-(\ell+1)}P_\ell(\cos\theta) & r\ge R \end{cases}$$

B.C. (iii)

B.C. (i): $A_\ell R^\ell P_\ell = -E_0 R\cos\theta + B_\ell R^{-(\ell+1)}P_\ell$

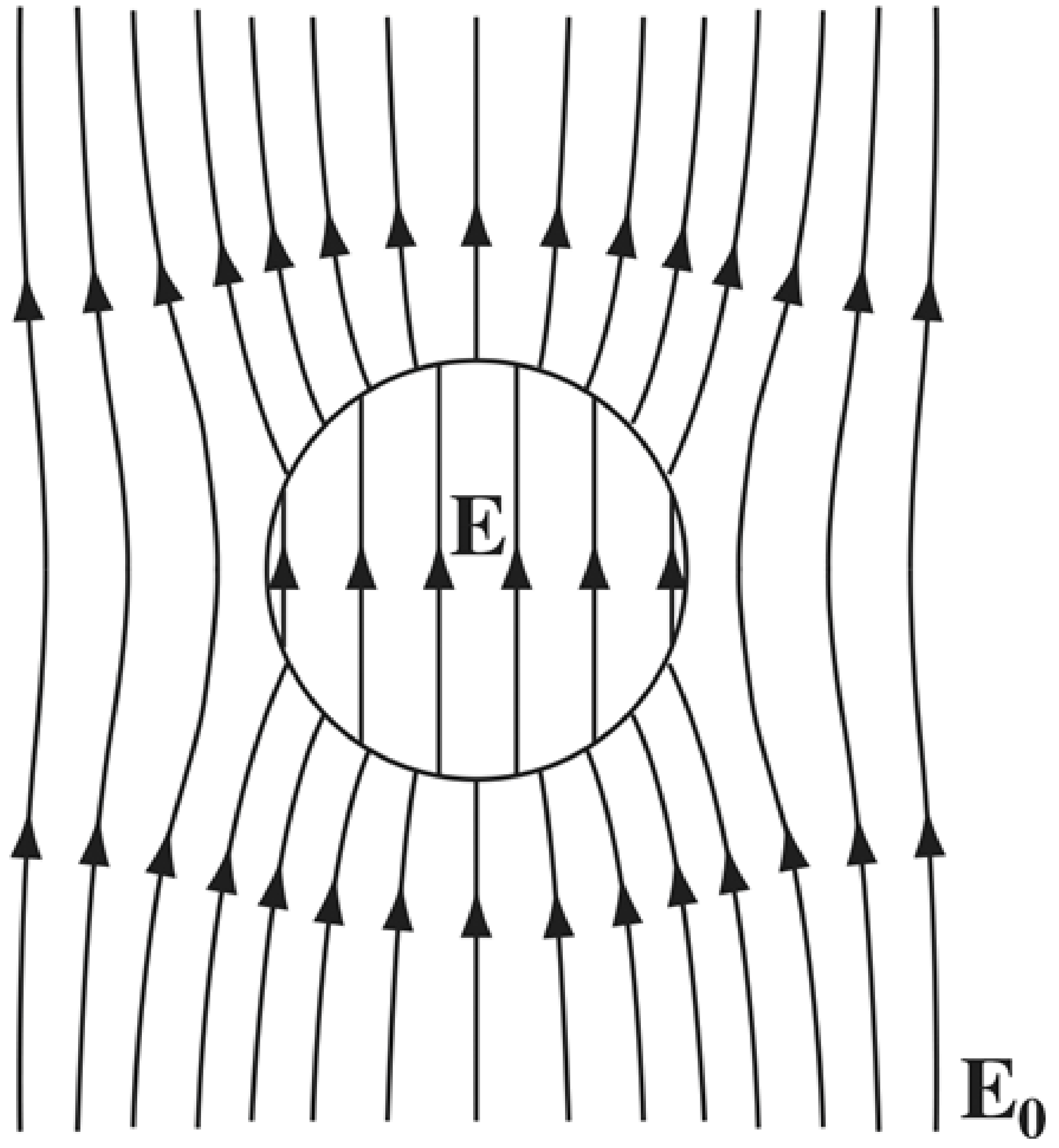
$$\Rightarrow \begin{cases} A_1 R = -E_0 R + B_1 R^{-2} & \ell = 1 \\ A_\ell R^\ell = B_\ell R^{-(\ell+1)} & \ell \neq 1 \end{cases}$$

B.C. (ii): $\varepsilon_r \ell A_\ell R^{\ell-1}P_\ell = -E_0\cos\theta - (\ell+1)B_\ell R^{-(\ell+2)}P_\ell$

$$\Rightarrow \begin{cases} \varepsilon_r A_1 = -E_0 - 2B_1 R^{-3} & \ell = 1 \\ \varepsilon_r \ell A_\ell R^{\ell-1} = -(\ell+1)B_\ell R^{-(\ell+2)} & \ell \neq 1 \end{cases}$$

$$\begin{cases} A_1 R = -E_0 R + B_1 R^{-2} & \ell = 1 \\ A_\ell R^\ell = B_\ell R^{-(\ell+1)} & \ell \neq 1 \end{cases} \qquad \begin{cases} \varepsilon_r A_1 = -E_0 - 2B_1 R^{-3} & \ell = 1 \\ \varepsilon_r \ell A_\ell R^{\ell-1} = -(\ell+1) B_\ell R^{-(\ell+2)} & \ell \neq 1 \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = -\dfrac{3E_0}{\varepsilon_r + 2}; B_1 = \dfrac{\varepsilon_r - 1}{\varepsilon_r + 2} R^3 E_0 & \ell = 1 \\ A_\ell = B_\ell = 0 & \ell \neq 1 \end{cases}$$



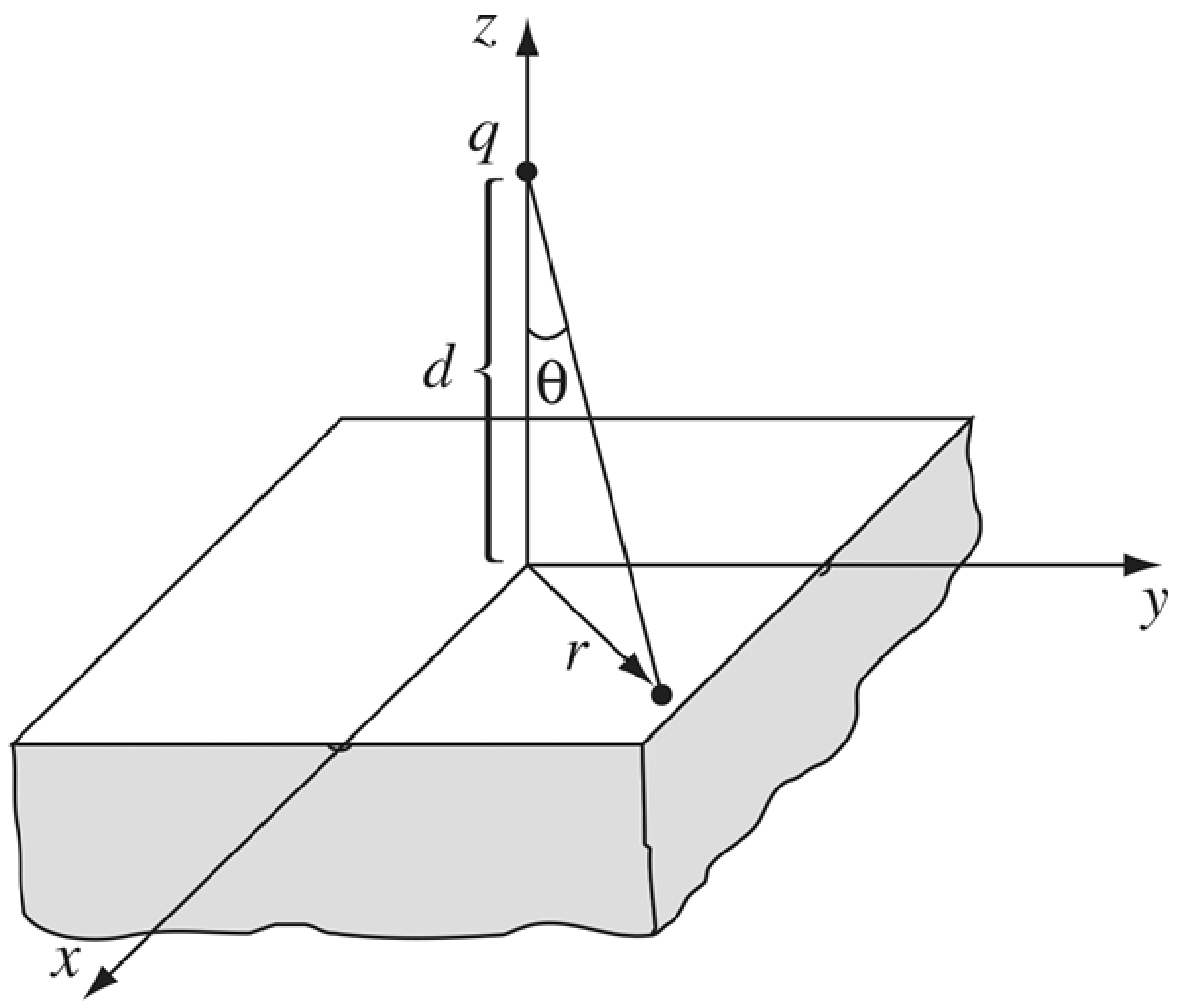
$$\begin{cases} V_{in}(r,\theta) = -\dfrac{3E_0}{\varepsilon_r + 2} r \cos\theta \\ V_{out}(r,\theta) = -E_0 r \cos\theta + \left(\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) R^3 E_0 r^{-2} \cos\theta \end{cases}$$

$$\mathbf{E}_{in} = -\nabla V_{in} = -\frac{3E_0}{\varepsilon_r + 2} \hat{\mathbf{z}} \quad \leftarrow \quad \text{uniform}$$

Partial Image Charge

Example 4.8 Suppose the entire region below the plane $z = 0$ is filled with uniform linear dielectric material of susceptibility χ_e. Calculate the force on a point charge q situated at distance d above the origin.

Sol: The surface bound charge on the xy plane is of opposite sign to q, so the force will be attractive.



$$z > 0 \quad V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_b}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

$$z < 0 \quad V = \frac{1}{4\pi\varepsilon_0}\left[\frac{(q+q_b)}{\sqrt{x^2+y^2+(z-d)^2}}\right] \leftarrow why?$$

Because $V_{above} = V_{below}$

$$D^{\perp}_{above} - D^{\perp}_{below} = 0 \;\Rightarrow\; \varepsilon_0 E^{\perp}_{above} = (1+\chi_e)\varepsilon_0 E^{\perp}_{below} \text{ when } z = 0$$

$$\Rightarrow\; -\varepsilon_0 \left.\frac{\partial V_{above}}{\partial z}\right|_{z=0^+} = -(1+\chi_e)\varepsilon_0 \left.\frac{\partial V_{below}}{\partial z}\right|_{z=0^-}$$

Partial Image Charge (Contd.)

$$-\varepsilon_0 \left.\frac{\partial V_{above}}{\partial z}\right|_{z=0^+} = -(1+\chi_e)\varepsilon_0 \left.\frac{\partial V_{below}}{\partial z}\right|_{z=0^-}$$

$$\frac{(q-q_b)d}{(x^2+y^2+d^2)^{3/2}} = \frac{(1+\chi_e)\left(q+q_b\right)d}{(x^2+y^2+d^2)^{3/2}} \quad \Rightarrow \quad q_b = \frac{-\chi_e}{\chi_e+2}q$$

$$\varepsilon_0 \mathbf{E}\big|_{z=0^+} = \varepsilon_0 \mathbf{E}\big|_{z=0^-} + \mathbf{P} \quad \Rightarrow \mathbf{P} = -\varepsilon_0\left(\left.\frac{\partial V}{\partial z}\right|_{z=0^+} - \left.\frac{\partial V}{\partial z}\right|_{z=0^-}\right)\hat{\mathbf{z}} = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{\left(x^2+y^2+d^2\right)^{3/2}}\hat{\mathbf{z}}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{\left(x^2+y^2+d^2\right)^{3/2}}$$

Double check!

$$q_b = 2\pi\int_{r=0}^{\infty} -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{\left(r^2+d^2\right)^{3/2}} r\,dr = -\frac{1}{2}\int_0^{\infty}\frac{\chi_e}{\chi_e+2}\frac{qd}{\left(r^2+d^2\right)^{3/2}}dr^2 = -\left(\frac{\chi_e}{\chi_e+2}\right)q$$

4.4.3 Energy in Dielectric systems

How to express the energy for a dielectric filled capacitor?

Suppose we bring in the free charge, a bit at a time. As ρ_f is increased by an amount $\Delta\rho_f$, the polarization will change and with it the bound charge distribution.

The work done on the incremental free charge is :

$$\Delta W = \int (\Delta\rho_f) V d\tau$$

$$\nabla \cdot \mathbf{D} = \rho_f \Rightarrow \Delta\rho_f = \nabla \cdot (\Delta\mathbf{D})$$ ← the resulting change in **D**

$$\Delta W = \int (\nabla \cdot \Delta\mathbf{D}) V d\tau = \int (\nabla \cdot \underbrace{\Delta\mathbf{D} V}_{0} - \nabla V \cdot \Delta\mathbf{D}) d\tau$$

surface integral vanish if we integrate over entire space.

$$\Delta W = \int \mathbf{E} \cdot \Delta\mathbf{D} d\tau = \frac{1}{2}\int \Delta(\varepsilon \mathbf{E}^2) d\tau \qquad \therefore W = \frac{1}{2}\int (\mathbf{E} \cdot \mathbf{D}) d\tau$$

Which Formula is Correct?

$$W = \frac{1}{2}\int (\varepsilon_0 \mathbf{E}\cdot\mathbf{E})\,d\tau$$ derived in Chap. 2

$$W = \frac{1}{2}\int (\mathbf{E}\cdot\mathbf{D})\,d\tau$$ derived in Chap. 4

speak to somewhat different question.

What do we mean by “the energy of a system“?

It is the work required to assemble the system.

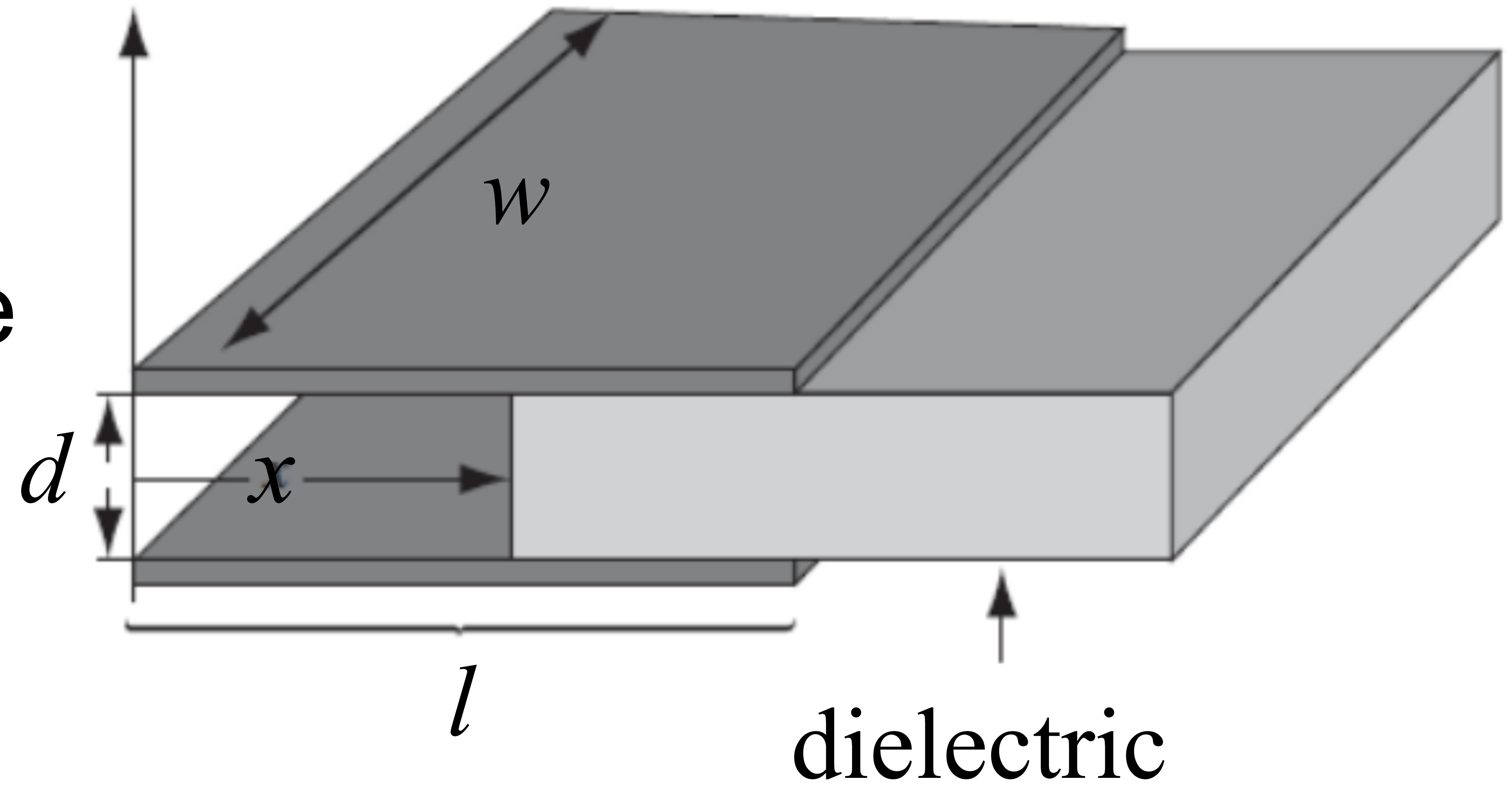
1) Bring in all the charges (free and bound), one by one, with tweezers, and glue each one down in its proper final position (Chap. 2).
2) Bring in the free charges, with the unpolarized dielectric in place, one by one, allowing the dielectric to respond as it sees fit (Chap. 4).

4.4.4 Forces on Dielectric

The dielectric is attracted into an electric field, just like conductor: the bound charge tends to accumulate near the free charge of the opposite sign.

How to calculate the forces on dielectrics?

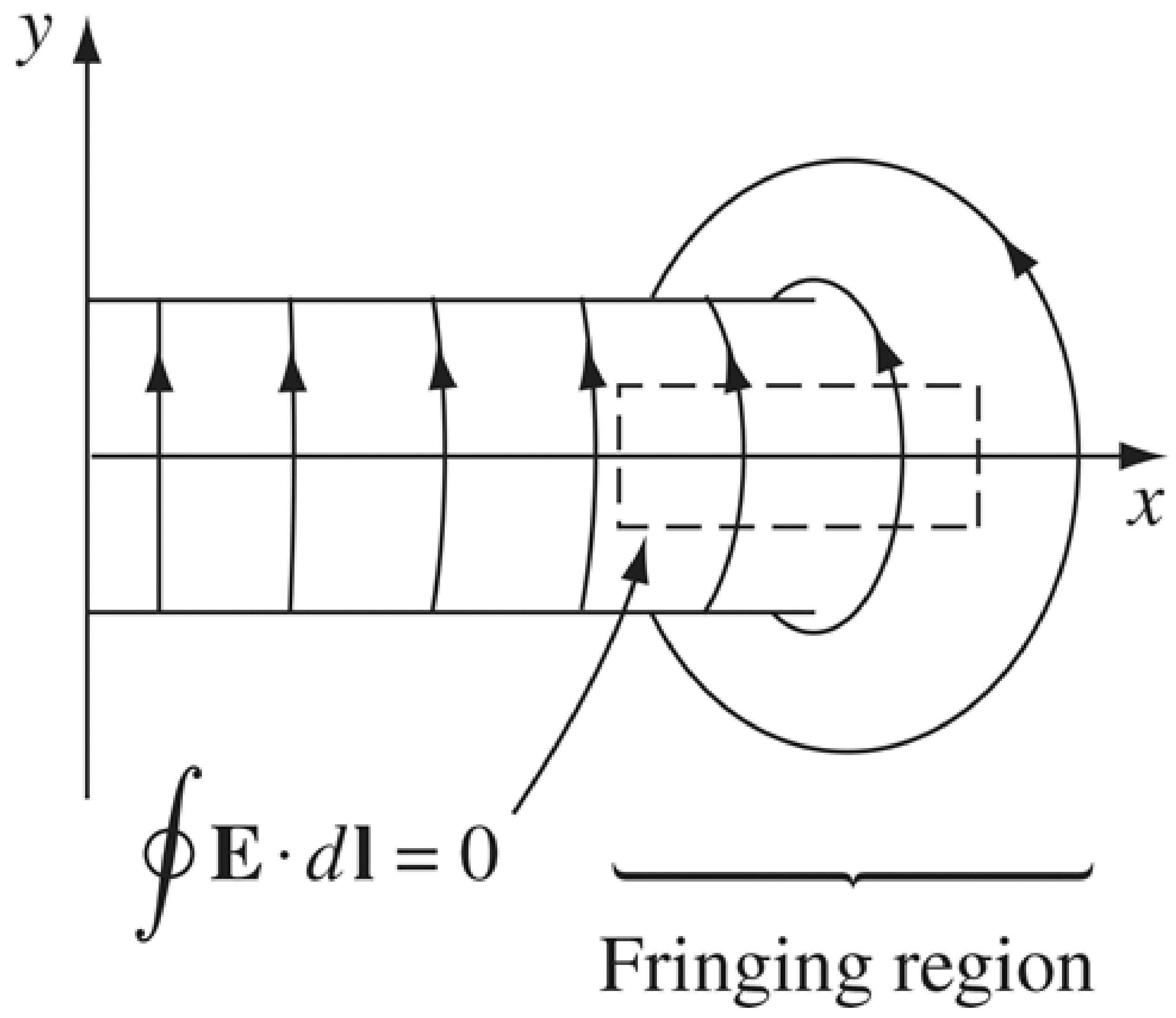
Consider the case of a slab of linear dielectric material, partially inserted between the plates of a parallel-plate capacitor.



If the field is perpendicular to the plates, no force would exert on the dielectric. Is that true?

The Fringing Field Effect

In reality a fringing field around the edges is responsible for the whole effect.
It is this nonuniform fringing field that pulls the dielectric into the capacitor.



Fringing field are difficult to calculate, so we adapt the following ingenious method.

The energy stored in the capacitor is: $W = \frac{1}{2} C V^2 = \frac{Q^2}{2C}$

The electric force on the slab is: $F = -\frac{dW}{dx}$

$$C = C_1 + C_2 = \frac{\varepsilon_0 w x}{d} + \frac{\varepsilon_0 \varepsilon_r w (\ell - x)}{d} = \frac{\varepsilon_0 w}{d}(\varepsilon_r \ell - \chi_e x)$$

Fixed charge

$$F=-\frac{dW}{dx}=\frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}=\frac{1}{2}V^2\frac{dC}{dx}=-\frac{\varepsilon_0\chi_e w}{2d}V^2$$

$F=-\frac{\varepsilon_0\chi_e w}{2d}V^2<0$ indicates that the force is in the negative x direction; the dielectric is **pulled into** the capacitor.

Fixed voltage

To maintain a constant voltage, *the battery must do work*.

work done by the battery

$$dW=Fdx+VdQ$$

$$F=\frac{dW}{dx}-V\frac{dQ}{dx}=\frac{1}{2}V^2\frac{dC}{dx}-V^2\frac{dC}{dx}=-\frac{1}{2}V^2\frac{dC}{dx}=\frac{\varepsilon_0\chi_e w}{2d}V^2$$

$F=\frac{\varepsilon_0\chi_e w}{2d}V^2>0$ The dielectric will be **pushed out of** the capacitor.

Homework of Chap. 4 (part II)

Problem 4.21 A certain coaxial cable consists of a copper wire, radius a, surrounded by a concentric copper tube of inner radius c (Fig. 4.26). The space between is partially filled (from b out to c) with material of dielectric constant ε_r, as shown. Find the capacitance per unit length of this cable.

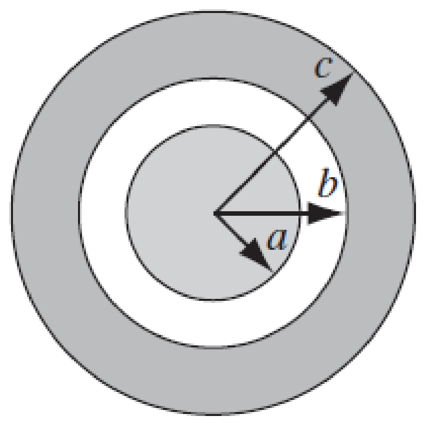


FIGURE 4.26

Problem 4.28 Two long coaxial cylindrical metal tubes (inner radius a, outer radius b) stand vertically in a tank of dielectric oil (susceptibility χ_e, mass density ρ). The inner one is maintained at potential V, and the outer one is grounded (Fig. 4.32). To what height (h) does the oil rise, in the space between the tubes?

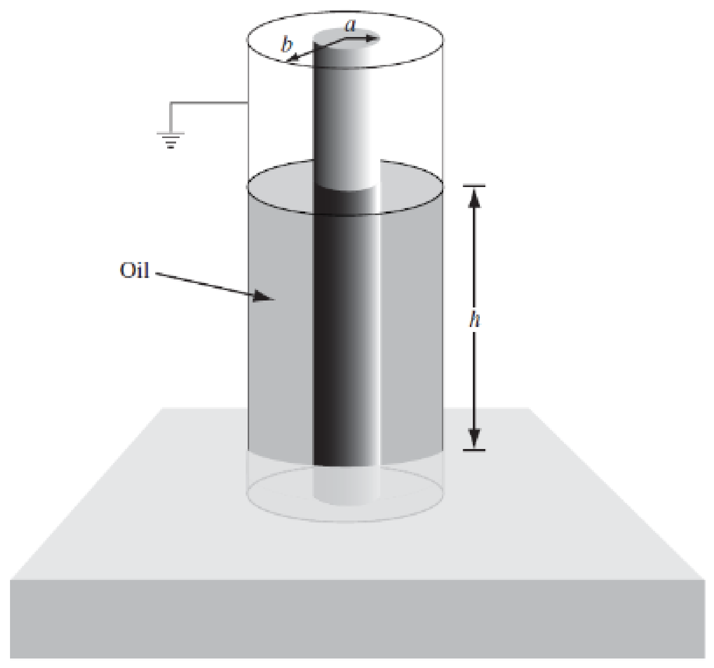


FIGURE 4.32

Problem 4.36 At the interface between one linear dielectric and another, the electric field lines bend (see Fig. 4.34). Show that

$$\tan\theta_2/\tan\theta_1 = \varepsilon_2/\varepsilon_1, \qquad (4.68)$$

assuming there is no *free* charge at the boundary. [*Comment*: Eq. 4.68 is reminiscent of Snell's law in optics.Would a convex "lens" of dielectric material tend to "focus," or "defocus," the electric field?]

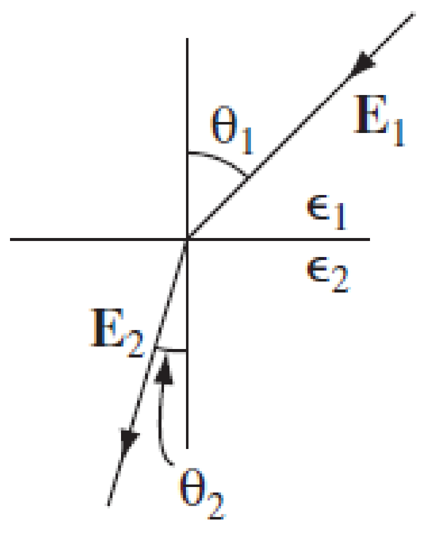


FIGURE 4.34

Homework of Chap. 4 (part II)

Problem 4.39 A conducting sphere at potential V_0 is half embedded in linear dielectric material of susceptibility χ_e, which occupies the region $z < 0$ (Fig. 4.35). *Claim*: the potential everywhere is exactly the same as it would have been in the absence of the dielectric! Check this claim, as follows:

(a) Write down the formula for the proposed potential $V(r)$, in terms of V_0, R, and r . Use it to determine the field, the polarization, the bound charge, and the free charge distribution on the sphere.

(b) Show that the resulting charge configuration would indeed produce the potential $V(r)$.

(c) Appeal to the uniqueness theorem in Prob. 4.38 to complete the argument.

(d) Could you solve the configurations in Fig. 4.36 with the same potential? If not, explain *why*.

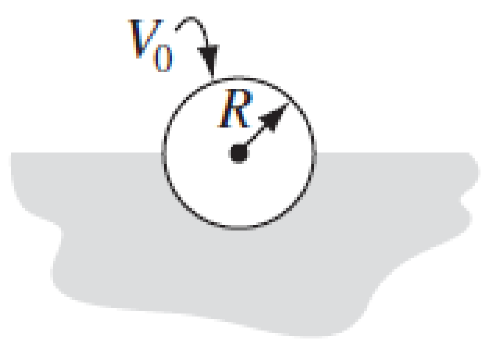


FIGURE 4.35

FIGURE 4.36