Chap.1 Supplement

For the EM Course Lectured by Prof. Tsun-Hsu Chang

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Homework exercises (Chap. 1)

1. $\nabla \cdot \left(r^3 \mathbf{r} \right)$

2. $\nabla^2 \left(\nabla \cdot \left(\dfrac{\mathbf{r}}{r^2} \right) \right)$

3. $\sum_{mn} \varepsilon_{imn} \varepsilon_{jmn} = \varepsilon_{imn} \varepsilon_{jmn} = 2\delta_{ij}$

4. $\sum_{ijk} \varepsilon_{ijk} \varepsilon_{ijk} = \varepsilon_{ijk} \varepsilon_{ijk} = 6$

5. Prove the product rules in page 23

6. If u, v are two scalar functions, prove that

$$\oint u \vec{\nabla} v \cdot d\vec{\ell} = \int_S \left(\vec{\nabla} u \right) \times \left(\vec{\nabla} v \right) \cdot d\mathbf{a}$$

7. If A is a 3 by 3 matrix and its i-th row j-th column component is denoted as A_{ij}, show that

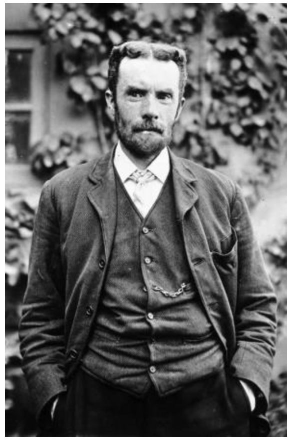
$$\det A = \sum_{ijk} \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$$

8. More problems in Griffiths:

1.12, 1.13, 1.25, 1.26, 1.36, 1.43, 1.47, 1.49, 1.64

More on "Vectors"

$e+\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0$	(1)	Gauss' Law
$\mu\alpha=\frac{dH}{dy}-\frac{dG}{dz}$ $\mu\beta=\frac{dF}{dz}-\frac{dH}{dx}$ $\mu\gamma=\frac{dG}{dx}-\frac{dF}{dy}$	(2)	Equivalent to Gauss' Law for magnetism
$P=\mu\left(\gamma\frac{dy}{dt}-\beta\frac{dz}{dt}\right)-\frac{dF}{dt}-\frac{d\Psi}{dx}$ $Q=\mu\left(\alpha\frac{dz}{dt}-\gamma\frac{dx}{dt}\right)-\frac{dG}{dt}-\frac{d\Psi}{dy}$ $R=\mu\left(\beta\frac{dx}{dt}-\alpha\frac{dy}{dt}\right)-\frac{dH}{dt}-\frac{d\Psi}{dz}$	(3)	Faraday's Law (with the Lorentz Force and Poisson's Law)
$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=4\pi p'$ $\quad p'=p+\frac{df}{dt}$ $\frac{d\alpha}{dz}-\frac{d\gamma}{dx}=4\pi q'$ $\quad q'=q+\frac{dg}{dt}$ $\frac{d\beta}{dx}-\frac{d\alpha}{dy}=4\pi r'$ $\quad r'=r+\frac{dh}{dt}$	(4)	Ampère-Maxwell Law
$P=-\xi p \quad Q=-\xi q \quad R=-\xi r$		Ohm's Law
$P=kf \quad Q=kg \quad R=kh$		The electric elasticity equation (**E** = **D**/ε)
$\frac{de}{dt}+\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}=0$		Continuity of charge



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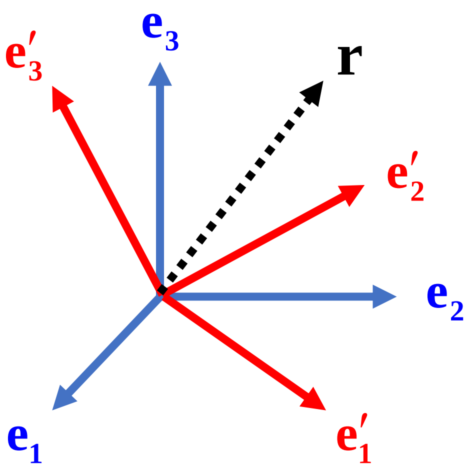
$\nabla\cdot\mathbf{D}=\rho$	(1)	Gauss' Law
$\nabla\cdot\mathbf{B}=0$	(2)	Gauss' Law for magnetism
$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$	(3)	Faraday's Law
$\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}+\mathbf{J}$	(4)	Ampère-Maxwell Law

Heaviside rewrote Maxwell's Equations in a form that involved only electric and magnetic fields. Maxwell's original equations had included both fields and potentials.

More on "Vectors"

- Vector is a quantity that has both magnitude and direction.
- Consider two orthonormal coordinate systems



$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_1'\mathbf{e}_1' + x_2'\mathbf{e}_2' + x_3'\mathbf{e}_3'$$

$$\Rightarrow \begin{cases} x_1' = x_1'(x_1, x_2, x_3) = x_1\mathbf{e}_1'\cdot\mathbf{e}_1 + x_2\mathbf{e}_1'\cdot\mathbf{e}_2 + x_3\mathbf{e}_1'\cdot\mathbf{e}_3 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ x_2' = x_2'(x_1, x_2, x_3) = x_1\mathbf{e}_2'\cdot\mathbf{e}_1 + x_2\mathbf{e}_2'\cdot\mathbf{e}_2 + x_3\mathbf{e}_2'\cdot\mathbf{e}_3 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x_3' = x_3'(x_1, x_2, x_3) = x_1\mathbf{e}_3'\cdot\mathbf{e}_1 + x_2\mathbf{e}_3'\cdot\mathbf{e}_2 + x_3\mathbf{e}_3'\cdot\mathbf{e}_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{cases}$$

$$\Rightarrow \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ or simply } x_i' = \sum_{j=1}^{3} a_{ij}x_j = a_{ij}x_j \Rightarrow a_{ij} = \frac{\partial x_i'}{\partial x_j}$$

a is an orthogonal matrix

Einstein Summation Convention: Sum over all the repeated dummy indices

$$x_i x_i = x_1^2 + x_2^2 + x_3^2$$

More on “Vectors”

- Vector is a quantity defined as

$$x_i' = \sum_{j=1}^{3} a_{ij} x_j = a_{ij} x_j$$

Namely, components of this quantity in different coordinates are related as above.

Q: Is “force,” e.g., Gravitational force, a vector?
How to prove this?

More on "Vectors"

$$x_i' = \sum_{j=1}^{3} a_{ij} x_j = a_{ij} x_j$$

Q: Is "force," e.g. Gravitational force, a vector?

Consider a gravitational potential in two different frames

$$V(x_1, x_2, x_3) \Rightarrow F_i' = -\frac{\partial V}{\partial x_i'} = -\left(\frac{\partial V}{\partial x_1}\frac{\partial x_1}{\partial x_i'} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial x_i'} + \frac{\partial V}{\partial x_3}\frac{\partial x_3}{\partial x_i'} \right) = a_{i1}F_1 + a_{i2}F_2 + a_{i3}F_3 = a_{ij}F_j$$

$$a_{ij} = \frac{\partial x_i'}{\partial x_j}$$

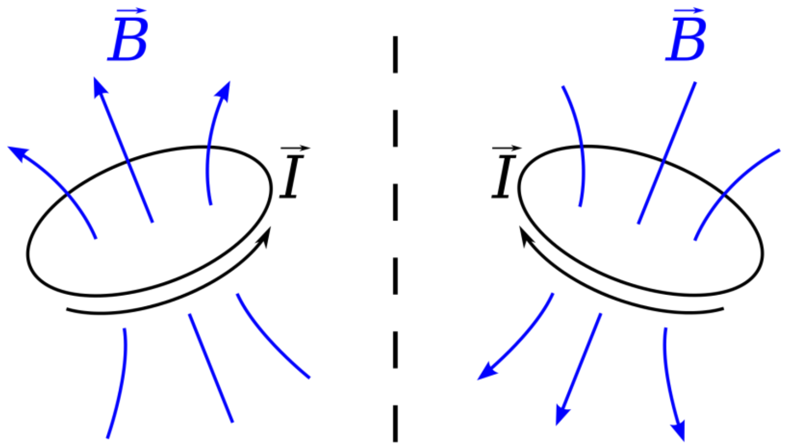
If a force is defined as the minus gradient of a potential, it is a vector—e.g., gravitational force, electric force, etc. Then, is the electric/magnetic field a vector? It turns out that the magnetic field (force) in general doesn't follow the described transformation!

More on "Vectors"

A pseudovector (or axial vector) is a quantity

1. defined as a function of some vectors and behaves like a vector in many situations
2. changed into its opposite under the inversion or the reflection.

Geometrically, the direction of a reflected pseudovector is opposite to its mirror image but with equal magnitude. In contrast, the reflection of a true (or polar) vector is the same as its mirror image.



More on "Vectors"

The transformation rules for vectors and pseudovectors can be compactly stated as

$$\mathbf{x}' = R\mathbf{x} \qquad \left(\text{Polar Vector}\right)$$

$$\mathbf{x}' = \left(\det R\right)\left(R\mathbf{x}\right) \qquad \left(\text{Pseudovector}\right)$$

Generally speaking, the cross product of vectors (V) and pseudovectors(PV) follows

$$V \times V = PV$$

$$PV \times PV = PV$$

$$V \times PV = V$$

$$PV \times V = V$$

For example, Angular momentum is the cross product of a displacement (a vector) and momentum (a vector) and is, therefore, a pseudovector.

Q: How about friction force? Is it a "vector"?

*Note: There are parity-violating vectors in the theory of weak interactions, which are neither polar vectors nor pseudovectors.

More on "Vectors"

Parity: a good symmetry in Electromagnetism

$$\mathbf{X} \xleftrightarrow{\text{P}} -\mathbf{X}$$

$$(x, y, z) \xleftrightarrow{\text{P}} -(x, y, z)$$

$$\begin{cases} P\left(Pf(\mathbf{x})\right) = P\left(\lambda f(-\mathbf{x})\right) = \lambda^2 f(\mathbf{x}) \\ P^2 f(\mathbf{x}) = f(\mathbf{x}) \end{cases} \Rightarrow \lambda = \pm 1$$

	Example	In EM
$\text{Scalar} \xrightarrow{P} \text{"+" Scalar}$	Temperature, $\mathbf{a}\cdot\mathbf{b}$	To be recognized later
$\text{P-scalar} \xrightarrow{P} \text{"}-\text{" P-Scalar}$	$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$	
$\text{Vector} \xrightarrow{P} \text{"}-\text{" Vector}$	Velocity, $\mathbf{a}$	$\mathbf{E}$ electric field
$\text{P-vector} \xrightarrow{P} \text{"+" P-Vector}$	$\mathbf{L} = \mathbf{r}\times\mathbf{p}$	$\mathbf{B}$ magnetic flux density

More on “Vectors”

Rank-0 tensor: Scalar

A vector (rank 1 tensor) has a form of $x_i' = \sum_{j=1}^{3} a_{ij} x_j$

A rank 2 tensor has a form of $T_{ij}' = \sum_{\ell=1}^{3} \sum_{k=1}^{3} a_{ik} a_{j\ell} T_{k\ell}$

A rank 2 tensor can be constructed by the direct product of two vectors:

$$
\begin{aligned}
T_{ij}' &= A_i' B_j' \\
&= \left(\sum_{k=1}^{3} a_{ik} A_k \right) \left(\sum_{\ell=1}^{3} a_{j\ell} B_\ell \right) \\
&= \sum_{k=1}^{3} \sum_{\ell=1}^{3} a_{ik} a_{j\ell} T_{k\ell} = a_{ik} a_{j\ell} T_{k\ell}
\end{aligned}
$$

$$
\begin{aligned}
x_1' &= \sum_{j=1}^{3} a_{1j} x_j \\
&= a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = a_{1j} x_j
\end{aligned}
$$

$$
\begin{aligned}
T_{11}' &= \sum_{k=1}^{3} \sum_{\ell=1}^{3} a_{1k} a_{1\ell} T_{k\ell} \\
&= a_{11} \left(a_{11} T_{11} + a_{12} T_{12} + a_{13} T_{13} \right) \\
&\quad + a_{12} \left(a_{11} T_{21} + a_{12} T_{22} + a_{13} T_{23} \right) \\
&\quad + a_{13} \left(a_{11} T_{31} + a_{12} T_{32} + a_{13} T_{33} \right) \\
&= a_{1k} a_{1\ell} T_{k\ell}
\end{aligned}
$$

More on "Vectors"

Consider the contracted tensor

$$W = T_{ii} = T_{11} + T_{22} + T_{33}$$

Under a transformation, it becomes

$$W' = T'_{ii} = a_{ik} a_{i\ell} T_{k\ell} = \left(a^T\right)_{ki} a_{i\ell} T_{k\ell}$$

$$= \left(a^T a\right)_{k\ell} T_{k\ell} = \delta_{k\ell} T_{k\ell} = T_{ii} = W = \textbf{Scalar!}$$

$$\therefore T_{ii} = T'_{ii} = T_{11} + T_{22} + T_{33} = T'_{11} + T'_{22} + T'_{33}$$

$$\Rightarrow A_i B_i = A'_i B'_i = A_1 B_1 + A_2 B_2 + A_3 B_3 = A'_1 B'_1 + A'_2 B'_2 + A_3 B'_3$$

$$\equiv \mathbf{A} \cdot \mathbf{B} \text{ (Inner product!)}$$

$$T'_{ij} = \sum_{k=1}^{3} \sum_{\ell=1}^{3} a_{ik} a_{j\ell} T_{k\ell}$$

$$= a_{ik} a_{j\ell} T_{k\ell}$$

$$\left(AB\right)_{ij} = \sum_{k} A_{ik} B_{kj}$$

$$= A_{ik} B_{kj}$$

$$a_{ik} a_{i\ell} = \sum_{i=1}^{3} a_{ik} a_{i\ell}$$

$$= \left(a^T\right)_{ki} a_{i\ell}$$

$$= I_{k\ell} = \delta_{k\ell}$$

$$W = A_i B_j \delta_{ij}$$

(1+1=2) **(2-2=0)**

The inner product of two vectors is a two-step operation: **direct product** and **contraction**!

More on "Vectors"

So far, we have defined the inner product, then what is **the origin of the outer product**?

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

Outer product

Inner product

It turns out that an anti-symmetric tensor is constructed to define the outer product of two vectors!

$$T_{ij} = A_i B_j - A_j B_i \Rightarrow T = \begin{pmatrix} 0 & T_{12} & T_{13} \\ -T_{12} & 0 & T_{23} \\ -T_{13} & -T_{23} & 0 \end{pmatrix}$$

To obtain a vector from this tensor, a Levi-Civita rank-3 tensor is introduced.

More on "Vectors"

Kronecker Delta Function

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$\delta_{11} = \delta_{22} = \delta_{33} = 1$$
$$\delta_{12} = \delta_{13} = \delta_{23} = \ldots = 0$$

$$\mathbf{A} \cdot \mathbf{B} = \sum_{i,j} A_i B_j \delta_{ij}$$

Levi-Civita Symbol

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{(ijk) is even permutation of (123)} \\ -1, & \text{(ijk) is odd permutation of (123)} \\ 0, & \text{otherwise} \end{cases}$$

$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$$
$$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$$
$$\varepsilon_{111} = \varepsilon_{112} = \varepsilon_{113} = \ldots = 0$$

$$\left(\mathbf{A} \times \mathbf{B}\right)_k = \sum_{i,j} \varepsilon_{kij} A_i B_j$$

More on "Vectors"

Let's work out the cross product as you learned in high school

$$
\begin{aligned}
\mathbf{C} &= \mathbf{A}\times\mathbf{B} \\
&= \left(A_x\mathbf{e_x} + A_y\mathbf{e_y} + A_z\mathbf{e_z}\right)\times\left(B_x\mathbf{e_x} + B_y\mathbf{e_y} + B_z\mathbf{e_z}\right) \\
&= \begin{vmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\
&= \underbrace{\left(A_yB_z - A_zB_y\right)}_{C_x}\mathbf{e_x} + \underbrace{\left(A_zB_x - A_xB_z\right)}_{C_y}\mathbf{e_y} + \underbrace{\left(A_xB_y - A_yB_x\right)}_{C_z}\mathbf{e_z}
\end{aligned}
$$

$$
\begin{aligned}
C_x &= A_yB_z - A_zB_y \\
&= \varepsilon_{123}A_2B_3 + \varepsilon_{132}A_3B_2
\end{aligned}
$$

$$
\begin{aligned}
C_y &= A_zB_x - A_xB_z \\
&= \varepsilon_{231}A_3B_1 + \varepsilon_{213}A_1B_3
\end{aligned}
$$

$$
C_i = \sum_{j,k}\varepsilon_{ijk}A_jB_k = \varepsilon_{ijk}A_jB_k
$$

More on “Vectors”

Therefore, the outer (cross) product is

$$(\mathbf{A}\times\mathbf{B})_\ell = A_i B_j \varepsilon_{\ell mn}\delta_{im}\delta_{jn} = \varepsilon_{\ell mn} A_m B_n$$

Example 1

$$\mathbf{C} = \nabla\times\mathbf{A}$$
$$\Rightarrow C_i = \sum_{j,k}\varepsilon_{ijk}\partial_j A_k = \varepsilon_{ijk}\partial_j A_k$$

Example 2

$$\nabla\cdot\mathbf{E} = \partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = \partial_i E_i$$
$$\nabla^2\mathbf{E} = (\nabla\cdot\nabla)\mathbf{E} \Rightarrow (\nabla^2\mathbf{E})_i = \partial_j\partial_j E_i$$

More on "Vectors"

Therefore, the outer (cross) product is

$$\left(\mathbf{A}\times\mathbf{B}\right)_\ell = A_i B_j \varepsilon_{\ell mn}\delta_{im}\delta_{jn} = \varepsilon_{\ell mn}A_m B_n$$

Example 3

$$\begin{aligned}\left(\nabla\times\nabla\phi\right)_i &= \varepsilon_{ijk}\partial_j\partial_k\phi \\ &= \frac{1}{2}\left(\varepsilon_{ijk}\partial_j\partial_k\phi+\varepsilon_{ikj}\partial_k\partial_j\phi\right) \\ &= \frac{1}{2}\varepsilon_{ijk}\left(\partial_j\partial_k\phi-\partial_k\partial_j\phi\right) \\ &= 0\end{aligned}$$

Example 4

$$\begin{aligned}\nabla\cdot\left(\nabla\times\mathbf{B}\right) &= \partial_i\varepsilon_{ijk}\partial_j B_k \\ &= \frac{1}{2}\varepsilon_{ijk}\left(\partial_i\partial_j B_k-\partial_j\partial_i B_k\right) \\ &= 0\end{aligned}$$

More on "Vectors"

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{(ijk) is even permutation of (123)} \\ -1, & \text{(ijk) is odd permutation of (123)} \\ 0, & \text{otherwise} \end{cases}$$

Useful relation between Levi-Civita symbol and Kronecker delta function

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

Quick Argument: consider the case for the Kronecker delta function being 1 and -1.

Furthermore, one can show that

$$\varepsilon_{imn}\varepsilon_{jmn} = 2\delta_{ij} \quad \text{and} \quad \varepsilon_{ijk}\varepsilon_{ijk} = 6$$

More on "Vectors"

Useful identity in Electromagnetism:

$$\nabla\times(\nabla\times\mathbf{E})=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}$$

Proof:

$$\begin{aligned}(\nabla\times(\nabla\times\mathbf{E}))_i &= \varepsilon_{ijk}\partial_j(\varepsilon_{kmn}\partial_m E_n)\\ &= \varepsilon_{ijk}\varepsilon_{kmn}\partial_j\partial_m E_n\\ &= (\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\partial_j\partial_m E_n\\ &= \partial_j\partial_i E_j-\partial_j\partial_j E_i\\ &= (\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E})_i\end{aligned}$$

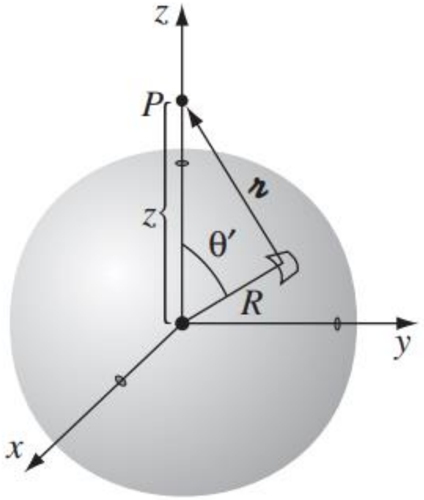
Quick Summary

So far, we've learned:

1. Why we need vector analysis in electromagnetism
2. Definition of Vector and Pseudovector
3. Introduction of tensor analysis
4. Definition of Inner/Outer product
5. Kronecker Delta Function and Levi-Civita Symbol

More on "Curvilinear Coordinates"

For many problems in electrodynamics, it will be more convenient to use different coordinate systems. These are curvilinear coordinates, and spherical and cylindrical coordinates are most used.



More on “Curvilinear Coordinates”

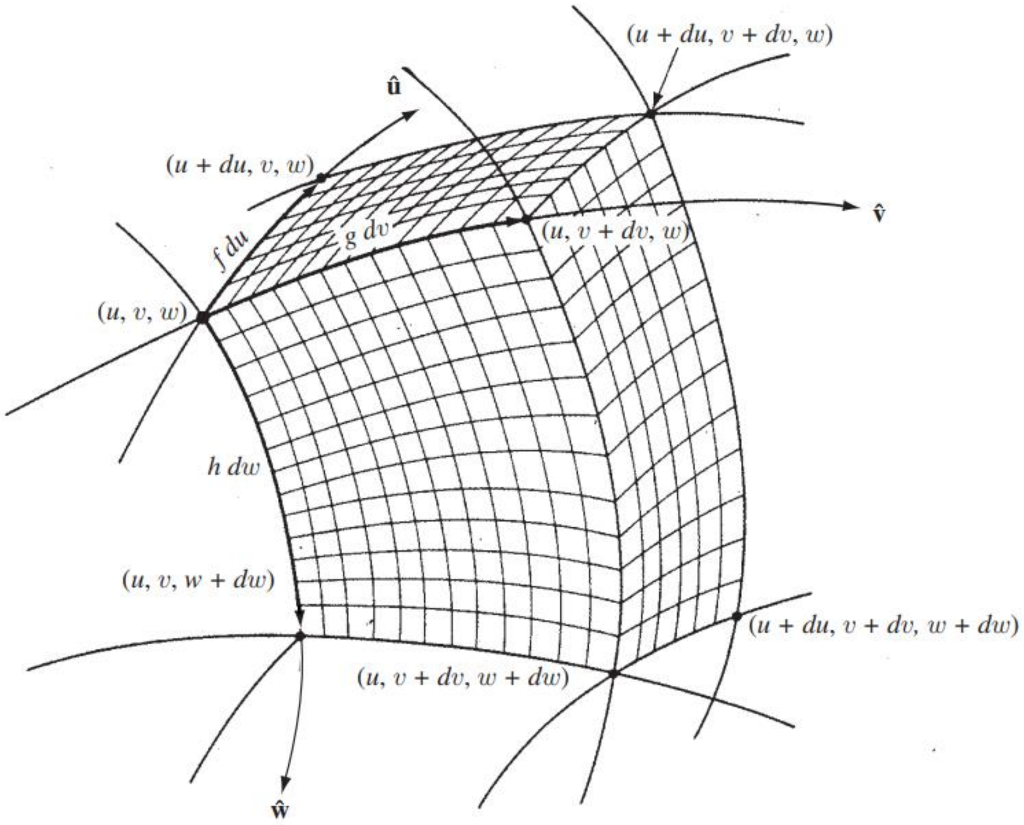
Let’s consider a general orthogonal system with three unit vectors

$$\hat{\mathbf{u}},\ \hat{\mathbf{v}},\ \hat{\mathbf{w}}$$

of the properties

1. Pointing in the direction of the increase of the corresponding coordinates
2. Mutually perpendicular
3. Functions of positions and therefore vary from point to point! (very important)
4. Any vector can be expressed in terms of these three unit vectors.

More on "Curvilinear Coordinates"



1. Pointing in the direction of the increase of the corresponding coordinates
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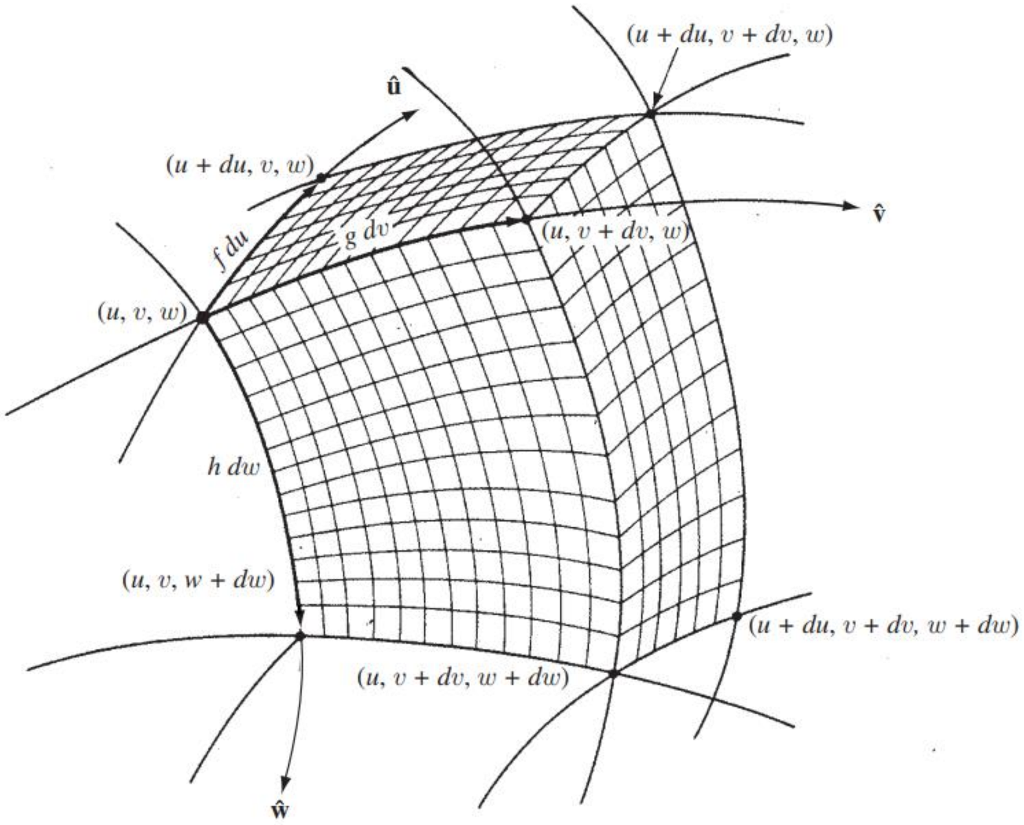
$$\hat{\mathbf{u}},\ \hat{\mathbf{v}},\ \hat{\mathbf{w}}$$

More on "Curvilinear Coordinates"



Infinitesimal displacement vector

$$d\mathbf{l} = (u+du, v+dv, w+dw) - (u, v, w)$$
$$= f du\hat{\mathbf{u}} + g dv\hat{\mathbf{v}} + h dw\hat{\mathbf{w}}$$

Scaling factor

$$dt = \frac{\partial t}{\partial u} du + \frac{\partial t}{\partial v} dv + \frac{\partial t}{\partial w} dw = \nabla t \cdot d\mathbf{l}$$
$$= (\nabla t)_u \, f du + (\nabla t)_v \, g dv + (\nabla t)_w \, h dw$$

$$\therefore \nabla t = \left(\hat{\mathbf{u}} \frac{1}{f} \frac{\partial}{\partial u} + \hat{\mathbf{v}} \frac{1}{g} \frac{\partial}{\partial v} + \hat{\mathbf{w}} \frac{1}{h} \frac{\partial}{\partial w} \right) t$$

General Form of Gradient

More on “Curvilinear Coordinates”

$$d\mathbf{l} = (u+du, v+dv, w+dw) - (u, v, w) = f du\,\hat{\mathbf{u}} + g dv\,\hat{\mathbf{v}} + h dw\,\hat{\mathbf{w}}$$

Scaling factor

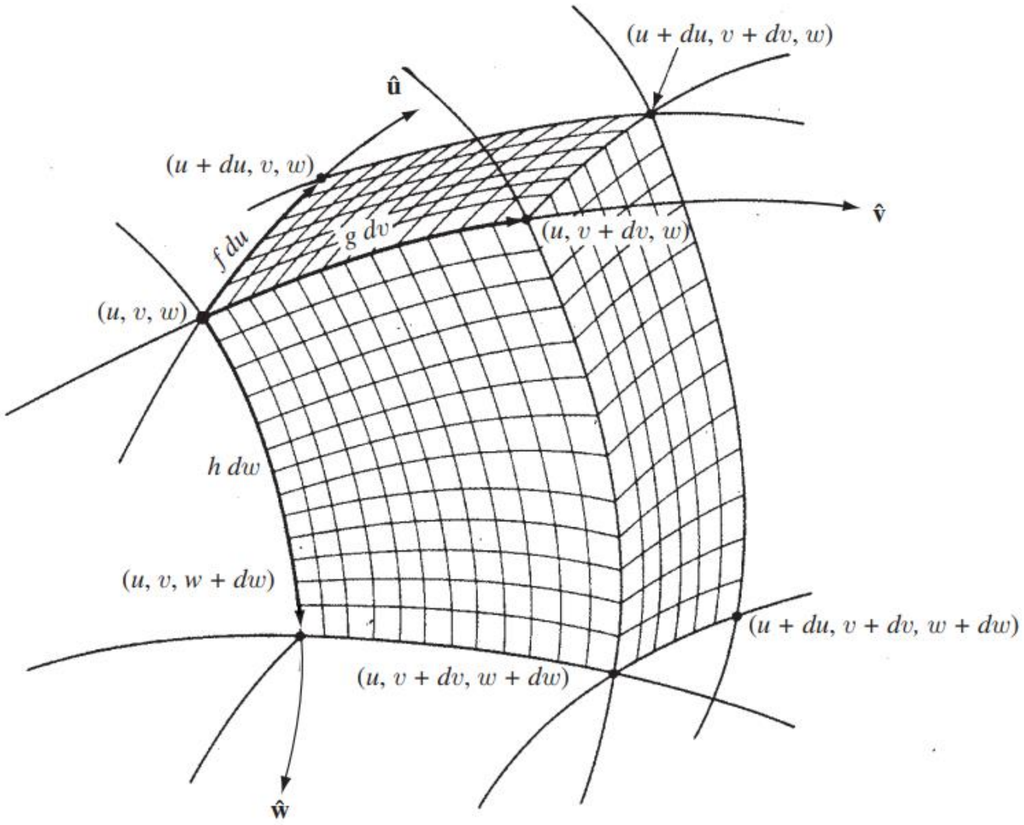
$$dt = \frac{\partial t}{\partial u}du + \frac{\partial t}{\partial v}dv + \frac{\partial t}{\partial w}dw = \nabla t \cdot d\mathbf{l} = (\nabla t)_u\, f du + (\nabla t)_v\, g dv + (\nabla t)_w\, h dw$$

$$\therefore \nabla t = \left(\hat{\mathbf{u}} \frac{1}{f}\frac{\partial}{\partial u} + \hat{\mathbf{v}} \frac{1}{g}\frac{\partial}{\partial v} + \hat{\mathbf{w}} \frac{1}{h}\frac{\partial}{\partial w} \right) t$$

General Form of Gradient

System	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	$r\sin\theta$
Cylindrical	s	ϕ	z	1	s	1

More on "Curvilinear Coordinates"



Infinitesimal volume element

$$d\tau = dl_u dl_v dl_w = (fgh)\,dudvdw$$

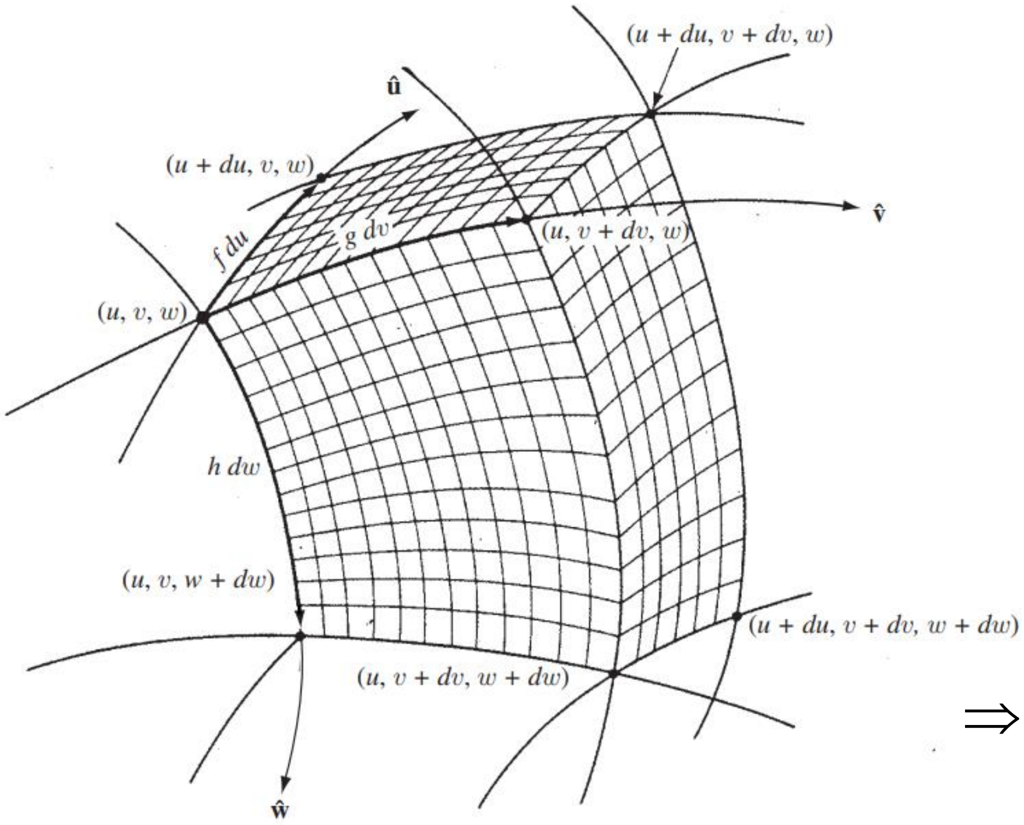
Front/Back surface

$$d\mathbf{a} = \mp ghdvdw\hat{\mathbf{u}}$$

$$\Rightarrow \sum_{front\&back} \mathbf{A}\cdot d\mathbf{a} = \left(A_{u+du} - A_u\right) ghdvdw$$

$$= \left[\frac{\partial}{\partial u}\left(ghA_u\right)\right] dudvdw = \frac{1}{fgh}\frac{\partial}{\partial u}\left(ghA_u\right) d\tau$$

More on "Curvilinear Coordinates"



Generalized to the other two components

$$\begin{cases} \hat{\mathbf{u}}: \dfrac{1}{fgh}\dfrac{\partial}{\partial u}\left(ghA_u\right)d\tau \\ \hat{\mathbf{v}}: \dfrac{1}{fgh}\dfrac{\partial}{\partial v}\left(fhA_v\right)d\tau \\ \hat{\mathbf{w}}: \dfrac{1}{fgh}\dfrac{\partial}{\partial w}\left(fgA_w\right)d\tau \end{cases}$$

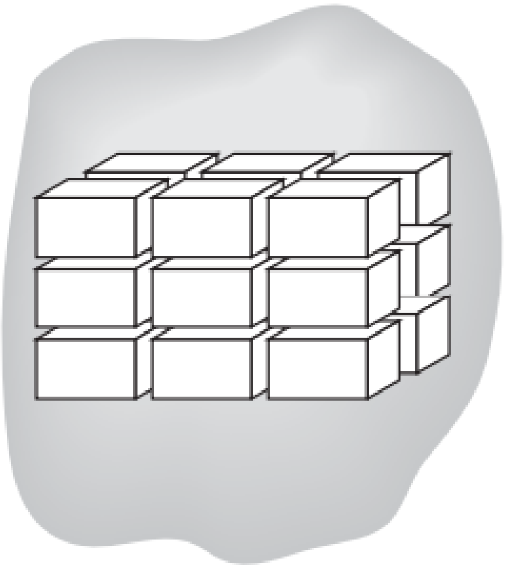
$$\Rightarrow \sum \mathbf{A}\cdot d\mathbf{a} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(ghA_u\right)+\frac{\partial}{\partial v}\left(fhA_v\right)+\frac{\partial}{\partial w}\left(fgA_w\right)\right]d\tau$$

General Form of Divergence

More on "Curvilinear Coordinates"

For any infinitesimal element

$$\sum \mathbf{A}\cdot d\mathbf{a} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(ghA_u\right)+\frac{\partial}{\partial v}\left(fhA_v\right)+\frac{\partial}{\partial w}\left(fgA_w\right)\right]d\tau \equiv \nabla\cdot\mathbf{A}\,d\tau$$

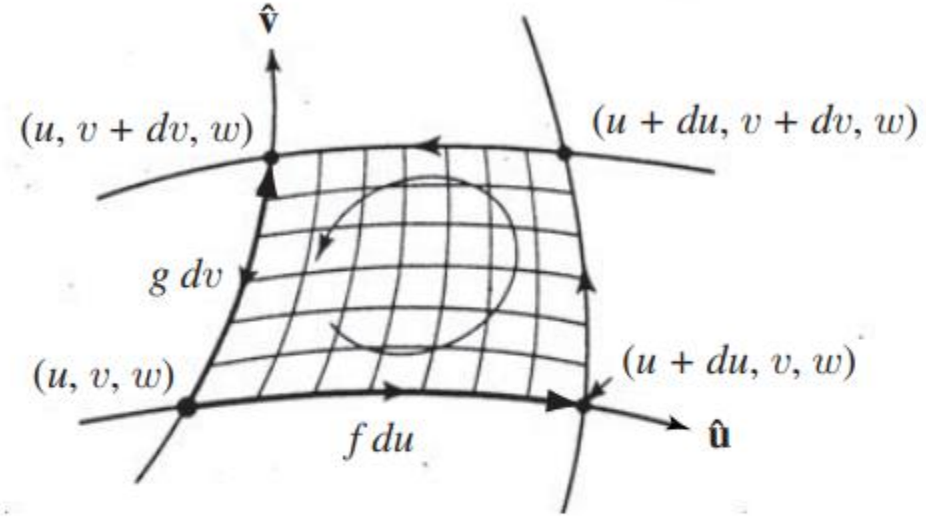


Extend to the finite region: $\oint \mathbf{A}\cdot d\mathbf{a} = \int \left(\nabla\cdot\mathbf{A}\right)d\tau$

This is known as the Divergence Theorem.

$$\nabla\cdot\mathbf{A} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(ghA_u\right)+\frac{\partial}{\partial v}\left(fhA_v\right)+\frac{\partial}{\partial w}\left(fgA_w\right)\right]$$

More on "Curvilinear Coordinates"



Increase u and v while holding w constant

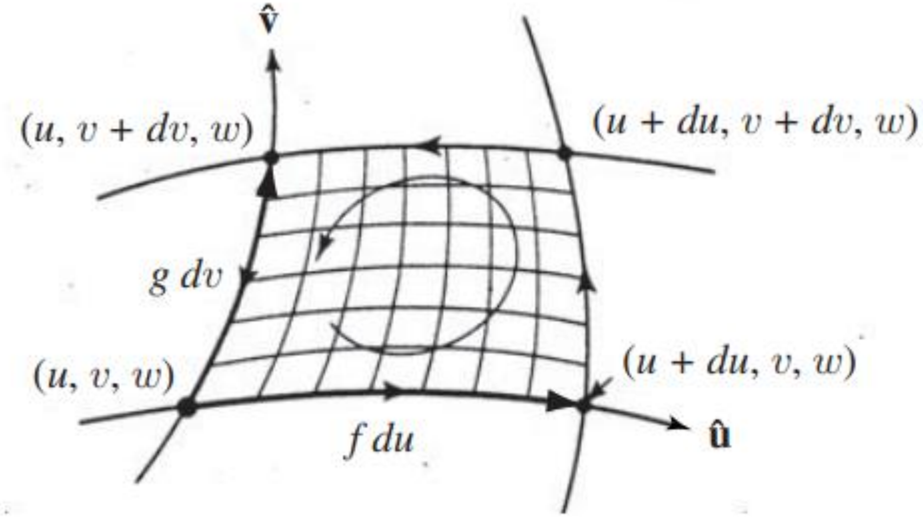
$$d\mathbf{a} = fg\,du\,dv\,\hat{\mathbf{w}}$$

Along the top / bottom segment

$$\mathbf{A}\cdot d\mathbf{l} = \mp\left(fA_u\right)du$$

$$\sum_{top\&bottom}\mathbf{A}\cdot d\mathbf{l} = \left[+\left(fA_u\right)\Big|_v - \left(fA_u\right)\Big|_{v+dv}\right]du = -\left[\frac{\partial}{\partial v}\left(fA_u\right)\right]du\,dv$$

More on "Curvilinear Coordinates"



$$d\mathbf{a} = fgdudv\hat{\mathbf{w}}$$

Similarly, the right and left sides yield

$$\left[\frac{\partial}{\partial u}(gA_v)\right]dudv$$

Therefore, the total is

$$\sum \mathbf{A}\cdot d\mathbf{l} = \left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]dudv$$

$$= \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{\mathbf{w}}\cdot d\mathbf{a}$$

More on "Curvilinear Coordinates"

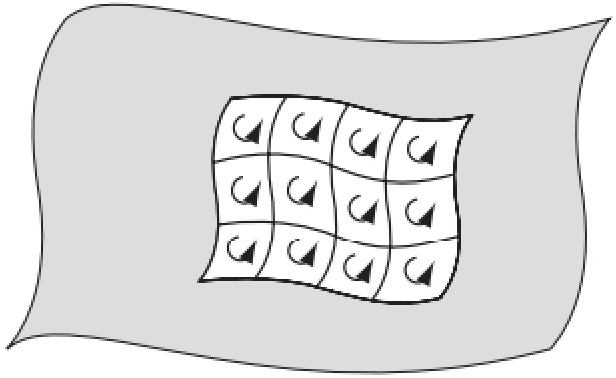
Construct the *u* and *v* components in the same way

General Form of Curl

$$\sum \mathbf{A}\cdot d\mathbf{l} = \left\{ \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{\mathbf{w}} + \frac{1}{gh}\left[\frac{\partial}{\partial v}(hA_w) - \frac{\partial}{\partial w}(gA_v)\right]\hat{\mathbf{u}} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(fA_u) - \frac{\partial}{\partial u}(hA_w)\right]\hat{\mathbf{v}} \right\}\cdot d\mathbf{a}$$

Again, extend to finite surfaces $\oint \mathbf{A}\cdot d\mathbf{l} = \int (\nabla\times\mathbf{A})\cdot d\mathbf{a}$

This is known as Stokes' Theorem.



$$\nabla\times\mathbf{A} = \frac{1}{gh}\left[\frac{\partial}{\partial v}(hA_w) - \frac{\partial}{\partial w}(gA_v)\right]\hat{\mathbf{u}} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(fA_u) - \frac{\partial}{\partial u}(hA_w)\right]\hat{\mathbf{v}} + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{\mathbf{w}}$$

$$= \frac{1}{fgh}\left[f\hat{\mathbf{u}}\begin{vmatrix} \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ gA_v & hA_w \end{vmatrix} + g\hat{\mathbf{v}}\begin{vmatrix} \frac{\partial}{\partial w} & \frac{\partial}{\partial u} \\ hA_w & fA_u \end{vmatrix} + h\hat{\mathbf{w}}\begin{vmatrix} \frac{\partial}{\partial u} & \frac{\partial}{\partial v} \\ fA_u & gA_v \end{vmatrix} \right] = \frac{1}{fgh}\begin{vmatrix} f\hat{\mathbf{u}} & g\hat{\mathbf{v}} & h\hat{\mathbf{w}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ fA_u & gA_v & hA_w \end{vmatrix}$$

More on "Curvilinear Coordinates"

Finally, the Laplacian of a scalar is the divergence of the gradient

$$\begin{cases} \nabla \cdot \mathbf{A} = \dfrac{1}{fgh}\left[\dfrac{\partial}{\partial u}(ghA_u) + \dfrac{\partial}{\partial v}(fhA_v) + \dfrac{\partial}{\partial w}(fgA_w) \right] \\ \nabla t = \left(\hat{\mathbf{u}}\dfrac{1}{f}\dfrac{\partial}{\partial u} + \hat{\mathbf{v}}\dfrac{1}{g}\dfrac{\partial}{\partial v} + \hat{\mathbf{w}}\dfrac{1}{h}\dfrac{\partial}{\partial w} \right) t \end{cases}$$

$$\Rightarrow \nabla^2 t = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial}{\partial u}t \right) + \frac{\partial}{\partial v}\left(\frac{fh}{g}\frac{\partial}{\partial v}t \right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial}{\partial w}t \right) \right]$$

Quick Summary

In this section, we've learned

$$\begin{cases}
\nabla t = \left(\hat{\mathbf{u}} \frac{1}{f} \frac{\partial}{\partial u} + \hat{\mathbf{v}} \frac{1}{g} \frac{\partial}{\partial v} + \hat{\mathbf{w}} \frac{1}{h} \frac{\partial}{\partial w} \right) t \\
\nabla \cdot \mathbf{A} = \frac{1}{fgh} \left[\frac{\partial}{\partial u} \left(ghA_u \right) + \frac{\partial}{\partial v} \left(fhA_v \right) + \frac{\partial}{\partial w} \left(fgA_w \right) \right] \\
\nabla \times \mathbf{A} = \frac{1}{fgh} \begin{vmatrix} f\hat{\mathbf{u}} & g\hat{\mathbf{v}} & h\hat{\mathbf{w}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ fA_u & gA_v & hA_w \end{vmatrix} \\
\nabla^2 t = \frac{1}{fgh} \left[\frac{\partial}{\partial u} \left(\frac{gh}{f} \frac{\partial}{\partial u} t \right) + \frac{\partial}{\partial v} \left(\frac{fh}{g} \frac{\partial}{\partial v} t \right) + \frac{\partial}{\partial w} \left(\frac{fg}{h} \frac{\partial}{\partial w} t \right) \right]
\end{cases}$$

System	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	$r \sin\theta$
Cylindrical	s	ϕ	z	1	s	1

Homework exercises (Chap. 1)

1. $\nabla\cdot\left(r^3\mathbf{r}\right)$

2. $\nabla^2\left(\nabla\cdot\left(\dfrac{\mathbf{r}}{r^2}\right)\right)$

3. $\sum_{mn}\varepsilon_{imn}\varepsilon_{jmn} = \varepsilon_{imn}\varepsilon_{jmn} = 2\delta_{ij}$

4. $\sum_{ijk}\varepsilon_{ijk}\varepsilon_{ijk} = \varepsilon_{ijk}\varepsilon_{ijk} = 6$

5. Prove the product rules in page 23

6. If u, v are two scalar functions, prove that

$$\oint u\vec{\nabla}v\cdot d\vec{\ell} = \int_S\left(\vec{\nabla}u\right)\times\left(\vec{\nabla}v\right)\cdot d\mathbf{a}$$

7. If A is a 3 by 3 matrix and its i-th row j-th column component is denoted as A_{ij}, show that

$$\det A = \sum_{ijk}\varepsilon_{ijk}A_{1i}A_{2j}A_{3k}$$

8. More problems in Griffiths:

1.12, 1.13, 1.25, 1.26, 1.36, 1.43, 1.47, 1.49, 1.64