EM
Tsun-Hsu Chang

Chapter 12 Electrodynamics and Relativity

12.1 The Special Theory of Relativity

Ether: Since mechanical waves require a medium to propagate, it was generally accepted that light also require a medium. This medium, called the ether, was assumed to pervade all matter and space in the universe.

“Absolute” frame: The Maxwell’s equation was inferred that the speed of light should equal c only with respect to ether. This means that the ether was a “preferred” or “absolute” reference frame.

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Ether

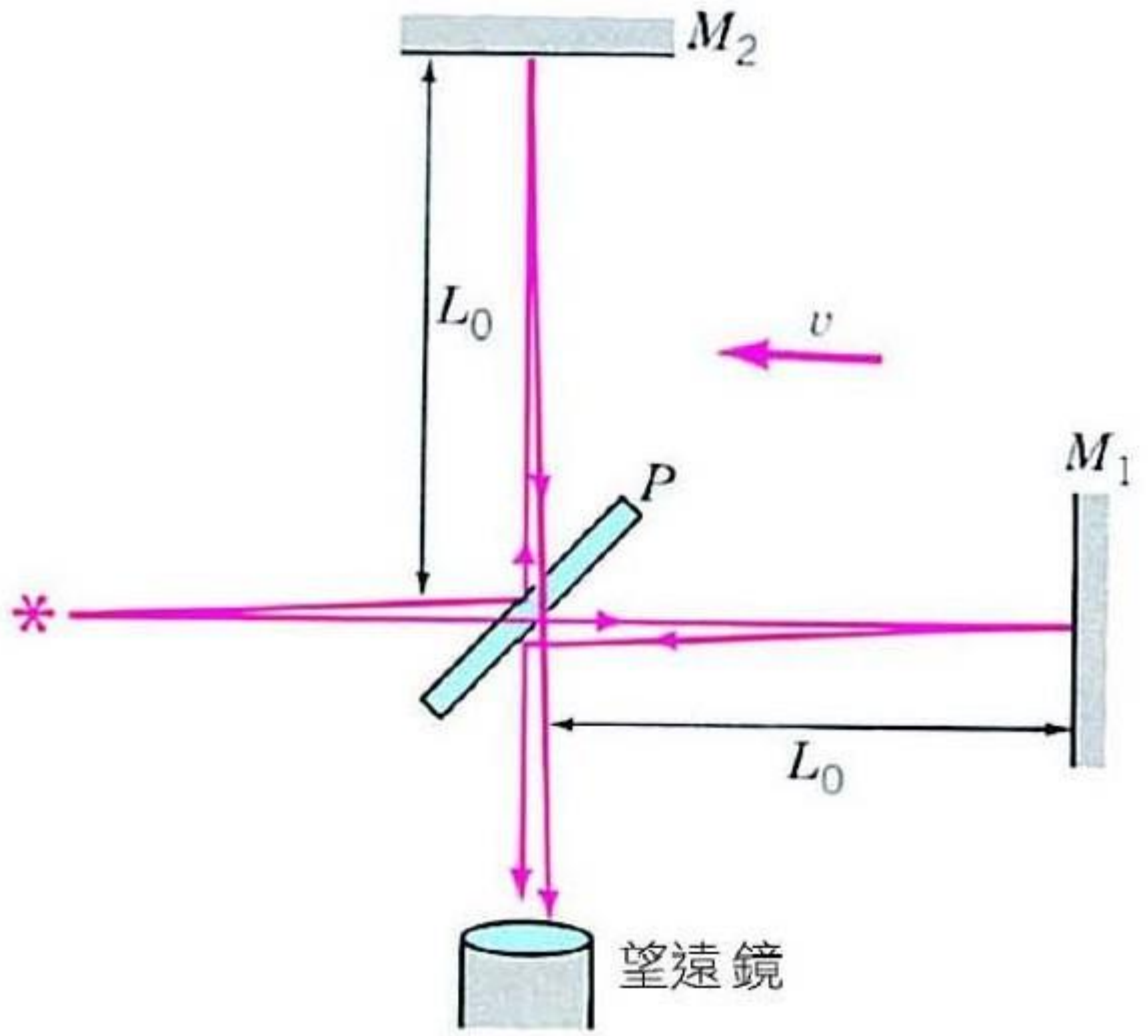
Properties of the ether: Since the light speed c is enormous, the ether had to be extremely rigid. So it did not impede the motion of light. For a substance so crucial to electro-magnetism, it was embarrassingly elusive. Despite the peculiar property just mentioned, no one could detect its ghostly presence.

Efforts to detect the ether: Michelson inspired by the Maxwell took the problem of detecting the ether as a challenge. He developed his interferometer and used it to try to detect the earth’s motion relative to the ether. The result were not conclusive.

The Michelson-Morley Experiment

Michelson and Morley wanted to detect the speed of the earth relative to the ether. If the earth were moving relative to the ether, there should be an "ether wind" blowing at the same speed relative to the earth but in the opposite direction.

Michelson-Morley interferometer: Use light speed variation to verify the existence of ether.



The Michelson-Morley Experiment (II)

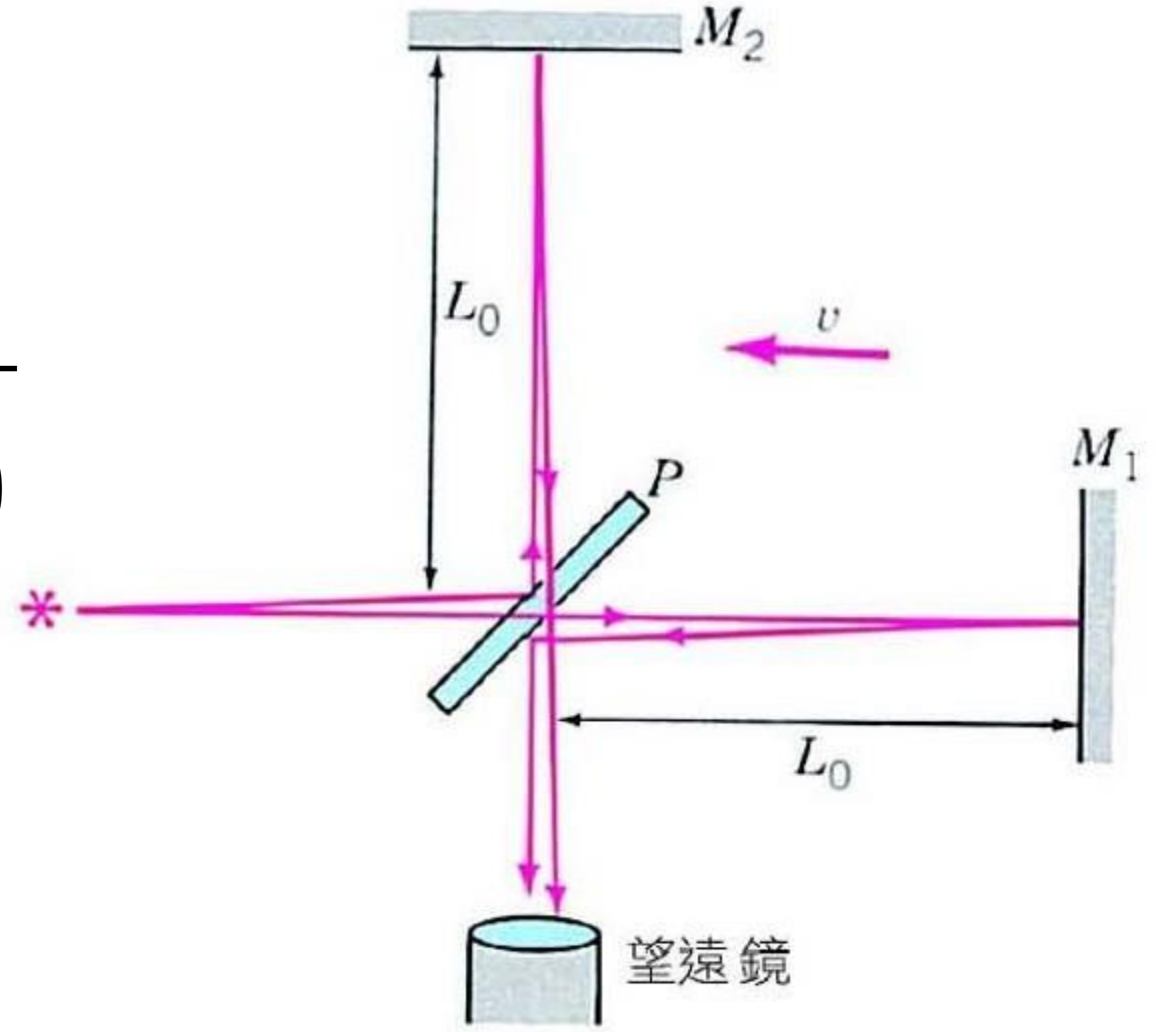
Parallel:

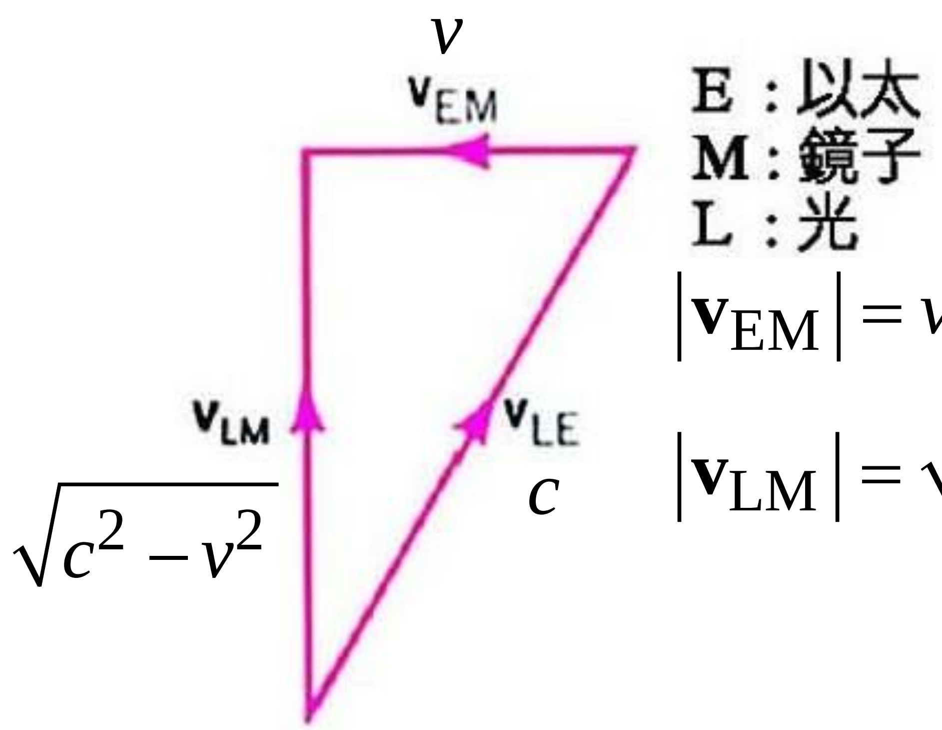
$$T_1 = \frac{L_0}{(c-v)} + \frac{L_0}{(c+v)} = \frac{(2L_0/c)}{(1-v^2/c^2)}$$

Perpendicular:

$$T_2 = \frac{2L_0}{(c^2-v^2)^{1/2}} = \frac{(2L_0/c)}{(1-v^2/c^2)^{1/2}}$$

$$\Delta T = T_1 - T_2 \cong \frac{L_0}{c}\left(\frac{v^2}{c^2}\right)$$





$|\mathbf{v}_{EM}| = v,\quad |\mathbf{v}_{LE}| = c$

$|\mathbf{v}_{LM}| = \sqrt{c^2-v^2}$

The Michelson-Morley Experiment (III)

Using $v = 30$ km/s, the expected shift was about 0.4 fringe. Even though they were able to detect shifts smaller than 1/20 of a fringe, ***they found nothing***.

Possibilities:

- The ether was dragged with the Earth. ✗
- No ether. ✓
- Constant light speed. ✓

The Two Postulates

The two postulates in the theory of special relativity are:

1. **The principle of relativity**: All physical laws have the same form in all inertia frames.
2. **The universal speed of light**: The speed of light in free space is the same in all inertial frames. It does not depend on the motion of the source or the observer.

Both postulates are restricted to inertial frames. This is why the theory is special.

- The principle of relativity extends the concept of covariance from mechanics to all physical laws.
- The constancy of the speed of light is difficult to accept at first.

All the experimental consequences have confirmed its correctness.

Some Preliminaries

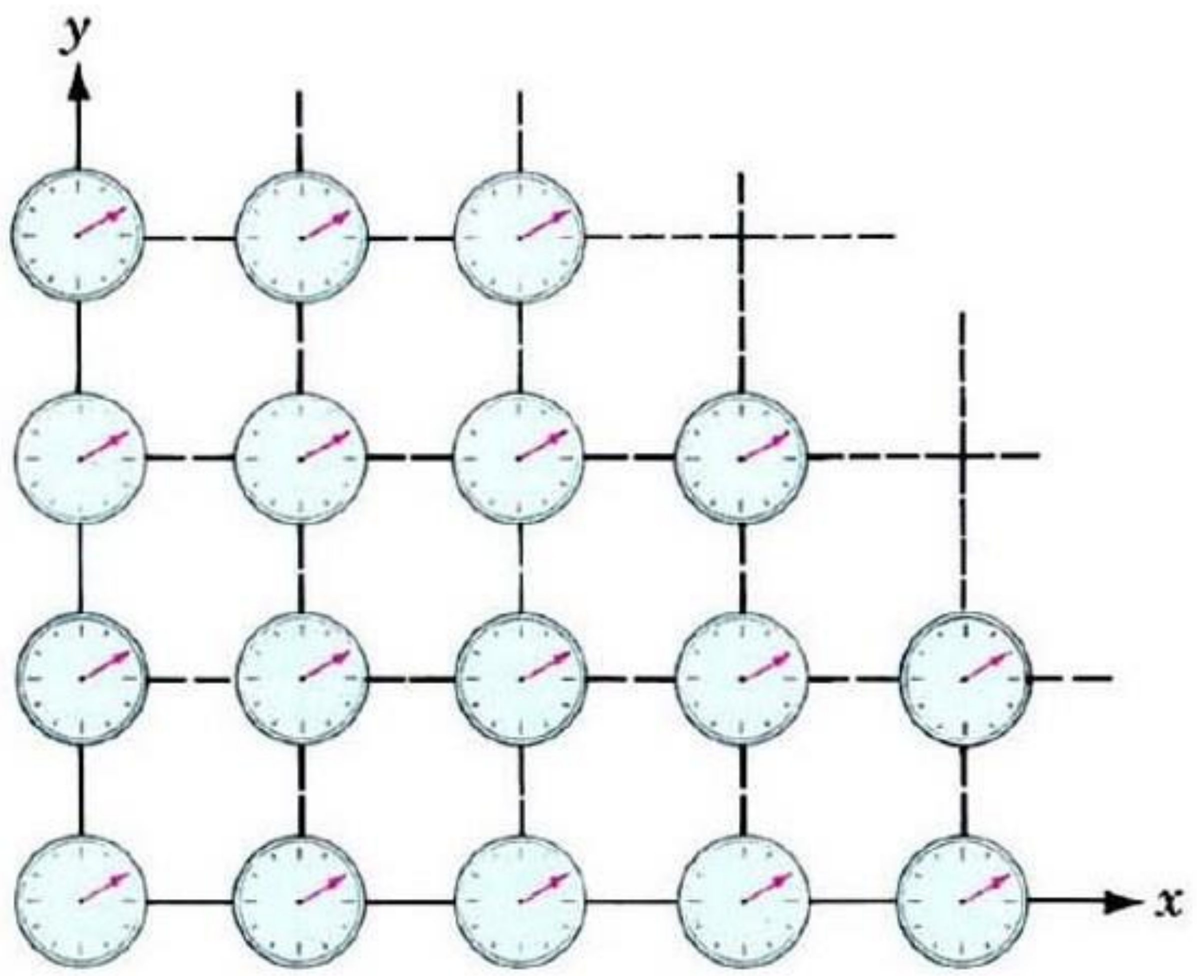
Event: Event is something that occurs at a single point in space at a single instant in time.

Observer: An observer is either a person, or an automatic device, with a clock and a meter stick. Each observer can record events only in the immediate vicinity.

Reference frame: A reference frame is a whole set of observers uniformly distributed in space. The frame in which an object is at rest is called its **rest frame**.

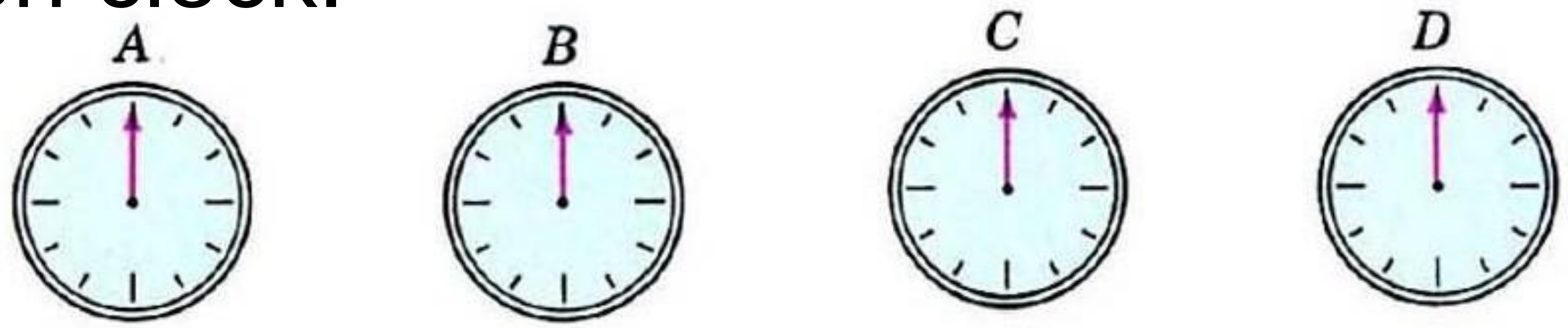
Synchronization of clocks: It is extremely important to define precisely what is meant by the time in a given reference frame. This requires a careful procedure for the synchronization of clocks.

Some Preliminaries (II)



A reference frame is assumed to consist of many observers uniformly spread through the space. Each observer has a meter stick and a clock to make measurements only in the immediate vicinity.

To synchronize four equally spaced clocks, a signal is sent out by clock *A* to trigger the other clocks---each of which has been set ahead by the amount of time it takes to travels from *A* to the given clock.

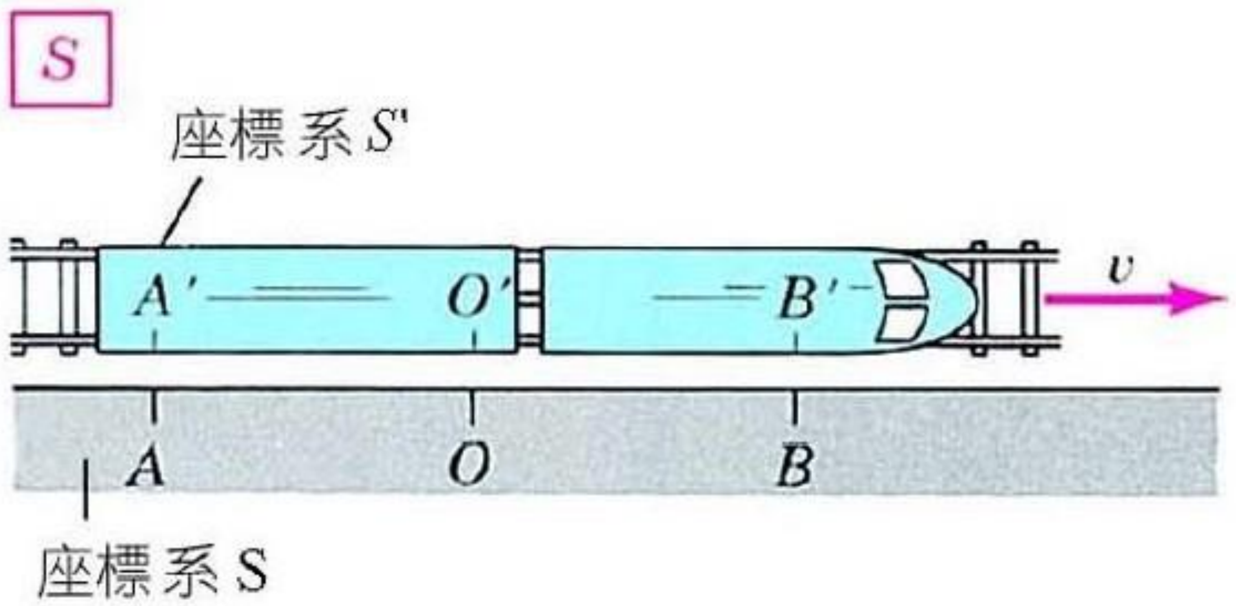


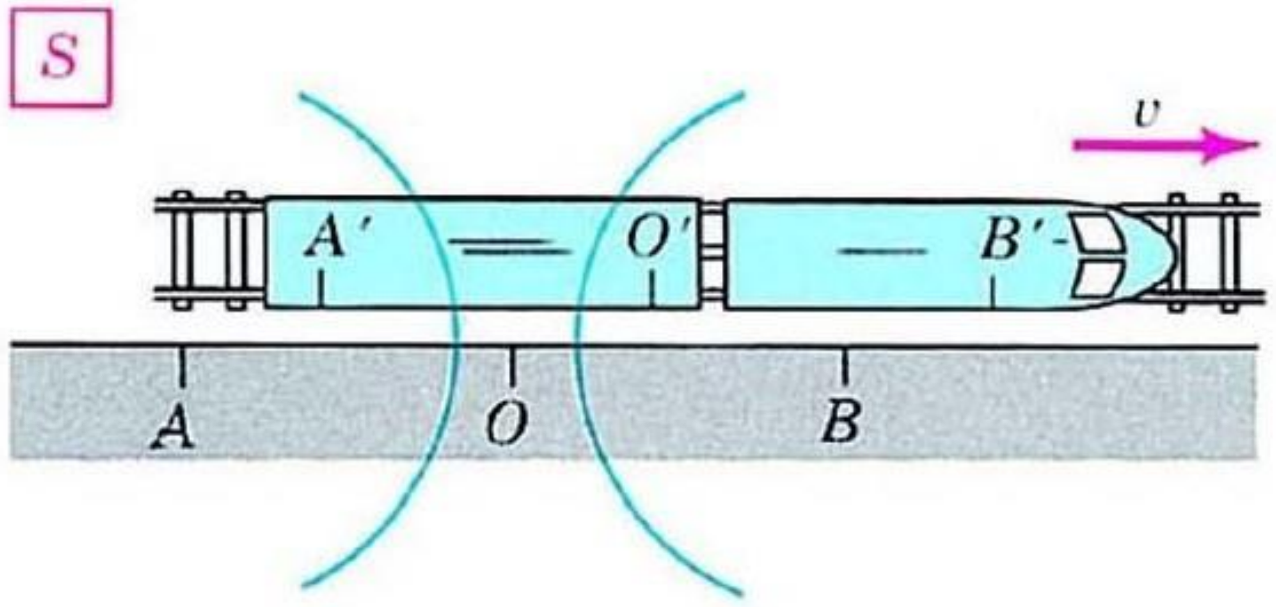
Relativity of Simultaneity

How can we determine whether two events at different locations are simultaneous?

Two events at different locations are simultaneous if an observer midway between them receives the flashes at the same instant.

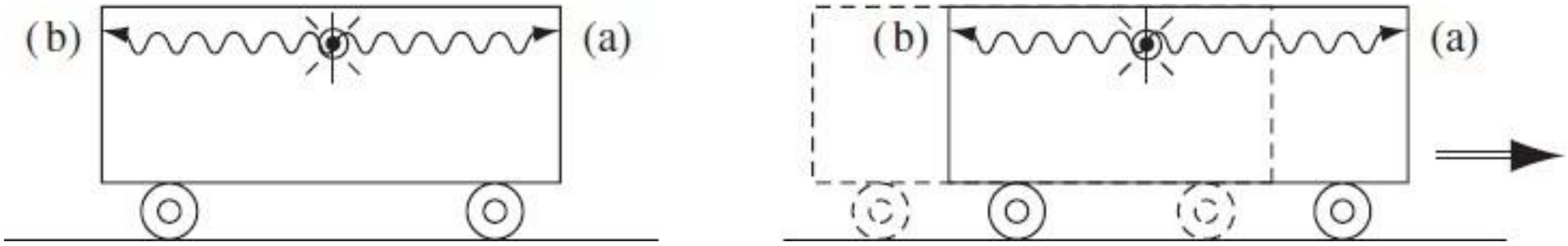
Relativity of Simultaneity: Spatially separated events that are simultaneous in one frame are not simultaneous in another, moving relative to the first.





Relativity of Simultaneity
(another example)

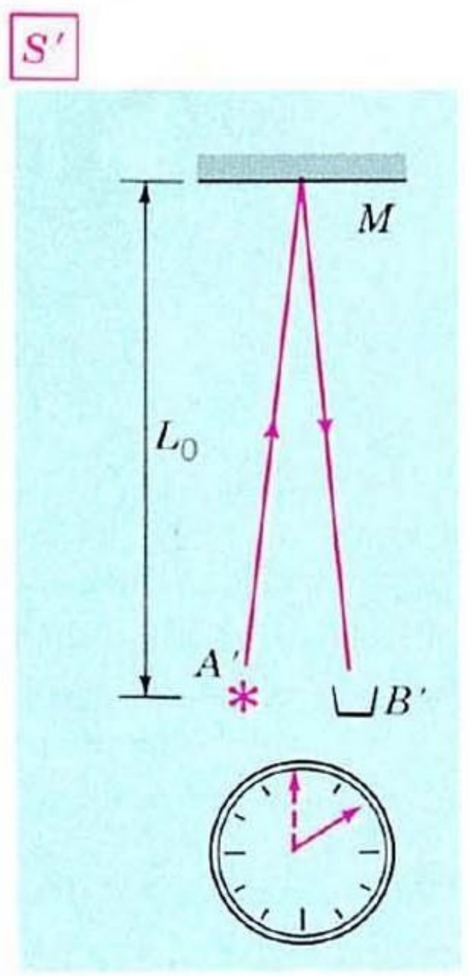
Two events that are simultaneous in one inertial system are not, in general, simultaneous in another.



Geometry of Relativity: Time Dilation

How does the relative motion of two frames affect the measured time interval between two events?

$$\tau = \frac{2L_0}{c}$$



A **proper time**, τ, is the time interval between two events as measured in the rest frame of a clock. In this frame both events occur at the same position. (Note: proper→own)

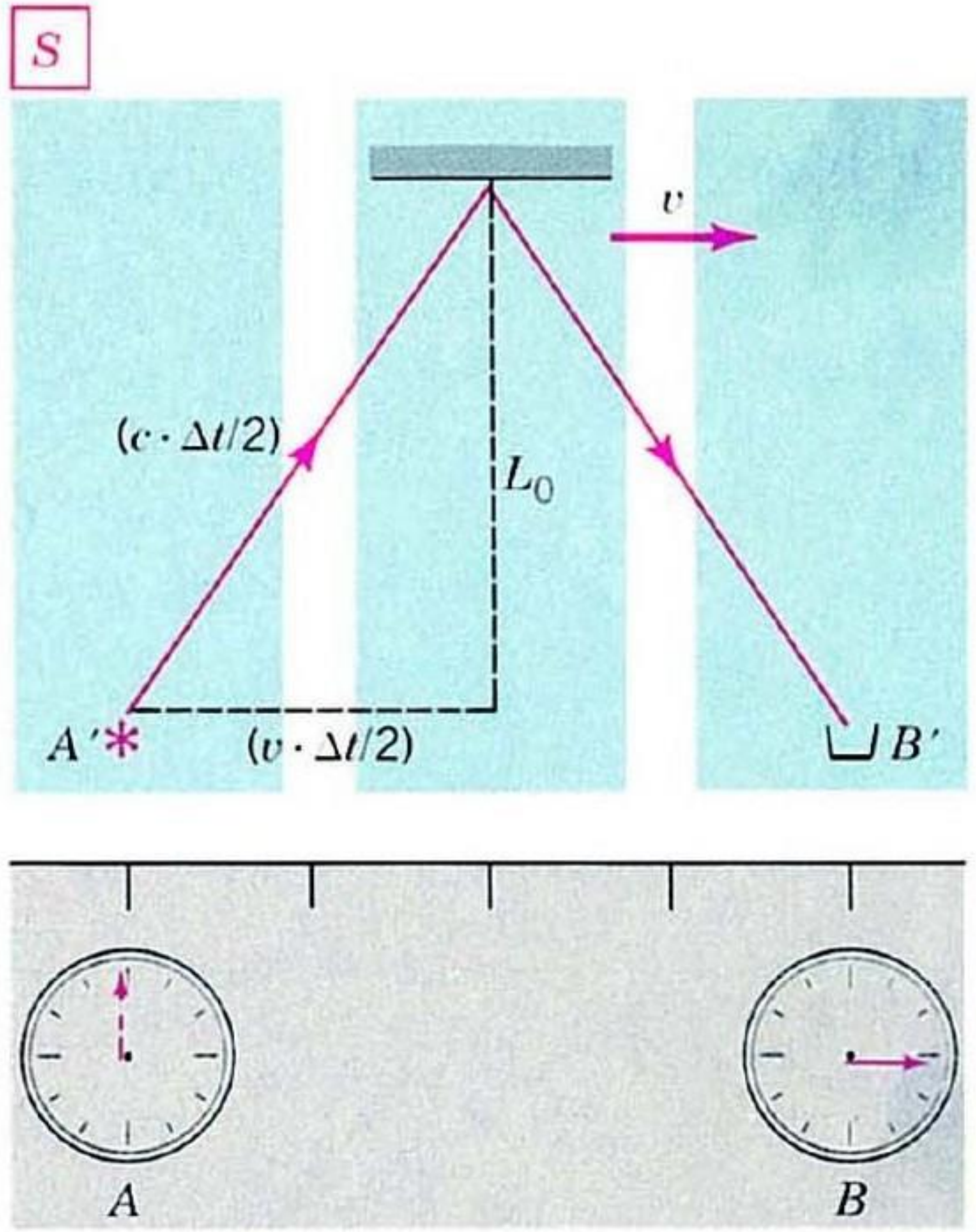
Time Dilation (II)

Now let us find the time interval recorded in the frame S, in which the clock has velocity v. The time interval Δt in frame S measured by two observers *A* and *B* at different positions.

$$\left(c\cdot\frac{\Delta t}{2}\right)^2 = L_0^2 + \left(v\cdot\frac{\Delta t}{2}\right)^2$$

$$T = \Delta t = \frac{2L_0}{c}\cdot\left(\frac{1}{\sqrt{1-v^2/c^2}}\right)$$

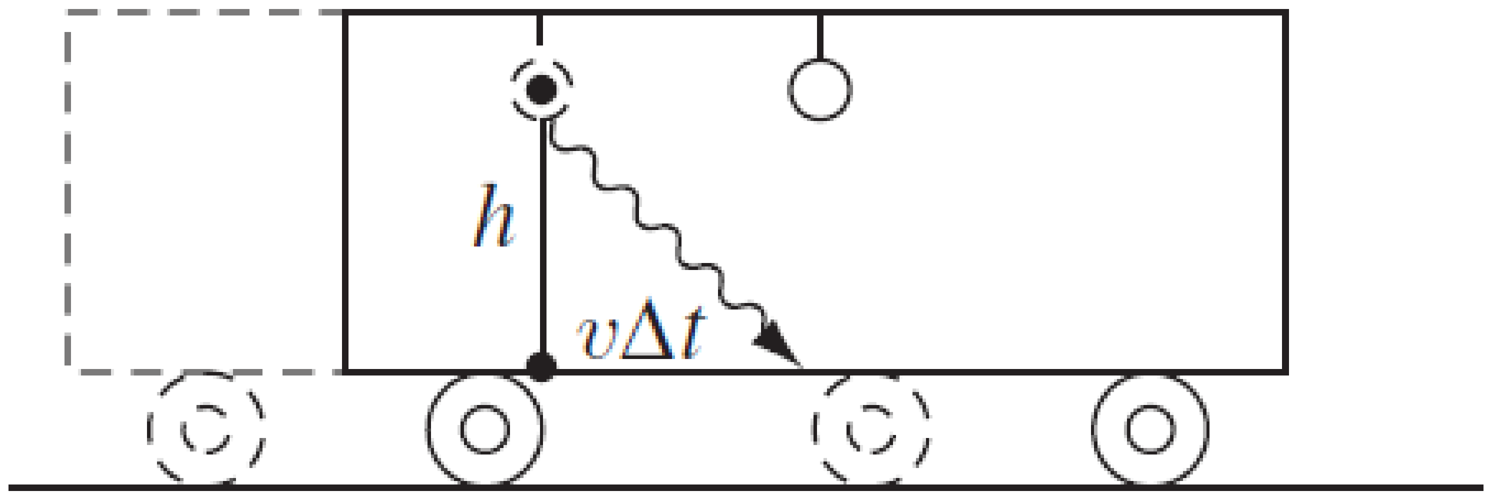
$$T = \gamma T_0 \quad \text{where } \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$



Note that we have used c as the speed of light in both frames---in accord with the second postulate.

Time Dilation (III)

Another example:



$$\tau = \frac{h}{c} \quad \text{(proper time)}$$

$$\Delta t = \frac{\sqrt{h^2 + (v\Delta t)^2}}{c} \quad \Rightarrow \quad \Delta t = \frac{1}{\sqrt{1 - v^2 / c^2}} \frac{h}{c}$$

$$\Delta t = \gamma\tau, \quad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2 / c^2}}$$

Moving clocks run slow.

Time Dilation (IV)

Since $\gamma > 1$, the time interval T measured in frame S (by two clocks) is greater than the proper time, T_0, registered by the clock in its rest frame S'. The effect is called time dilation.

Two spatially separated clocks, *A* and *B*, record a greater time interval between two events than the proper time recorded by a single clock that moves from *A* to *B* and is present at both events.

$$\gamma = \frac{1}{\sqrt{1 - v^2 / c^2}}$$

v/c	γ
0.6	5/4
0.8	5/3
0.98	5
0.995	10
0.9965	12
0.9992	25

*有些 γ 的數值已經加以化約。

Example of Time Dilation

Experimental evidence (muon decay):

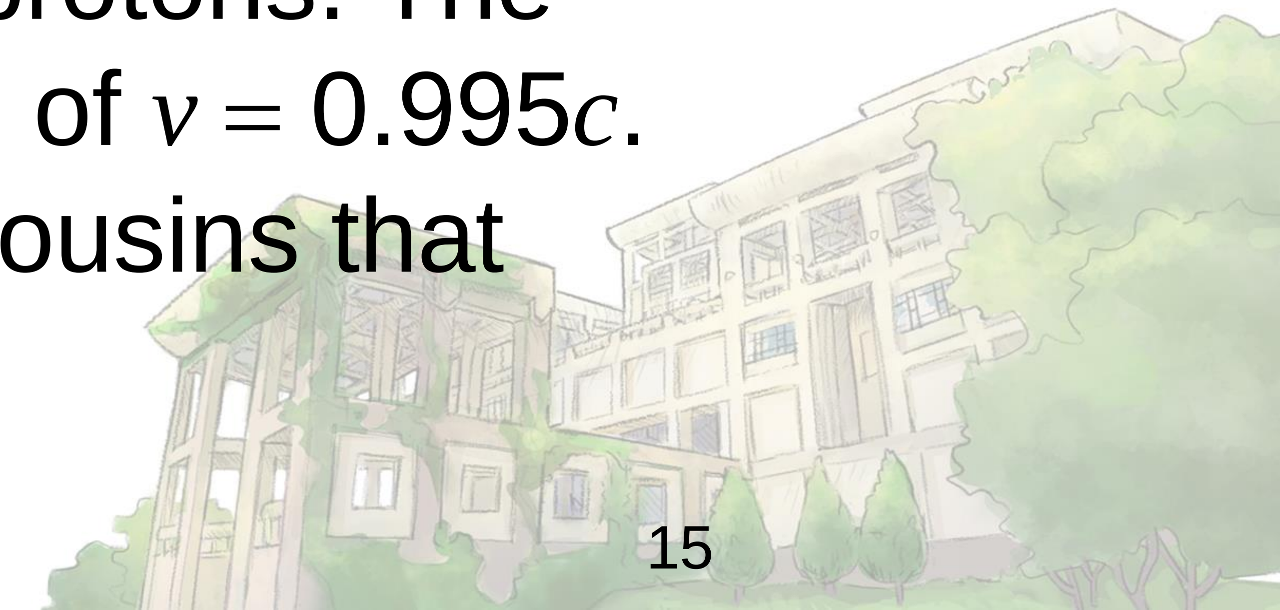
The reality of time dilation was verified in an experiment performed in 1941.

Rest frame at ground: An elementary particle, the muon (μ), decays into other particle. The particle decay rate is

$$N = N_0 e^{-t/\tau}$$

where $\tau = 2.2\ \mu\text{s}$ is called the mean lifetime.

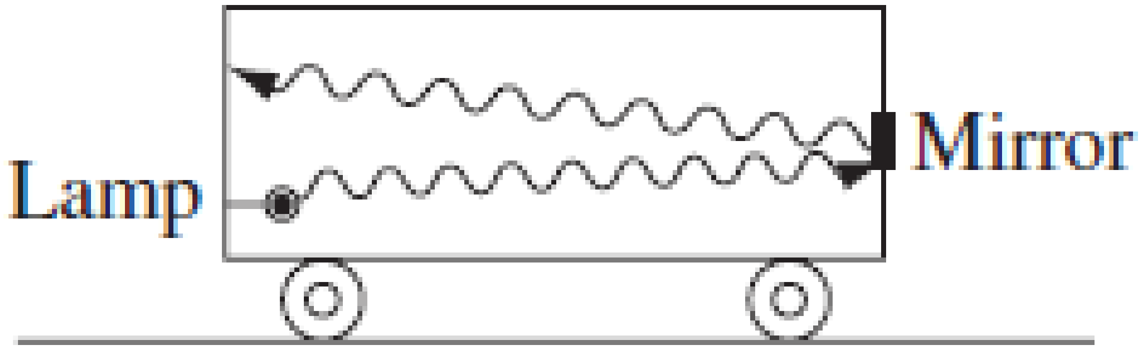
Moving frame at the upper atmosphere: Muons are produced from the bombardment of cosmic ray protons. The muon generated with this method has the speed of $v = 0.995c$. The mean lifetime is 10 times longer than their cousins that decay at rest in the laboratory.



Geometry of Relativity: Length Contraction

Consider a rod *AB* at rest in frame *S*, as shown below. The distance between its ends is its proper length L_0:

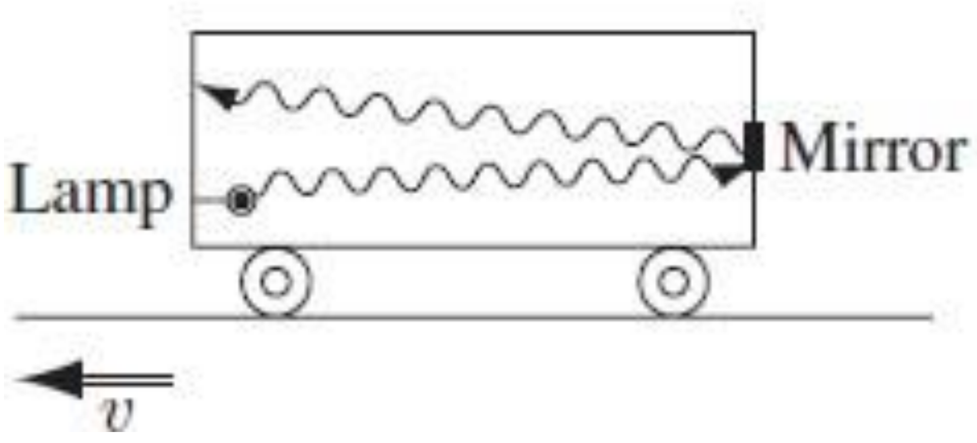
The **proper length**, L_0, of an object is the space interval between its ends measured in the rest frame of the object.

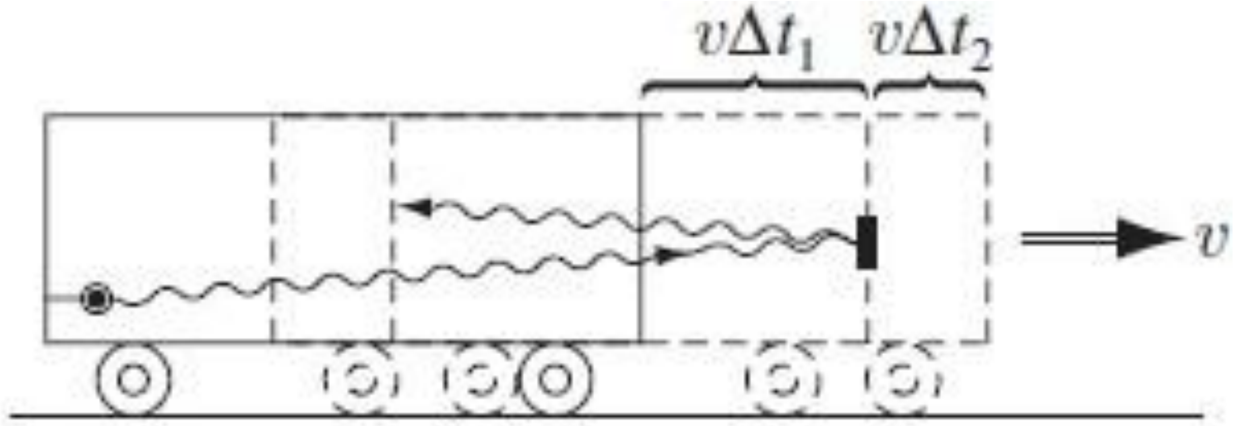


$$L_0 = \frac{c\tau}{2} \quad (L_0\text{: proper length})$$

Length Contraction (II)

Another example:





$$L_0 = \frac{c\tau}{2} \quad (L_0\text{: proper length})$$

$$\Delta t_1 = \frac{L + v\Delta t_1}{c}, \quad \Delta t_2 = \frac{L - v\Delta t_2}{c}$$

$$\Delta t_1 = \frac{L}{c-v}, \quad \Delta t_2 = \frac{L}{c+v} \quad \Rightarrow \Delta t = \Delta t_1 + \Delta t_2 = 2\frac{L}{c}\frac{1}{1-v^2/c^2}$$

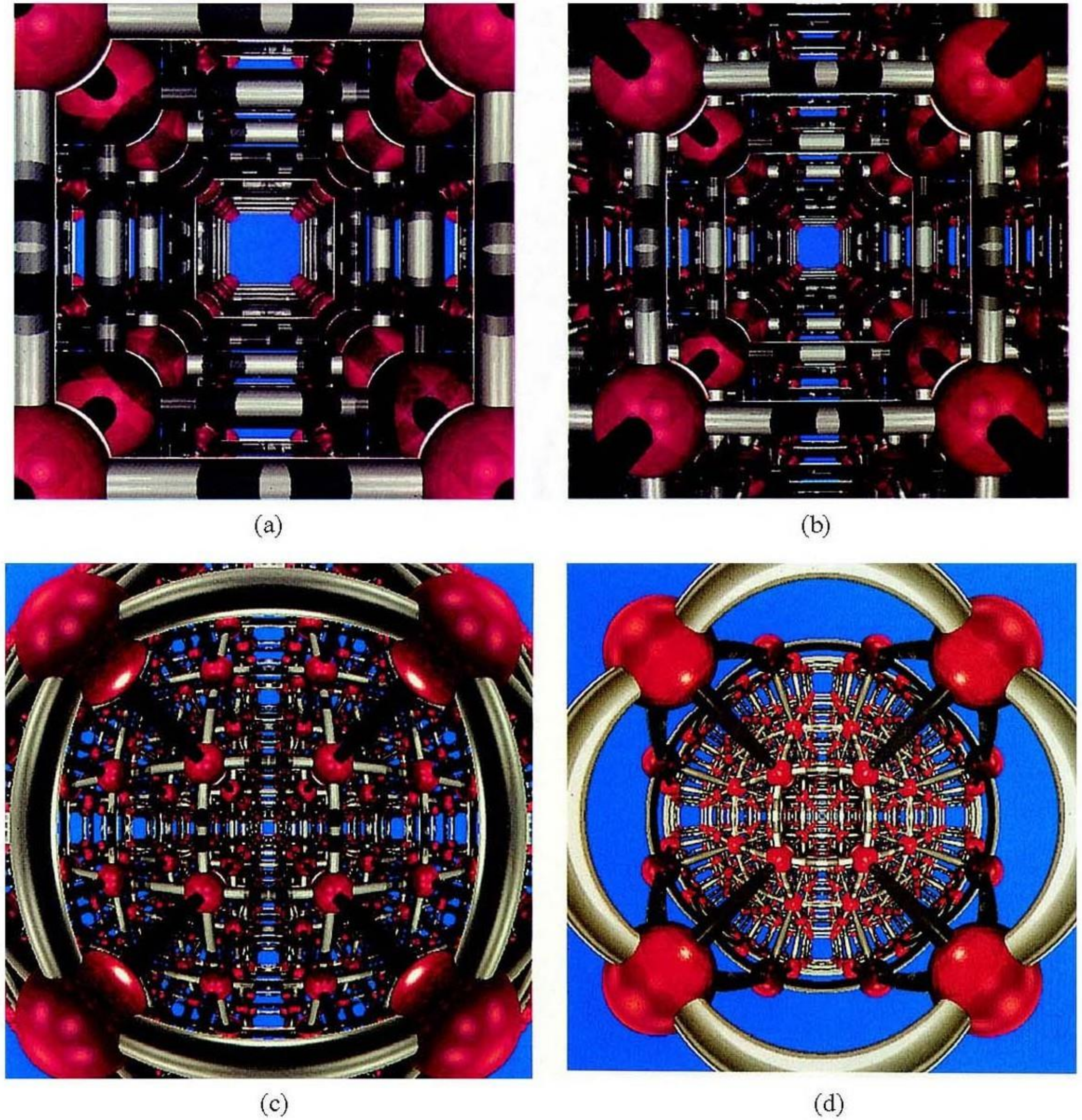
$$L = \frac{c}{2}\frac{1}{\gamma^2}\Delta t = \frac{c}{2}\frac{1}{\gamma^2}\gamma\tau = \frac{1}{\gamma}L_0$$

Moving objects are shortened.

Effects of Length Contraction (I)

$v = 0.0\,c$

$v = 0.5\,c$

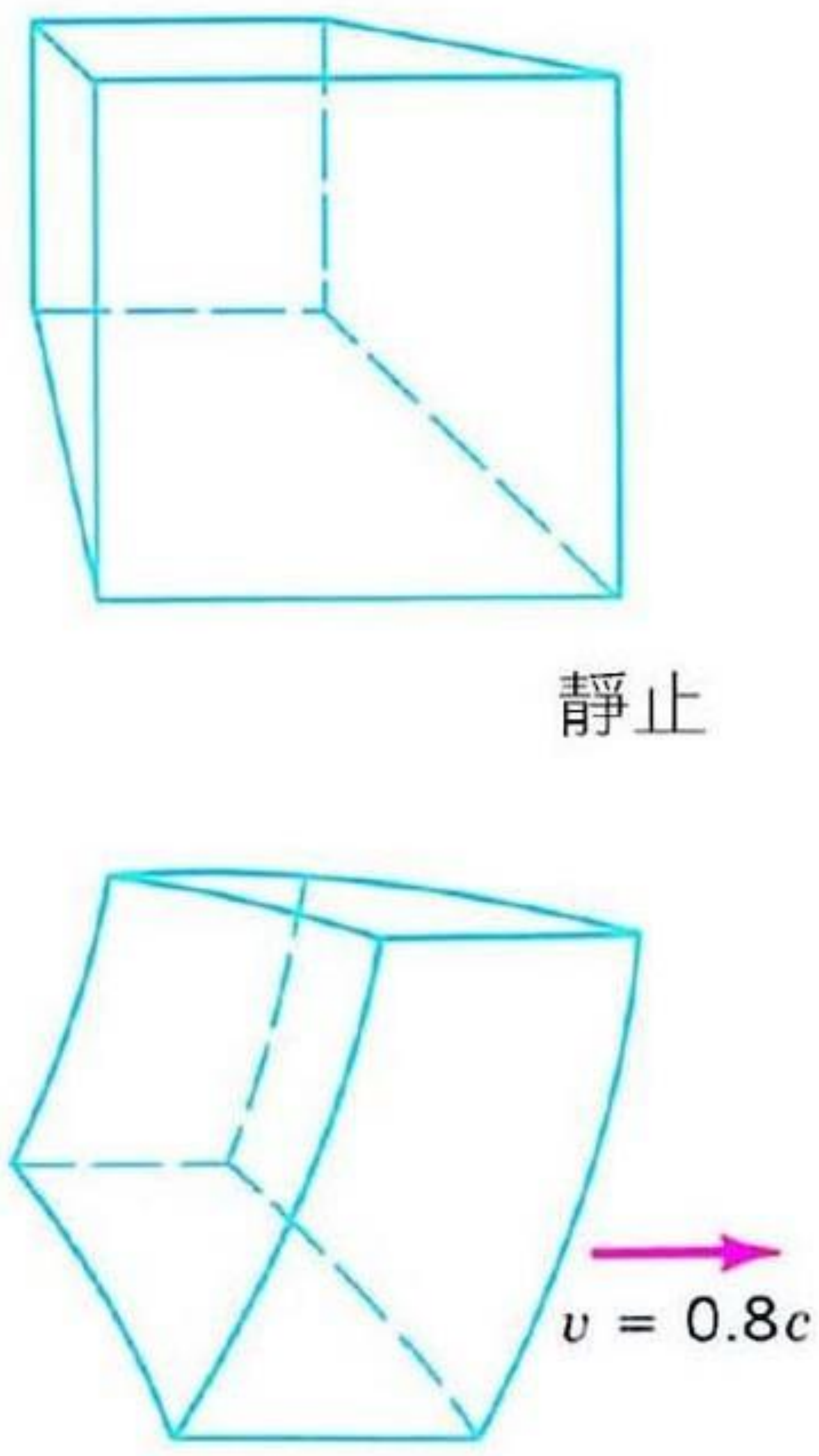


$v = 0.95\,c$

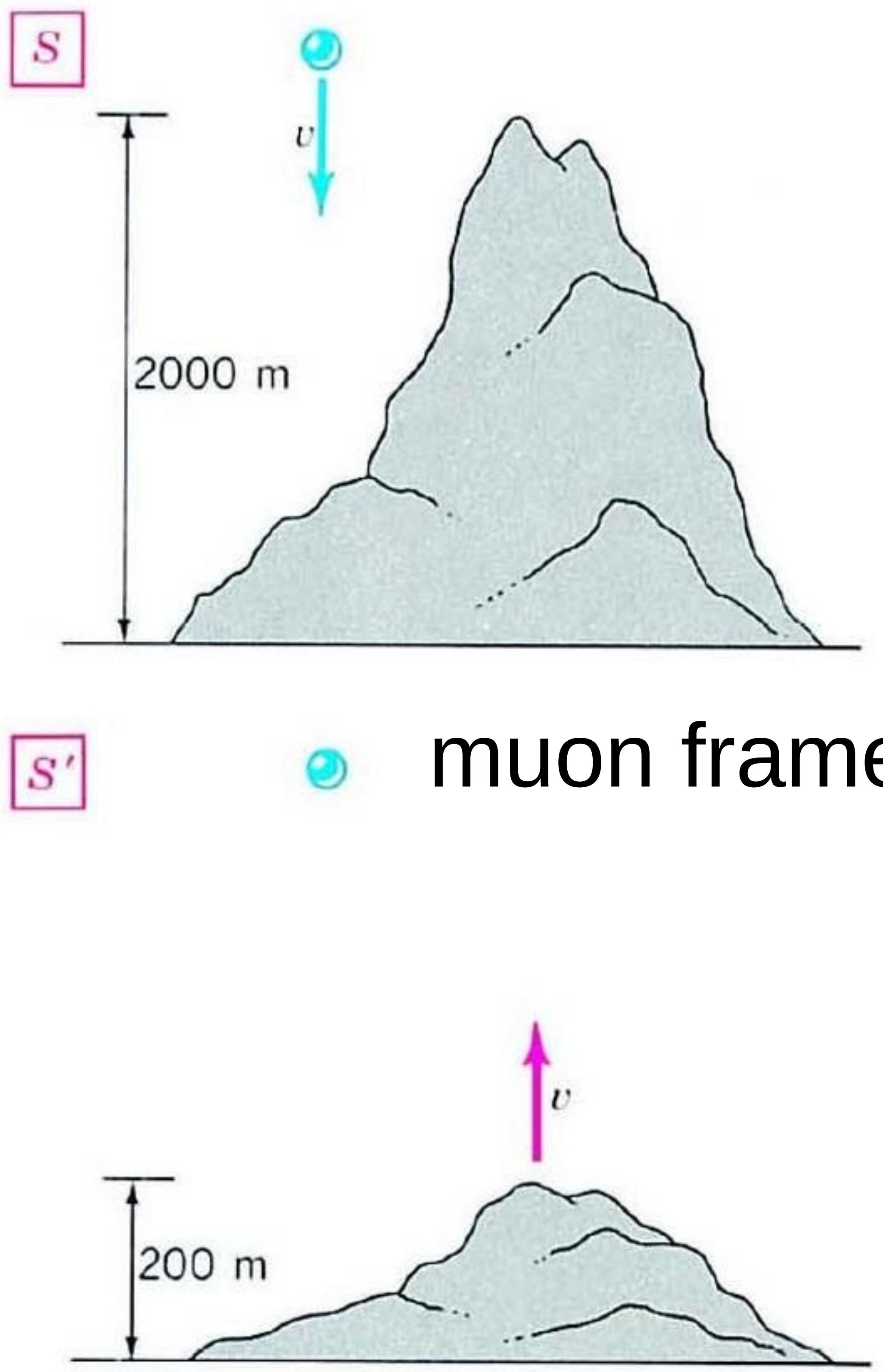
$v = 0.99\,c$

Length Contraction Effects (II)

distortion



rest frame



muon frame

The Twin Paradox

Nothing in the theory of relativity catches the imagination more than the so-called the twin paradox.

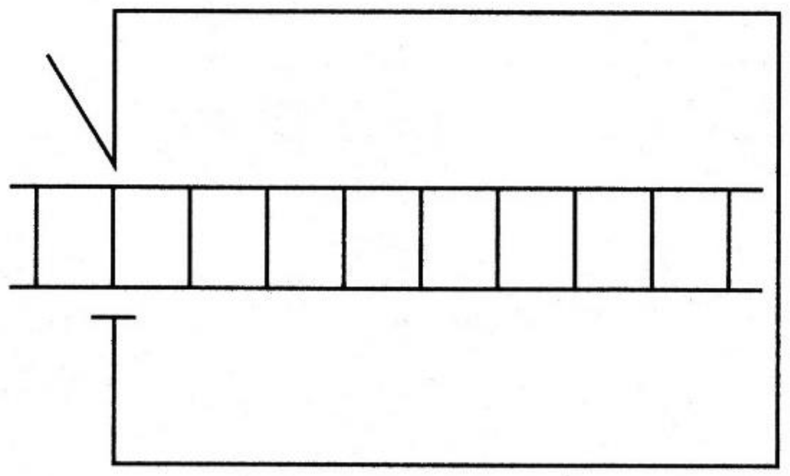
Twin *A* stays on earth while twin *B* travels at high speed to a nearby star. When *B* returns, they both find that *A* has aged more than *B*.

The paradox arises because of the apparent symmetry of the situation: In *B*'s frame, it is *A* that leaves and returns, so one should also find that *B* has aged more than *A*.

$$\left.\begin{array}{l} A > B \\ B > A \end{array}\right\} ?$$

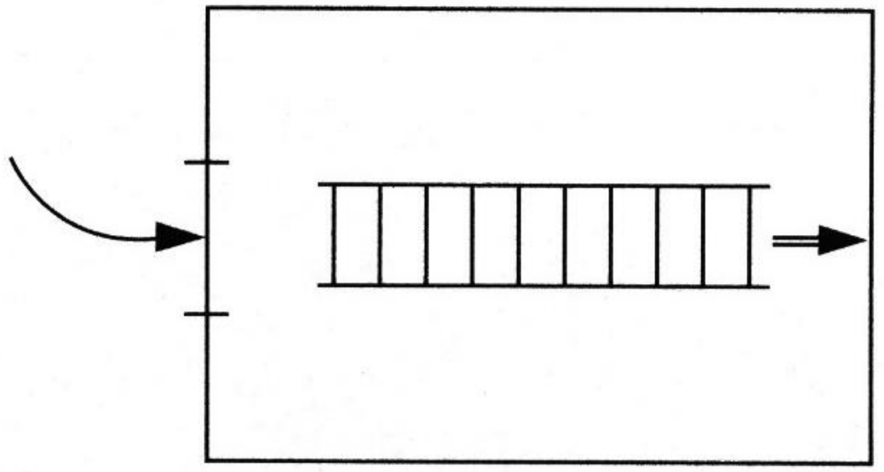
What's going on?

The Barn and Ladder Paradox



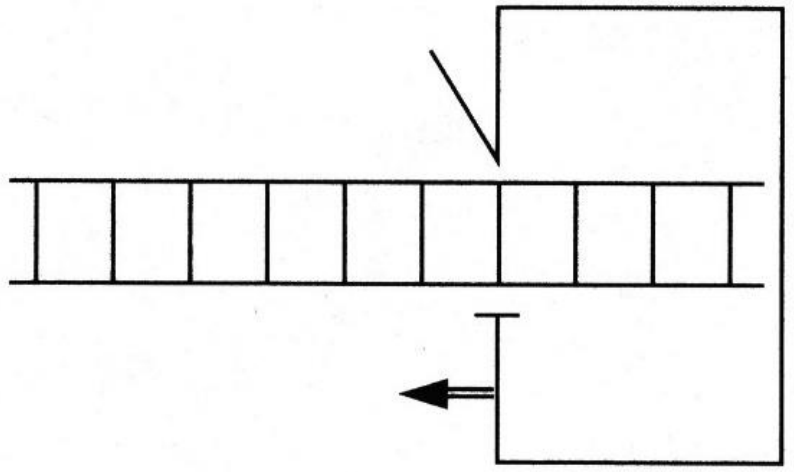
(a)

before



(b)

farmer’s view



(c)

ladder’s view

Who’s right?

The Galilean and Lorentz Transformation

Given:

1. Two reference frames S and S'. S' moves along the common x axis with speed v relative to S.
2. S and S' coincide at $t = t' = 0$.
3. An event occurs at point (x, y, z) and at time t as observed in S.

The Galilean Transformation (valid for $v \ll c$)

$$x' = x - vt,\ y' = y,\ z' = z,\ t' = t$$

Wiki: A Galilean transformation is used to transform between the coordinates of two reference frames which differ only by constant relative motion within the constructs of Newtonian physics.

The Galilean and Lorentz Transformation (ii)

The Lorentz Transformation (valid for any v)

(i) At time t, origin of S' is at a distance vt away from the origin of S, as observed in S.

(ii) x' is a proper length in S', so in S it is x'/γ.

(i)+(ii) → $x = vt + x'/\gamma$ → $x' = \gamma(x - vt)$

(iii) By symmetry $x = \gamma(x' + vt')$ and eliminating x', we have

→ $t' = \gamma(t - vx/c^2)$.

(iv) No length contraction perpendicular to the direction of relative motion → $y = y'$ and $z = z'$.

$$x' = \gamma(x - vt); \quad y' = y; \quad z' = z; \quad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

The Lorentz Transformation

The laws of electromagnetism are not *covariant* with respect to the Galiliean transformation. However, with the Lorentz transformation they are covariant. The space and time are related shown as follows:

In terms of the rest frame

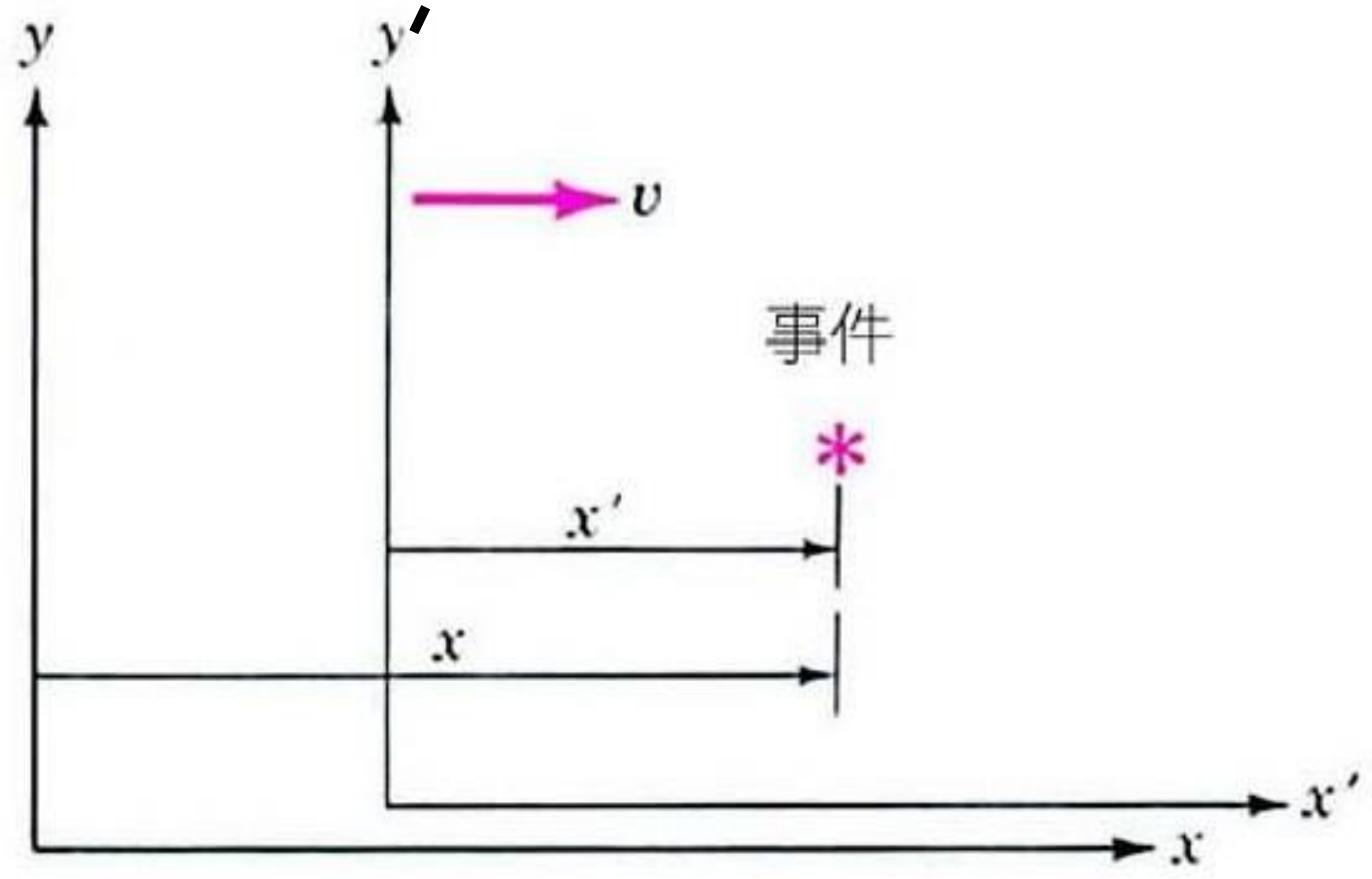
$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{vx}{c^2})$$

In terms of the moving frame

$$x = \gamma(x' + vt')$$

$$t = \gamma(t' + \frac{vx'}{c^2})$$



Ex. 12.4 Time dilation

$$dx' = \gamma(dx - vdt) = 0 \quad \Rightarrow \quad dx = vdt$$

$$dt' = \gamma(dt - \frac{vdx}{c^2}) = \gamma dt(1 - \frac{v^2}{c^2}) = \frac{1}{\gamma} dt$$

$$dt = \gamma dt'$$

Ex. 12.5 Length contraction

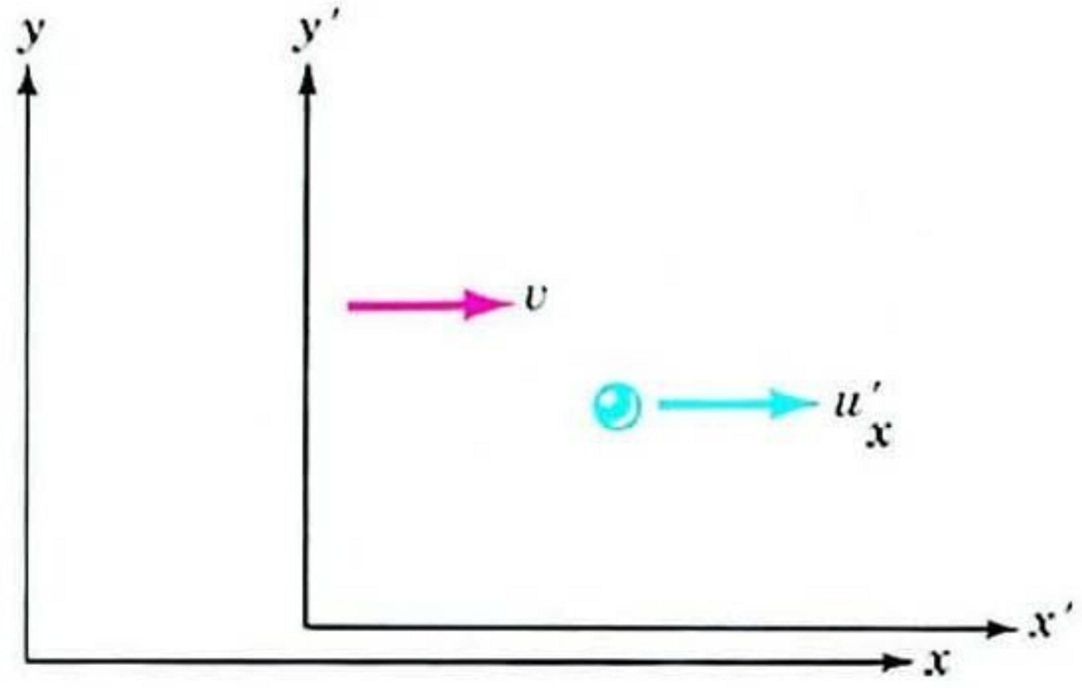
$$dt = \gamma(dt' + \frac{vdx'}{c^2}) = 0 \quad \Rightarrow \quad dt' = -\frac{vdx'}{c^2}$$

$$dx = \gamma(dx' + vdt') = \gamma dx'(1 - \frac{v^2}{c^2}) = \frac{1}{\gamma} dx'$$

Ex. 12.6 Einstein's velocity addition rule

$$dx = \gamma(dx' + vdt') = \gamma dt'(u'_x + v)$$

$$dt = \gamma(dt' + \frac{vdx'}{c^2}) = \gamma dt'(1 + \frac{u'_x v}{c^2})$$



Taking the ratio of these equations we find

$$u_x = \frac{u'_x + v}{1 + u'_x v / c^2}$$

An extreme case

when $u'_x = c$, we have $u_x = \frac{c + v}{1 + cv / c^2} = c$

The Structure of Space-time: (i) Four-vectors

$$x^0 \equiv ct,\ x^1 = x,\ x^2 = y,\ x^3 = z,\ \text{and } \beta = \frac{v}{c}$$

$$\left.\begin{aligned} \bar{x}^0 &= \gamma(x^0 - \beta x^1), \\ \bar{x}^1 &= \gamma(x^1 - \beta x^0) \\ \bar{x}^2 &= x^2 \\ \bar{x}^3 &= x^3 \end{aligned}\right\} \text{the Lorentz transformations}$$

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{the Lorentz transformation matrix}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad \bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu$$

Covariant Vector, Contravariant Vector, and Invariant Quantity

the covariant vector (row): a_μ

$$a_\mu = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \end{pmatrix}$$
$$\equiv \begin{pmatrix} -a^0 & a^1 & a^2 & a^3 \end{pmatrix}$$

the contravariant vector (column): a^μ

$$a^\mu = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$$

invariant quantity under Lorentz transformation

the Einstein summation convention

$$a_\mu b^\mu = \sum_{v=0}^{3} a_\mu b^\mu = \begin{pmatrix} -a^0 & a^1 & a^2 & a^3 \end{pmatrix} \begin{pmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{pmatrix}$$

$$a_\mu b^\mu = a^\mu b_\mu$$

$$= -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 = -\bar{a}^0 \bar{b}^0 + \bar{a}^1 \bar{b}^1 + \bar{a}^2 \bar{b}^2 + \bar{a}^3 \bar{b}^3$$

Jackson

Metric/Norm Tensor (I)

What is the difference between x^{α} and x_{α} ? $\Rightarrow$ Metric/norm tensor.

In special theory of relativity, Lorentz transformation of the four-dimensional coordinates follow from the invariance of two events:

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \qquad (11.67)$$

$$(ds)^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta} \qquad (11.68)$$

$g_{\alpha\beta} = g_{\beta\alpha}$ is called the *metric tensor* and diagonal in *flat* space-time.

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1 \qquad (11.69)$$

The contravariant metric tensor $g^{\alpha\beta}$ is defined as the normalized co-factor of $g_{\alpha\beta}$. For flat spac-time it is the same:

$$g^{\alpha\beta} = g_{\alpha\beta} \qquad (11.70)$$

Jackson

Metric/Norm Tensor (II)

The contraction of the contravariant and covariant metric tensors give the Kronecker delta in four dimensions:

$$g_{\alpha\gamma}g^{\gamma\beta} = \delta_{\alpha}^{\beta} \qquad (11.71)$$

where $\delta_{\alpha}^{\beta} = 0$ for $\alpha \neq \beta$ and $\delta_{\alpha}^{\alpha} = 1$ for $\alpha = 0, 1, 2, 3.$

$$x_{\alpha} = g_{\alpha\beta}x^{\beta} \qquad (11.72)$$

$$x^{\alpha} = g^{\alpha\beta}x_{\beta} \qquad (11.73)$$

With the metric tensor, a contravariant vector and a co-variant vector can be expressed as:

$$A^{\alpha} = (A^0, \mathbf{A}), \quad \text{and} \quad A_{\alpha} = (A^0, -\mathbf{A}) \qquad (11.75)$$

The Invariant Interval

Two events A and B occur at $\left(x_A^0, x_A^1, x_A^2, x_A^3\right)$ and $\left(x_B^0, x_B^1, x_B^2, x_B^3\right)$, respectively.

The displacement 4-vector: $\Delta x^\mu \equiv x_A^\mu - x_B^\mu$

The interval between two events: $I \equiv \Delta x_\mu \Delta x^\mu = -c^2 t^2 + d^2$

timelike	$I < 0$	$(c^2 t^2 > d^2)$
spacelike	$I > 0$	$(c^2 t^2 < d^2)$
lightlike	$I = 0$	$(c^2 t^2 = d^2)$

t : the time difference between the two events.

d : their spacial separation.

12.2 Relativistic Mechanics

12.2.1 Proper Time and Proper Velocity

How to define the velocity?

Imagine you are on a flight to Moon, and the pilot announces that the plane's velocity relative to ground is $4/5c$.

$$\mathbf{u} = \frac{d\mathbf{l}}{dt} \qquad \text{the ordinary velocity}$$

You might be more interested in the distance covered per unit proper time.

$$dt = \gamma d\tau \quad \Rightarrow \quad \frac{1}{d\tau} = \gamma \frac{1}{dt}$$

$$\boldsymbol{\eta} \equiv \frac{d\mathbf{l}}{d\tau} = \frac{1}{\sqrt{1 - u^2 / c^2}} \mathbf{u} \qquad \text{the proper velocity}$$

Which definition is more preferable/useful?

Proper Velocity 4-Vector (4-velocity)

Proper time τ is invariant, whereas "ordinary" time t depends on the particular reference frame.

Proper velocity has an enormous advantage over ordinary velocity: it transforms simply.

$$\eta^{\mu} \equiv \frac{dx^{\mu}}{d\tau}$$

The numerator, dx^{μ}, is a displacement 4-vector;

The denominator, $d\tau$, is invariant.

$$\left.\begin{aligned} \bar{\eta}^0 &= \gamma(\eta^0 - \beta\eta^1), \\ \bar{\eta}^1 &= \gamma(\eta^1 - \beta\eta^0) \\ \bar{\eta}^2 &= \eta^2 \\ \bar{\eta}^3 &= \eta^3 \end{aligned}\right\}$$

More generally, $\bar{\eta}^{\mu} = \Lambda^{\mu}_{\nu}\eta^{\nu}$

$$\eta^0 = \frac{dct}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1-u^2/c^2}}$$

12.2.2 Relativistic Energy and Momentum

How to define the momentum?

In classical mechanics momentum is mass times velocity, but immediately a question arise: Should we use *ordinary* velocity or *proper* velocity? There is no prior reason to favor one over the other.

$$\mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} \qquad \text{the relativistic momentum}$$

$$= m_{\text{rel}}\mathbf{u} \qquad m_{\text{rel}}\text{: the relativistic mass}$$

$$p^0 = m\eta^0 = mc\frac{dt}{d\tau} = \frac{mc}{\sqrt{1-u^2/c^2}} = \frac{E}{c}$$

$$\text{where } E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}} \qquad \text{relativistic energy}$$

Kinetic Energy

How to define the kinetic energy?

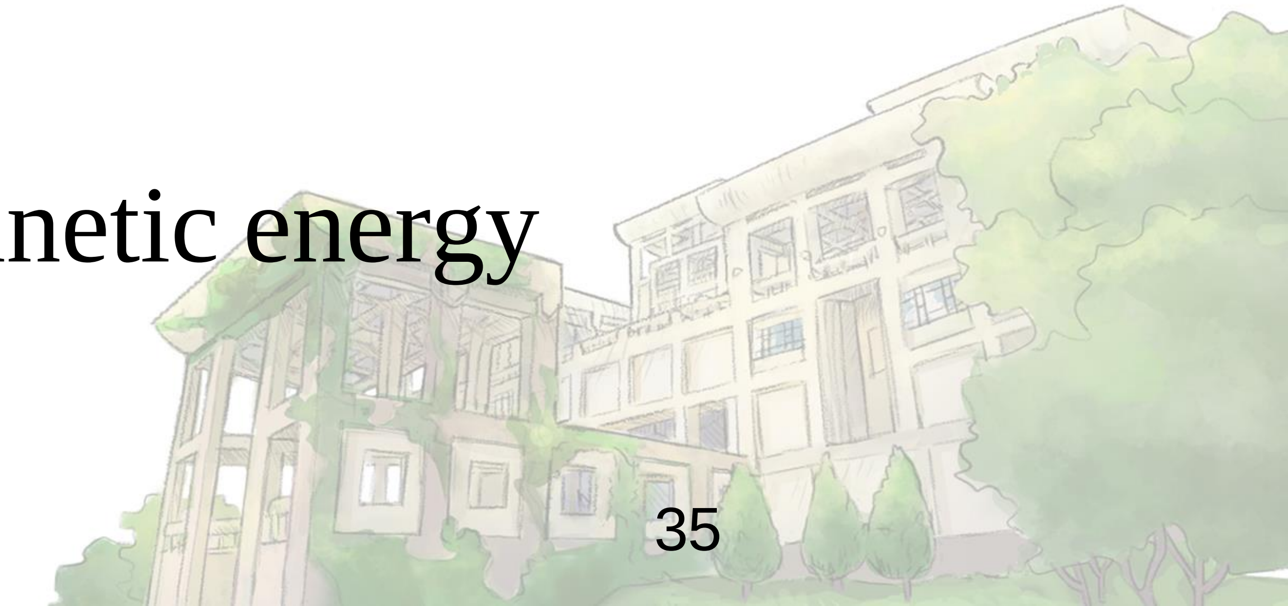
The **relativistic kinetic energy** is the **total energy** minus the **rest energy**:

$$E_{\text{kin}} = E - E_{res} = \frac{mc^2}{\sqrt{1-u^2/c^2}} - mc^2$$

$$= mc^2 \left(1 + \frac{1}{2}\frac{u^2}{c^2} + \frac{3}{8}\frac{u^4}{c^4} + \ldots - 1\right)$$

$$= \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \ldots$$

$K = \frac{1}{2}mu^2$ the classical defination of the kinetic energy



Conservation and Invariant

Conserved quantity: same value before and after some process.
Invariant quantity: same value in all inertia frames.

Charge is both conserved and invariant.
Energy is conserved, but not invariant.
Momentum is conserved, but not invariant.

Velocity is neither conserved nor invariant.

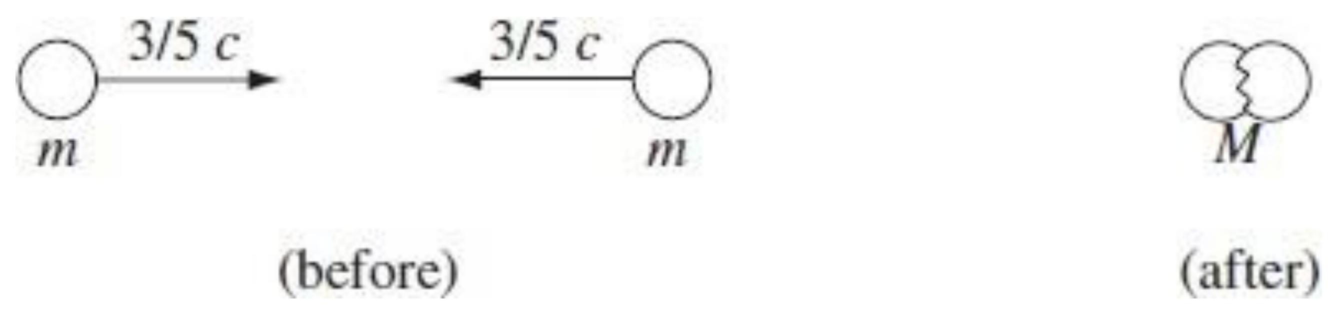
Invariant: $p^{\mu} p_{\mu} = -(p^0)^2 + (p \cdot p) = -m^2 c^2$

$$\Rightarrow \quad \frac{E^2}{c^2} - p^2 = m^2 c^2$$

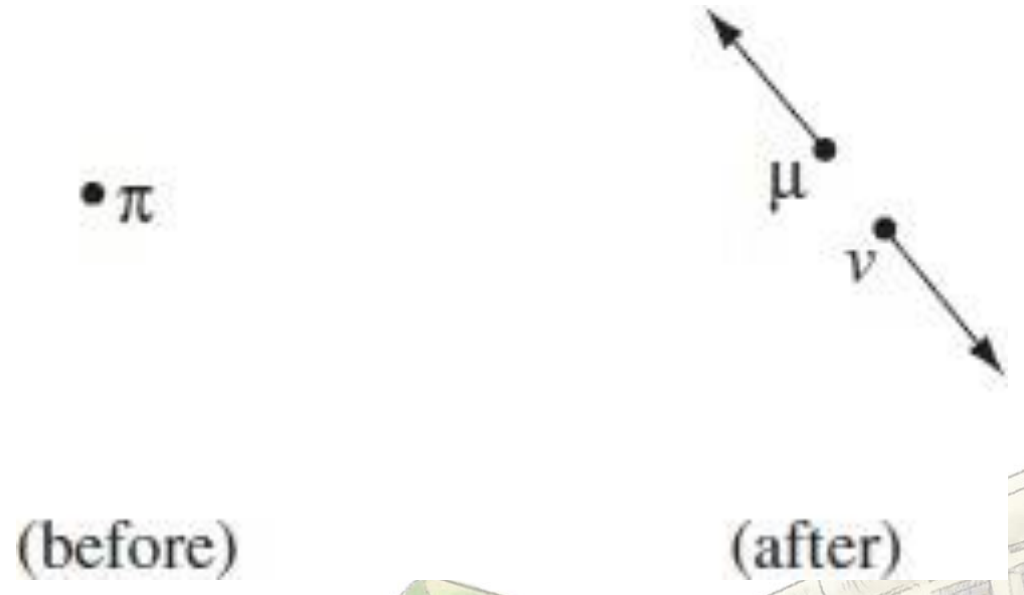
12.2.3 Relativistic Kinematics

Explore some applications of the conservation law to particle decays and collisions.

Example 12.7 Two lumps of clay, each of (rest) mass m, collide head-on at $3/5c$. They stick together. Question: what is the mass (M) of the composite lump?



Example 12.8 A pion at rest decays into a muon and a neutrino. Find the energy of the outgoing muon, in terms of the two masses, m_π and m_μ (assume m_μ=0)



Does it make any sense?

Massless Particle: Photon

In classical mechanics there is no such thing as a massless particle.

In special relativity, $\mathbf{p}$ and E are still proportional to m. If $u = c$, then the zero numerator is balanced by a zero in the denominator, leaving $\mathbf{p}$ and E indeterminate (zero over zero).

$$\text{When } u = c \text{ and } m = 0, \quad \Rightarrow \begin{cases} \mathbf{p} = \dfrac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} = \dfrac{0}{0} \\ E = \dfrac{mc^2}{\sqrt{1-u^2/c^2}} = \dfrac{0}{0} \end{cases}$$

A massless particle could carry energy and momentum, provided it always travels at the speed of light.

$$E = pc = h\nu \qquad \text{photon}$$

The Compton Scattering (Example 12.9)

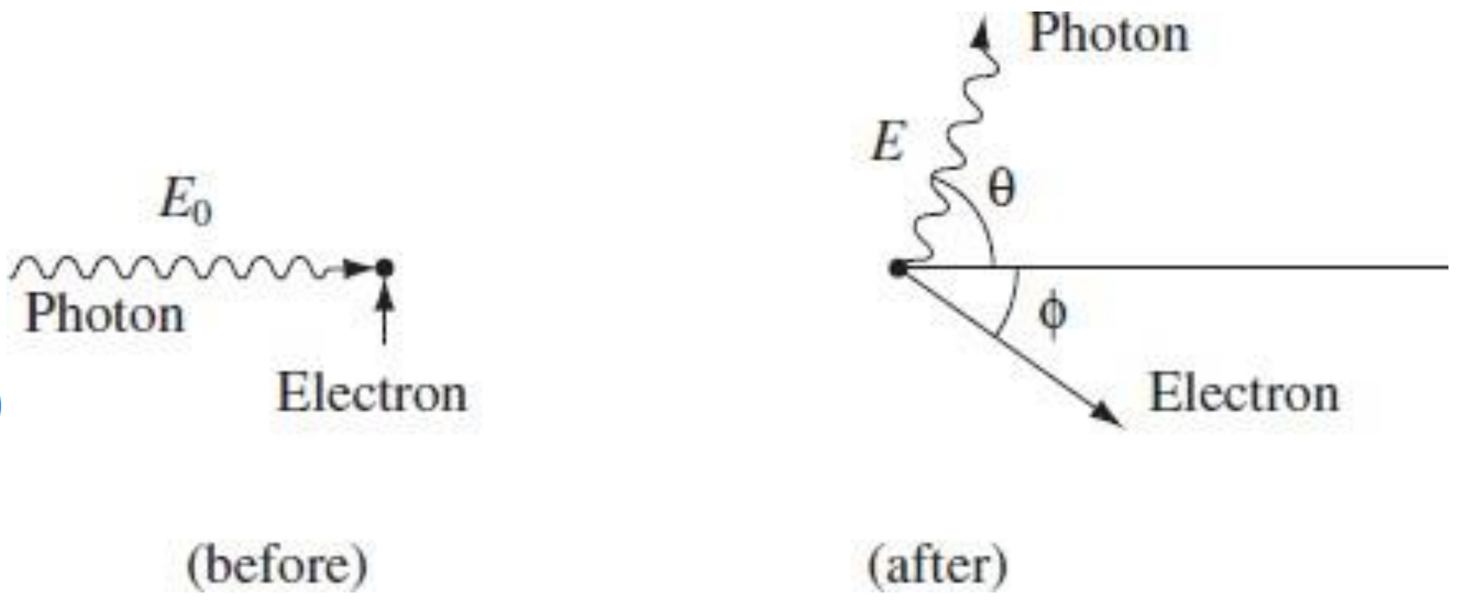
A photon of wavelength λ and energy hf "bounces" off an electron, initially at rest with the rest mass m_0. Find the wavelength λ' of the outgoing photon, as a function of the scattering angle θ.

An electromagnetic wave carries moment given by: $p_\lambda = \dfrac{hf}{c} = \dfrac{h}{\lambda}$

Conservation of linear momentum:

$$\begin{cases} p_x: \quad p_\lambda = p_{\lambda'}\cos\theta + p\cos\phi \\ p_y: \quad 0 = p_{\lambda'}\sin\theta - p\sin\phi \end{cases}$$

$$(p_\lambda - p_{\lambda'}\cos\theta)^2 + (p_{\lambda'}\sin\theta)^2 = p^2 \qquad (1)$$



Conservation of energy:

$$\begin{cases} hf + m_0c^2 = hf' + \gamma m_0c^2 \quad \Rightarrow \quad (p_\lambda - p_{\lambda'}) = (\gamma - 1)m_0c \\ p^2 = \gamma^2 m_0^2 v^2 = \dfrac{1-1+\beta^2}{1-\beta^2}m_0^2c^2 = (\gamma^2-1)m_0^2c^2 = [(\gamma-1)m_0c]^2 + 2[(\gamma-1)m_0c]m_0c \end{cases}$$

$$(p_\lambda - p_{\lambda'})^2 + 2(p_\lambda - p_{\lambda'})m_0c = p^2 \qquad (2)$$

The Compton Scattering (ii)

For known X-ray frequency and final particle momentums
We can further solve these two equations.

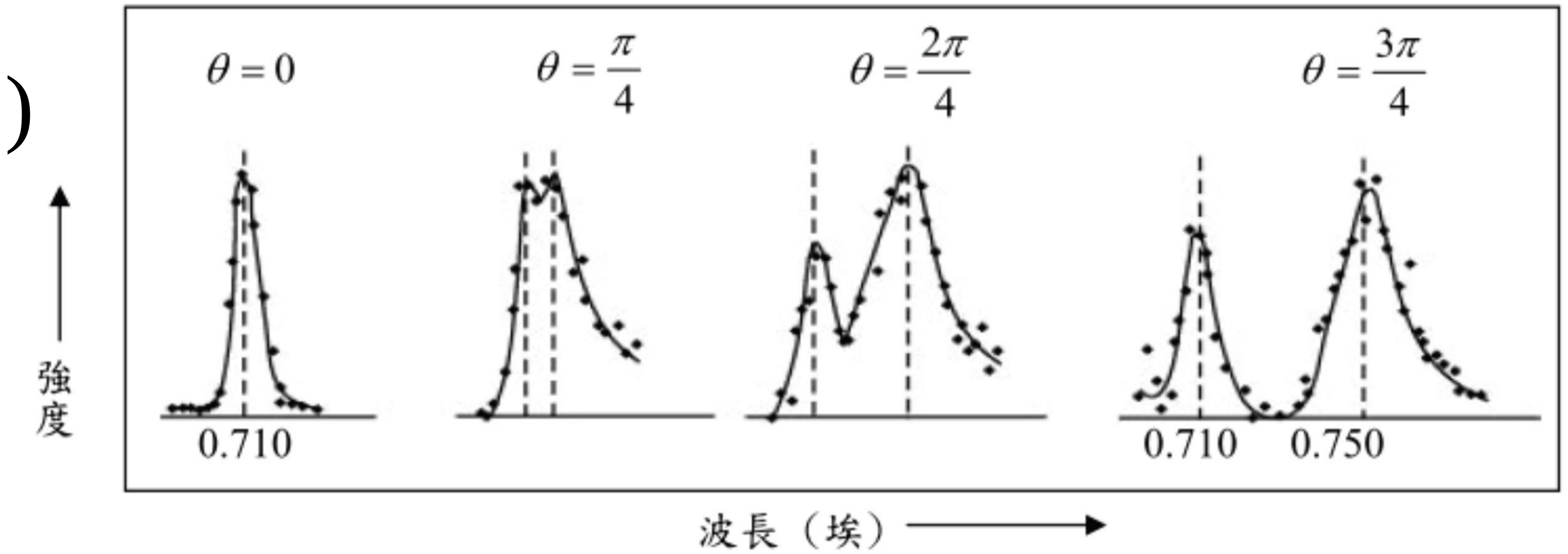
$$\left.\begin{aligned}(p_\lambda - p_{\lambda'}\cos\theta)^2 + (p_{\lambda'}\sin\theta)^2 = p^2 \\ (p_\lambda - p_{\lambda'})^2 + 2(p_\lambda - p_{\lambda'})m_0c = p^2\end{aligned}\right\}\ \begin{aligned}&\text{2 eqs., 3 unknown} \\ &p_\lambda,\ p_{\lambda'},\ \text{and } p\end{aligned}$$

Further solving these two equations, we obtain

$$(p_\lambda - p_{\lambda'})m_0c = p_\lambda p_{\lambda'}(1-\cos\theta)$$

$$\frac{1}{p_{\lambda'}} - \frac{1}{p_\lambda} = \frac{1}{m_0c}(1-\cos\theta)$$

$$\Rightarrow \Delta\lambda = \frac{h}{m_0c}(1-\cos\theta)$$



$$\frac{h}{m_0c} = 0.00243 \text{ nm is called the Compton wavelength.}$$

12.2.4 Relativistic Dynamics
Newton's laws

Newton's *first* law is built into the principle of relativity.

Newton's *second* law retains its validity in relativistic mechanics, provided we use the relativistic momentum.

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad \text{where } \mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} = m_{\text{rel}}\mathbf{u}$$

Newton's *third* law does not, in general, extend to the relativistic domain due to the relativity of simultaneity.

Only in the case of contact interactions, where the two forces are applied at the same physical point, can the third law be retained.

Work-Energy Theorem

The work-energy theorem ("the net work done on a particle equals the increase in its kinetic energy") holds relativistically.

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \text{ where } \mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} = m_{\text{rel}}\mathbf{u}$$

$$W \equiv \int \mathbf{F}\cdot d\mathbf{l} = \int \frac{d\mathbf{p}}{dt}\cdot d\mathbf{l} = \int \frac{d\mathbf{p}}{dt}\cdot\frac{d\mathbf{l}}{dt}dt = \int \frac{d}{dt}\left(\frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}\right)\cdot\mathbf{u}dt$$

$$\frac{d}{dt}\left(\frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}\right)\cdot\mathbf{u} = \frac{1}{1-u^2/c^2}\left(\sqrt{1-u^2/c^2}\,\frac{dm\mathbf{u}}{dt} - m\mathbf{u}\frac{1}{\sqrt{1-u^2/c^2}}\frac{-udu}{c^2dt}\right)\cdot\mathbf{u}$$

$$= \frac{1}{(1-u^2/c^2)^{\frac{3}{2}}}\left((1-\frac{u^2}{c^2})mu\frac{du}{dt} + \frac{u^2}{c^2}mu\frac{du}{dt}\right) = \frac{1}{(1-u^2/c^2)^{\frac{3}{2}}}\left(mu\frac{du}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right) = \frac{dE}{dt} \quad\Rightarrow\quad W = \int\frac{dE}{dt}dt = E_{\text{final}} - E_{\text{initial}}$$

The Ordinary Force and The Minkowski Force

The *ordinary* force: **F** is the derivative of momentum with respect to ordinary time, transformation is ugly (both the numerator and denominator must be transformed).

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma(dt - \frac{\beta}{c}dx)} = \frac{dp_y \big/ dt}{\gamma(1 - \frac{\beta}{c}u_x)} = \frac{F_y}{\gamma(1 - \frac{\beta}{c}u_x)}$$

$$\bar{F}_z = \frac{d\bar{p}_z}{d\bar{t}} = \frac{dp_z}{\gamma(dt - \frac{\beta}{c}dx)} = \frac{dp_z \big/ dt}{\gamma(1 - \frac{\beta}{c}u_x)} = \frac{F_z}{\gamma(1 - \frac{\beta}{c}u_x)}$$

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma(dp_x - \beta dp^0)}{\gamma(dt - \frac{\beta}{c}dx)} = \frac{\frac{dp_x}{dt} - \beta\frac{dp^0}{dt}}{(1 - \frac{\beta}{c}u_x)} = \frac{\frac{dp_x}{dt} - \frac{\beta}{c}\frac{dE}{dt}}{(1 - \frac{\beta}{c}u_x)}$$

The *Minkowski* force: **K** is the derivative of momentum with respect to proper time.

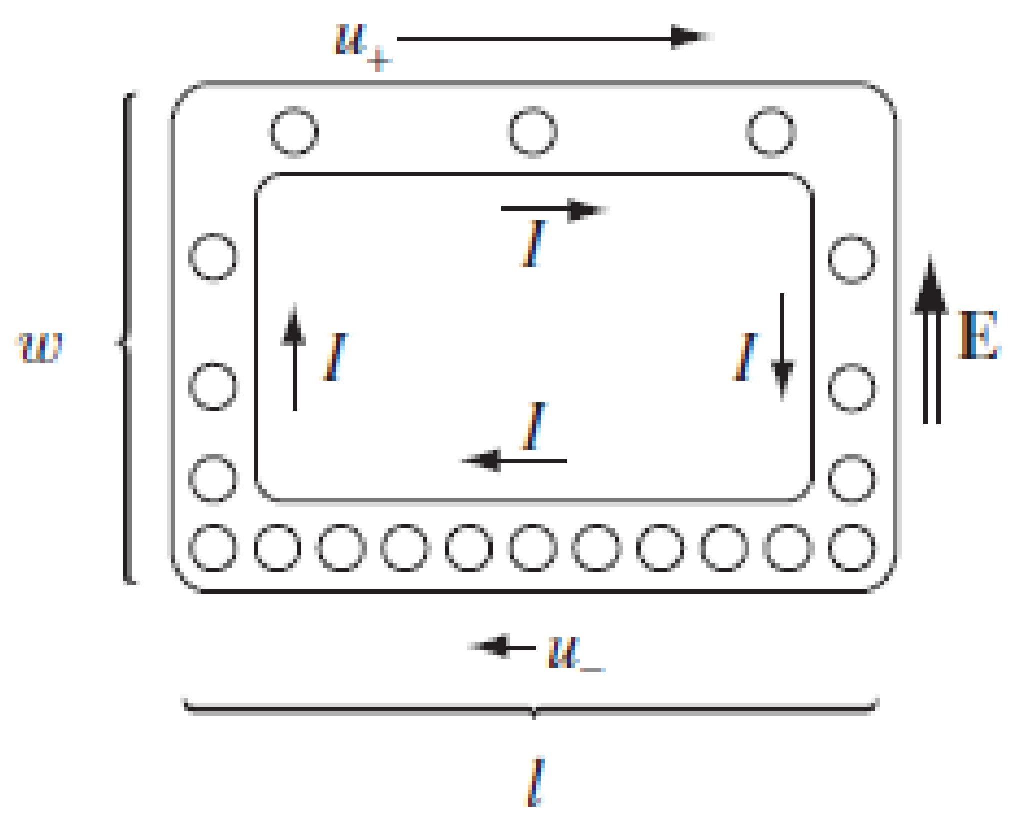
$$\mathbf{K} \equiv \frac{d\mathbf{p}}{d\tau} = \frac{dt}{d\tau}\frac{d\mathbf{p}}{dt} = \frac{\mathbf{F}}{\sqrt{1 - u^2 / c^2}}, \qquad K^\mu \equiv \frac{dp^\mu}{d\tau}$$

Example 12.12: Hidden momentum

As a model for a magnetic dipole $\mathbf{m}$, consider a rectangular loop of wire carrying a steady current. Picture the current as a stream of noninteracting positive charges that move freely within the wire. When a uniform electric field $\mathbf{E}$ is applied, the charges accelerate in the left segment and decelerate in the right one. Find the total momentum of all charges in the loop.

Solution:

The current is the same in all four segments $I = \lambda u$.



$$I = \frac{eN_+}{l}u_+ = \frac{eN_-}{l}u_- \quad \text{so} \quad N_\pm u_\pm = \frac{Il}{e}$$

Relativistic momentum is

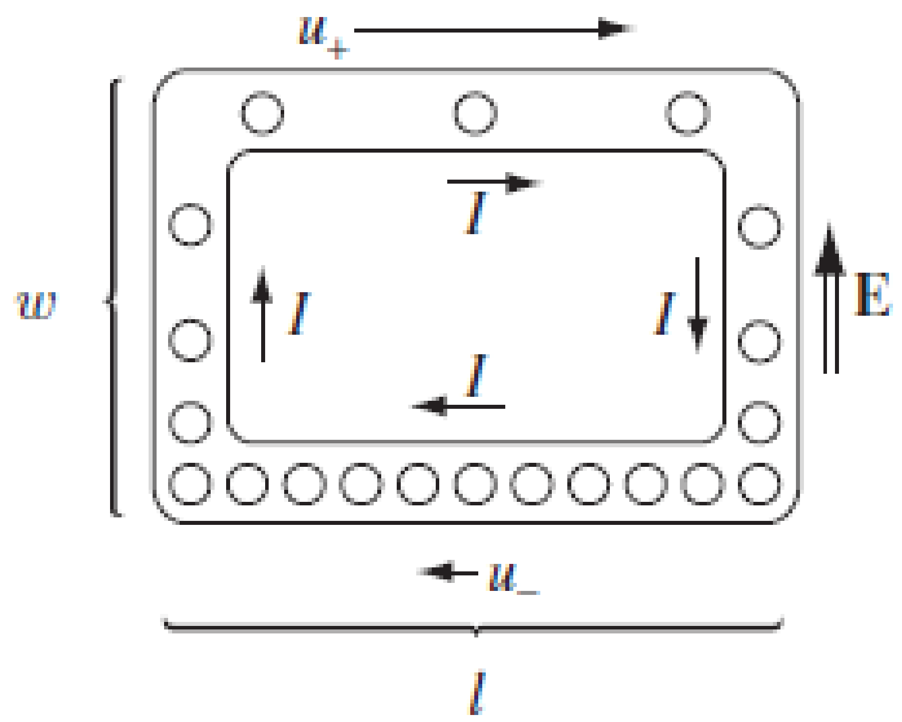
$$p = \gamma_+ m N_+ u_+ - \gamma_- m N_- u_- = (\gamma_+ - \gamma_-) m \frac{Il}{e} \neq 0$$

Hidden momentum (relativistic effect)

The gain in energy (γmc^2) is equal to the work done by the electric force $\mathbf{E}$.

$$p = (\gamma_+ - \gamma_-)m\frac{Il}{e}$$

$$(\gamma_+ - \gamma_-)mc^2 = eEw \quad \Rightarrow \quad p = \frac{IlEw}{c^2}$$



Ilw is the magnetic dipole moment of the loop as vectors $\mathbf{m}$ points into the page, and $\mathbf{p}$ is to the right, so

$$\mathbf{p} = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E})$$

A magnetic dipole in an electric field carries linear momentum, even though it is not moving.

This so-called *hidden momentum* is strictly relativistic, and purely mechanical. (See Ex. 8.3.)

A more realistic model for a current-carrying wire can be found in the supplement.

See V. Hnizdo, *Am. J. Phys*. **65**, 92 (1997).

12.3 Relativistic Electrodynamics
12.3.2 How the Fields Transform

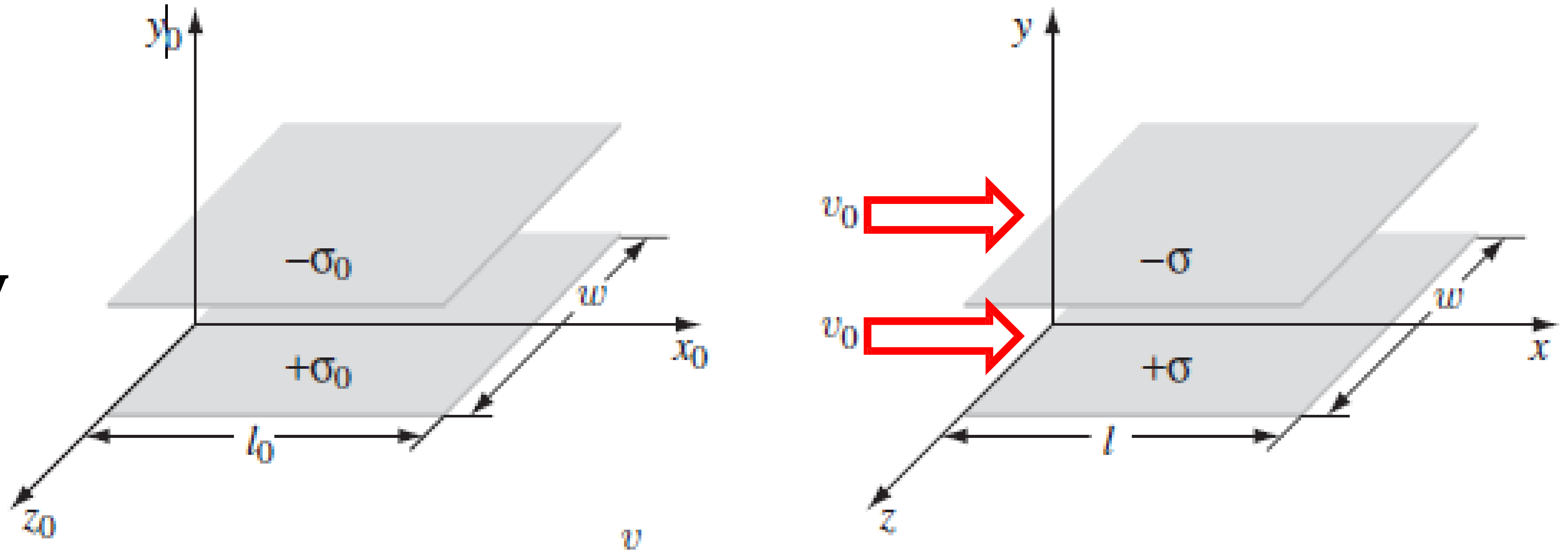
We have learned, in various special cases, that
one observer's electric field is another's magnetic field.

What are the general transformation rules for electromagnetic fields?

Let's start with "***Charge invariant***".

Consider the simplest possible electric field.

$$\mathbf{E}_0 = \frac{\sigma_0}{\varepsilon_0}\hat{\mathbf{y}}$$

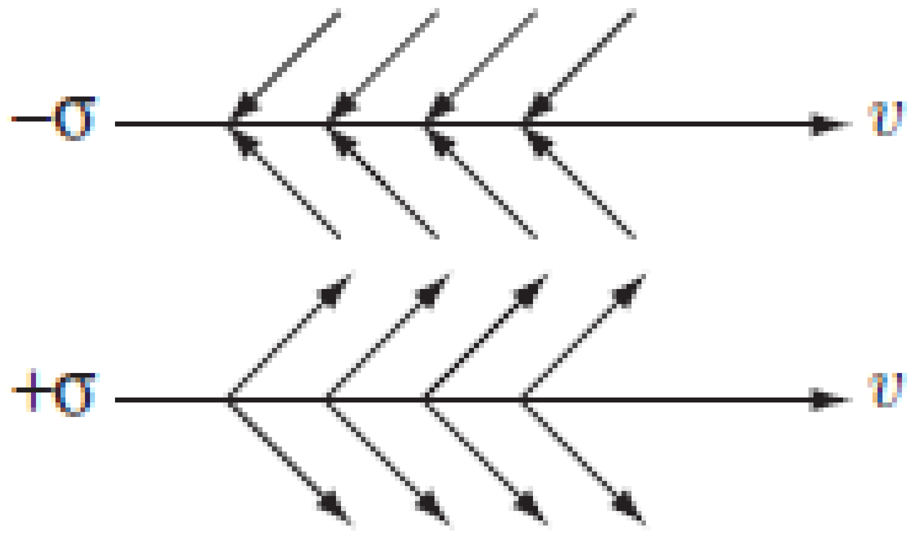


(a) (b)

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{y}}$$

The Transformation of The Electric Field

Are you sure that the field is still perpendicular to the plates? Yes.



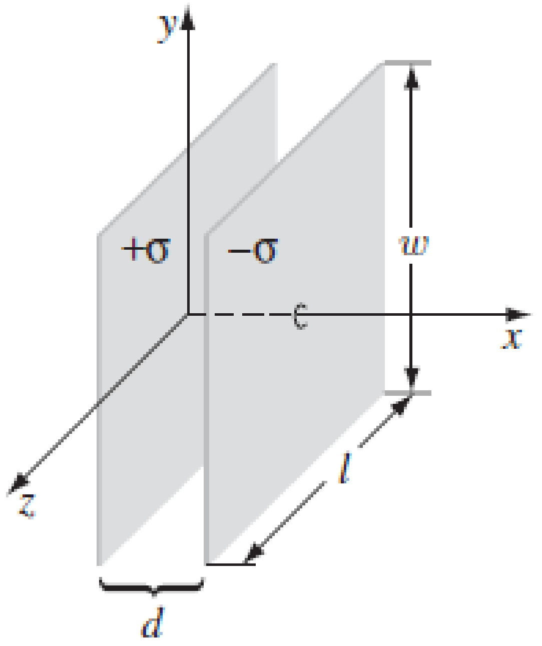
The total charge on each plate is invariant.

$$Q = \sigma_0 l_0 w_0 = \sigma l w \quad \text{where } l = \sqrt{1 - v_0^2 / c^2}\, l_0 \text{ and } w = w_0$$

$$\sigma = \frac{1}{\sqrt{1 - v_0^2 / c^2}} \sigma_0 = \gamma_0 \sigma_0 \quad \Rightarrow \quad \mathbf{E}^{\perp} = \gamma_0 \mathbf{E}_0^{\perp}$$

← perpendicular components

What if the field of a moving plane tilted, say, in the direction of motion?



$$Q = \sigma_0 l_0 w_0 = \sigma l w,$$

where $l = l_0$ and $w = w_0$

$$\sigma = \sigma_0 \quad \Rightarrow \quad \mathbf{E}^{//} = \mathbf{E}_0^{//}$$

← parallel components

Example 12.13: The E-field of a moving point charge.

A point charge q is at rest at the origin in system S_0. Question: What is the electric field of this same charge in system S, which moves to the right at speed v_0 relative to S_0?

Solution:

$$\mathbf{E}_0 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_0^2}\hat{\mathbf{r}}_0$$

$$\begin{cases} E_{x0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qx_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} \\ E_{y0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qy_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} \\ E_{z0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qz_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} \end{cases} \Rightarrow \begin{cases} E_x = E_{x0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qx_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} \\ E_y = \gamma_0 E_{y0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\gamma_0 qy_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} \\ E_z = \gamma_0 E_{z0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\gamma_0 qz_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} \end{cases}$$

Very efficient as compared with Chap.10 Eq. 10.68. (10-40)

The Transformation of The Magnetic Field

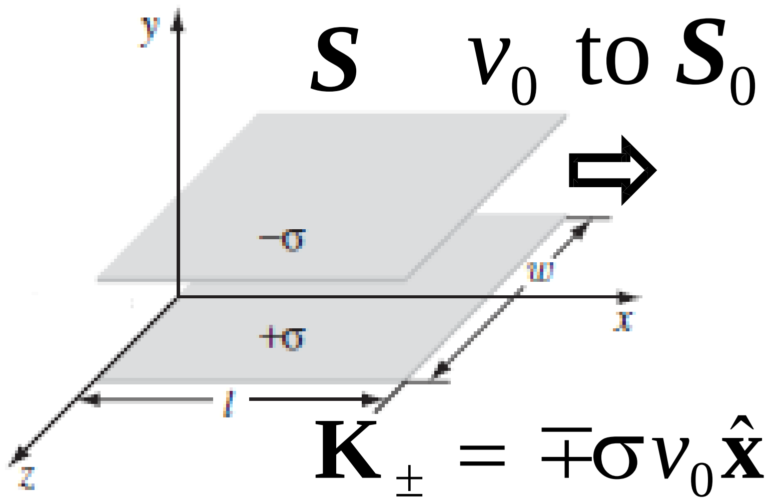
To derive the general rule we must start out in a system with both electric and magnetic fields.

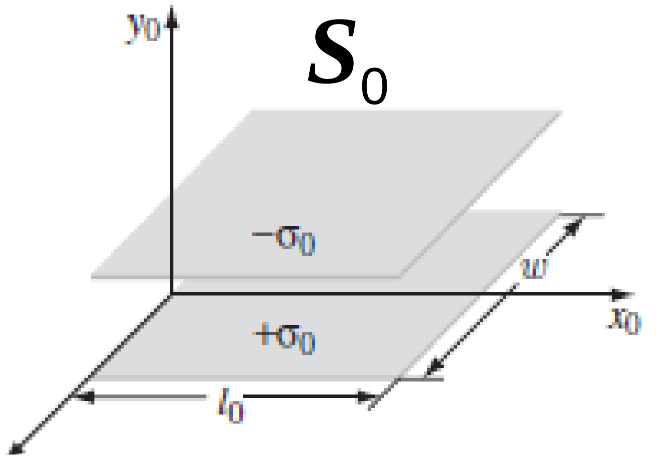
$$E_y = \frac{\sigma}{\varepsilon_0} \quad \text{and} \quad B_z = -\mu_0 \sigma v_0$$

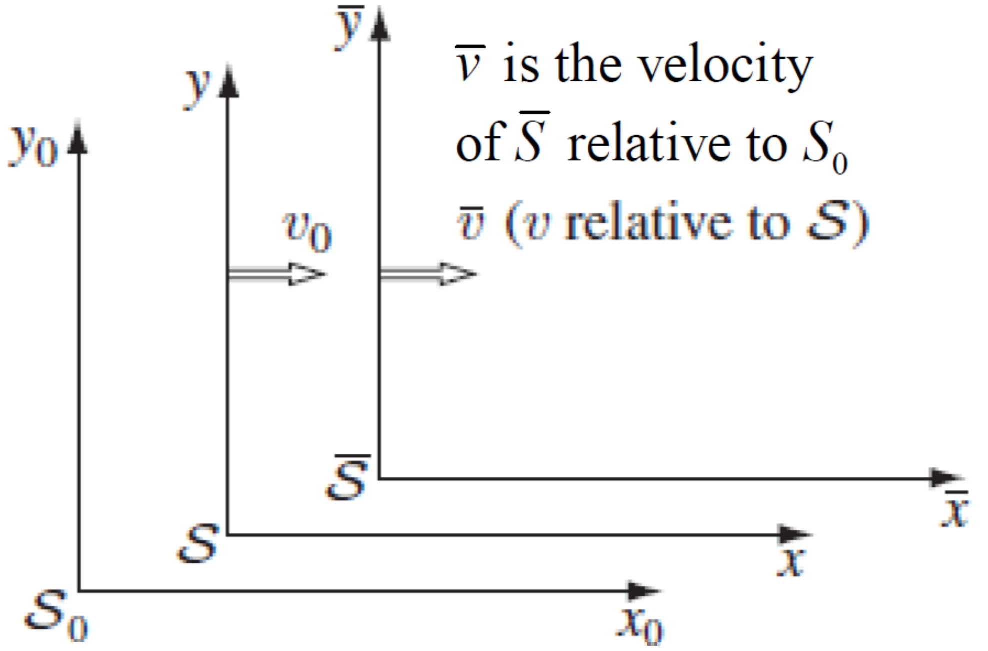
In a third system, $\bar{S}$, traveling to the right with speed v relative to S, the field would be

$$\bar{E}_y = \frac{\bar{\sigma}}{\varepsilon_0} \quad \text{and} \quad \bar{B}_z = -\mu_0 \overline{\sigma v}$$

$$\bar{v} = \frac{v + v_0}{1 + v v_0 / c^2}, \quad \bar{\gamma} = \frac{1}{\sqrt{1 - \bar{v}^2 / c^2}}, \quad \bar{\sigma} = \bar{\gamma} \sigma_0$$

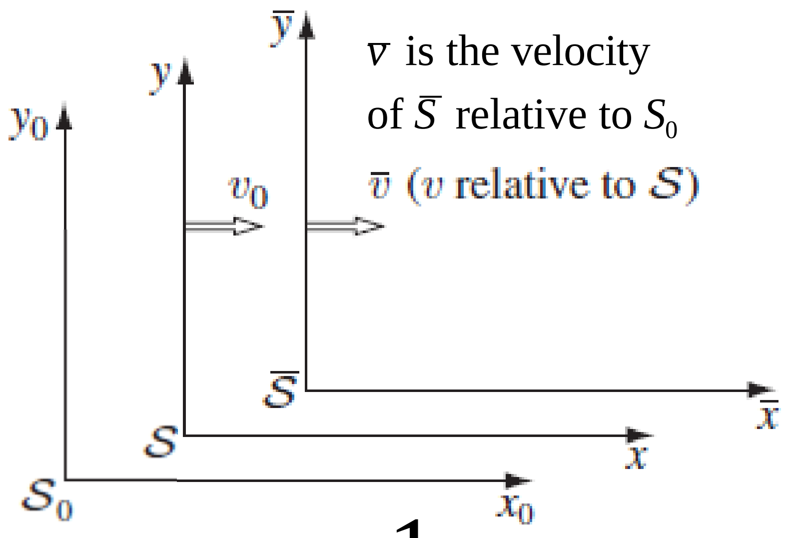






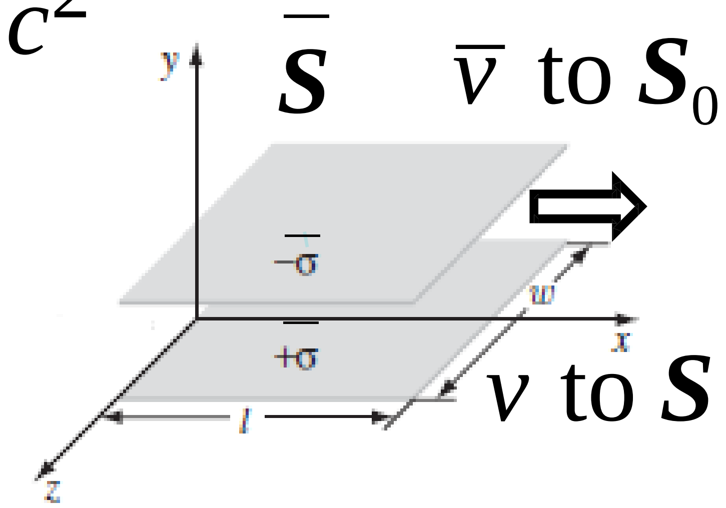
How to express $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ in terms of $\mathbf{E}$ and $\mathbf{B}$?

Contd.



$$\bar E_y = \frac{\bar\gamma \sigma_0}{\varepsilon_0} = \left(\frac{\bar\gamma}{\gamma_0}\right)\frac{\sigma}{\varepsilon_0},$$

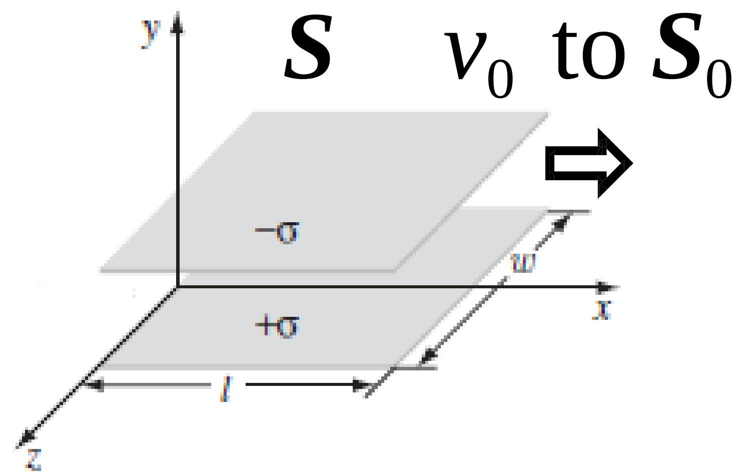
where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$, $\bar\gamma = \dfrac{1}{\sqrt{1-\bar v^2/c^2}}$, $\gamma_0 = \dfrac{1}{\sqrt{1-v_0^2/c^2}}$



With a little algebra.

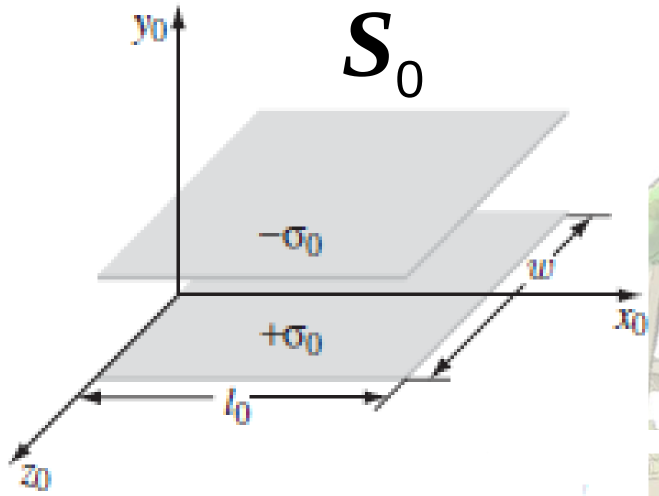
⇓

$$\frac{\bar\gamma}{\gamma_0} = \frac{\sqrt{1-v_0^2/c^2}}{\sqrt{1-\bar v^2/c^2}} = \gamma\left(1+\frac{v v_0}{c^2}\right) \text{ and } c^2 = \frac{1}{\varepsilon_0 \mu_0},$$

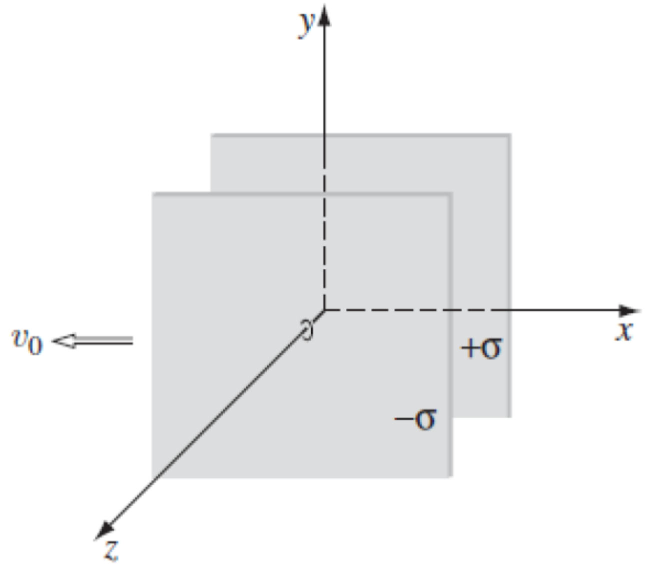


$$\Rightarrow \bar E_y = \gamma(E_y - vB_z)$$

$$\Rightarrow \bar B_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

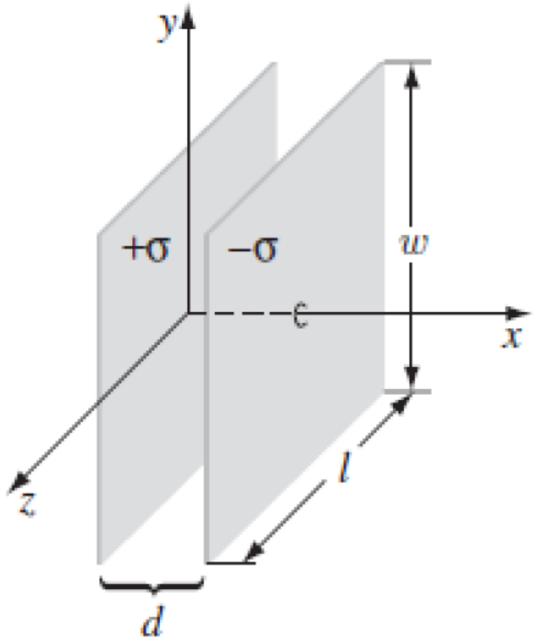


The Transformation of The Magnetic Field



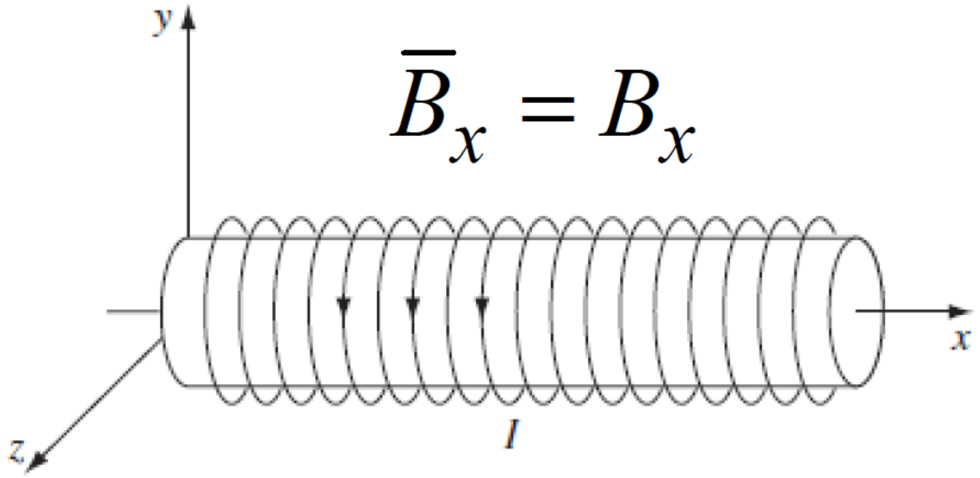
$$\Rightarrow \quad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\Rightarrow \quad \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z)$$



$$\bar{E}_x = E_x$$

$$\bar{B}_x = B_x$$



$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \quad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z), \quad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

Two Special Cases

$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \qquad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z), \quad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

1. If $\mathbf{B} = 0$ in S, then

$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma E_y, \qquad \bar{E}_z = \gamma E_z$$

$$\bar{B}_x = 0, \quad \bar{B}_y = \gamma\frac{v}{c^2}E_z, \quad \bar{B}_z = -\gamma\frac{v}{c^2}E_y$$

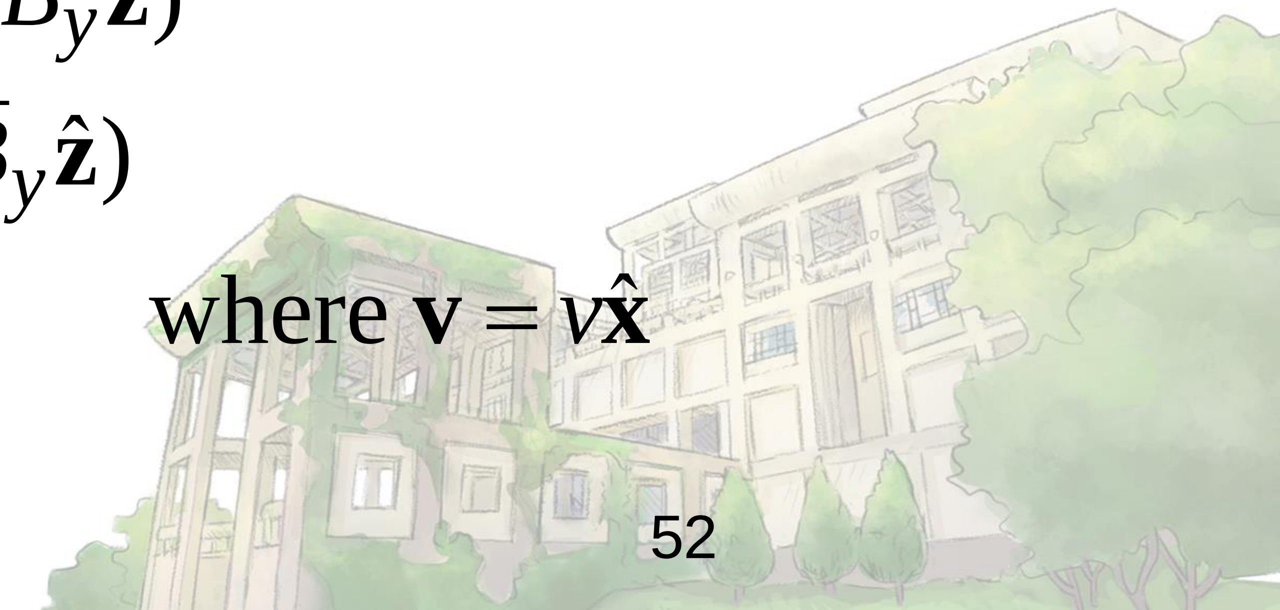
$$\bar{\mathbf{B}} = \frac{v}{c^2}\gamma E_z\hat{\mathbf{y}} - \frac{v}{c^2}\gamma E_y\hat{\mathbf{z}} = \frac{v}{c^2}\bar{E}_z\hat{\mathbf{y}} - \frac{v}{c^2}\bar{E}_y\hat{\mathbf{z}} = -\frac{1}{c^2}(\mathbf{v}\times\bar{\mathbf{E}}) \qquad \text{where } \mathbf{v} = v\hat{\mathbf{x}}$$

2. If $\mathbf{E} = 0$ in S, then

$$\bar{E}_x = 0, \quad \bar{E}_y = -\gamma vB_z, \quad \bar{E}_z = \gamma vB_y$$

$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma B_y, \qquad \bar{B}_z = \gamma B_z$$

$$\bar{\mathbf{E}} = -\gamma v(B_z\hat{\mathbf{y}} - B_y\hat{\mathbf{z}}) = -v(\bar{B}_z\hat{\mathbf{y}} - \bar{B}_y\hat{\mathbf{z}}) = \mathbf{v}\times\bar{\mathbf{B}} \qquad \text{where } \mathbf{v} = v\hat{\mathbf{x}}$$



12.3.3 The Field Tensor

E and **B** certainly do not transform like the spatial parts of the two 4-vectors (4-velocity and 4-momentum).

What sort of an object is this, which has six components and transforms according previous results?

Answer: Antisymmetric, second-rank tensor.

$$t^{\mu\nu} = \begin{Bmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{Bmatrix}$$

$$t^{\mu\nu} = t^{\nu\mu}$$ (symmetric tensor, 10 distinct components)

$$t^{\mu\nu} = -t^{\nu\mu}$$ (antisymmetric tensor, 6 distinct components)

$$t^{\mu\nu} = \begin{Bmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{Bmatrix}$$

The Tensor Transformation

$$\bar{a}^{\nu} = \Lambda^{\nu}_{\lambda} a^{\lambda} \qquad \text{4-vector transformation}$$

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma} t^{\lambda\sigma} \qquad \text{tensor transformation}$$

$$\Lambda = \begin{Bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

Work out the following transformation:

$$\bar{t}^{01} = t^{01} \quad \bar{t}^{02} = \gamma(t^{02} - \beta t^{12}) \quad \bar{t}^{03} = \gamma(t^{03} + \beta t^{31})$$

$$\bar{t}^{23} = t^{23} \quad \bar{t}^{31} = \gamma(t^{31} + \beta t^{03}) \quad \bar{t}^{12} = \gamma(t^{12} - \beta t^{02})$$

By direct comparison, we find:

$$\bar{E}_x = E_x \quad \bar{E}_y = \gamma(E_y - vB_z) \quad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x \quad \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z) \quad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

Now we can construct the **field tensor** $F^{\mu\nu}$

The Field Tensor and The Dual Tensor

$$F^{01} \equiv \frac{E_x}{c},\; F^{02} \equiv \frac{E_y}{c},\; F^{03} \equiv \frac{E_z}{c},\; F^{12} \equiv B_z,\; F^{31} \equiv B_y,\; F^{23} \equiv B_x.$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$ the **field tensor**

There was a different way of imbedding **E** and **B** in an antisymmetric tensor.

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$ the **dual tensor**

12.3.4 Electrodynamics in Tensor Notation

Reformulate the laws of electrodynamics (Maxwell's equations and the Lorentz force law) in relativistic notation.

How the sources of the fields, ρ and $\mathbf{J}$, transform?

$$\rho = \frac{Q}{V} \quad \text{and} \quad \mathbf{J} = \rho\mathbf{u}, \quad \text{where } \rho_0 = \frac{Q}{V_0} \text{ (the proper charge density)}$$

$$\rho = \rho_0 \frac{V_0}{V} = \gamma\rho_0, \quad \text{where } V = \sqrt{1-u^2/c^2}V_0 \text{ (length contraction)}$$

$$\mathbf{J} = \rho\mathbf{u} = \gamma\rho_0\mathbf{u} = \rho_0(\gamma\mathbf{u}) = \rho_0\boldsymbol{\eta}, \quad \text{where } \boldsymbol{\eta} = \gamma\mathbf{u} \text{ (proper velocity)}$$

The current density 4-vector: $J^\mu = (c\rho,\ J_x,\ J_y,\ J_z)$

$$x^\mu = (ct,\ x,\ y,\ z)$$

Conservation of charge:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \sum_{i=0}^{3} \frac{\partial J^i}{\partial x^i}$$

$$-\frac{\partial \rho}{\partial t} = -\frac{\partial (c\rho)}{\partial (ct)} = -\frac{\partial J^0}{\partial x^0}$$

$$\frac{\partial J^\mu}{\partial x^\mu} = 0$$

Maxwell's Equations in Tensor Notation (i)

Maxwell's equations can be written in the following forms.

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^u$$

Gauss's law
Ampere's law with Maxwell's correction

$$\mu = 0 \quad \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} = \mu_0 J^0$$

$$\frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \mu_0 c\rho \quad \Rightarrow \quad \underline{\underline{\nabla \cdot \mathrm{E} = \frac{\rho}{\varepsilon_0}}}$$

$$\mu = 1 \quad \frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} = \mu_0 J^1$$

$$-\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x \quad \rightarrow \quad \left(-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right)_x = \mu_0 (\mathbf{J})_x$$

$$+\mu = 2 \text{ and } 3 \quad \left(-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right)_{y,z} = \mu_0 \mathbf{J}_{y,z} \quad \Rightarrow \underline{\underline{\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}}}$$

Maxwell's Equations in Tensor Notation (ii)

Maxwell's equations can be written in the following forms.

$$\frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0$$

Gauss's law for magnetic field
Faraday's law

$$\mu = 0 \quad \frac{\partial G^{0\nu}}{\partial x^{\nu}} = \frac{\partial G^{00}}{\partial x^{0}} + \frac{\partial G^{01}}{\partial x^{1}} + \frac{\partial G^{02}}{\partial x^{2}} + \frac{\partial G^{03}}{\partial x^{3}} = 0$$

$$\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) = 0 \qquad \Rightarrow \qquad \underline{\underline{\nabla \cdot \mathbf{B} = 0}}$$

$$\mu = 1 \quad \frac{\partial G^{1\nu}}{\partial x^{\nu}} = \frac{\partial G^{10}}{\partial x^{0}} + \frac{\partial G^{11}}{\partial x^{1}} + \frac{\partial G^{12}}{\partial x^{2}} + \frac{\partial G^{13}}{\partial x^{3}} = 0$$

$$-\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z} = 0 \quad \rightarrow \quad \left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right)_x = 0$$

$$+\mu = 2 \text{ and } 3 \qquad \left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right)_{y,z} = 0 \quad \Rightarrow \quad \underline{\underline{\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0}}$$

$$\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial F_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial F_{\nu\lambda}}{\partial x^{\mu}} = 0 \;(\lambda, \mu, \nu = 0-3) \qquad (11.143)$$

set $(\lambda,\mu,\nu) = (1,2,3) \Rightarrow \nabla \cdot \mathbf{B} = 0$

set $(\lambda,\mu,\nu) = (0,1,2),\ (0,1,3),\ \text{and}\ (0,2,3) \Rightarrow \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0.$

Jackson

The Minkowski Force and Relativistic Potentials

The Minkowski force on a charge q is given by

$$\mathbf{K} = \frac{1}{\sqrt{1-u^2/c^2}}\, q[\mathbf{E} + (\mathbf{u}\times\mathbf{B})] = \frac{1}{\sqrt{1-u^2/c^2}}\,\mathbf{F}$$

The electric and magnetic fields can be expressed in terms of a scalar potential and a vector potential.

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

$$A^{\mu} = (V/c, A_x, A_y, A_z) \quad \text{4-vector potential}$$

$$F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} \quad \text{the definition of the field tensor}$$

$$\frac{\partial A^{\mu}}{\partial x^{\mu}} = 0 \quad \text{the Lorentz gauge}$$

Homework of Chap.12

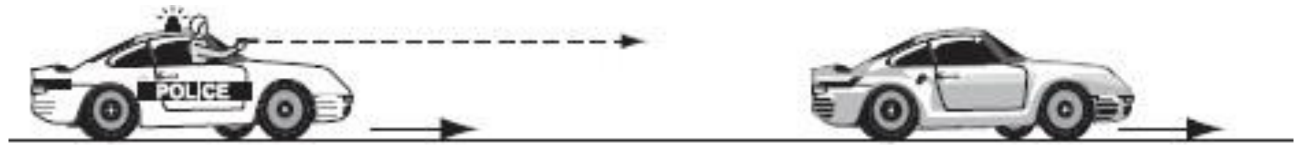
Problem 12.3

(a) What's the percent error introduced when you use Galileo's rule, instead of Einstein's, with $v_{AB} = 5$ mi/h and $v_{BC} = 60$ mi/h?

(b) Suppose you could run at half the speed of light down the corridor of a train going three-quarters the speed of light. What would your speed be relative to the ground?

(c) Prove, using Eq. 12.3, that if $v_{AB} < c$ and $v_{BC} < c$ then $v_{AC} < c$. Interpret this result.

Problem 12.4 As the outlaws escape in their getaway car, which goes $3c/4$, the police officer fires a bullet from the pursuit car, which only goes $c/2$ (Fig. 12.3). The muzzle velocity of the bullet (relative to the gun) is $c/3$. Does the bullet reach its target (a) according to Galileo, (b) according to Einstein?



Problem 12.6 Every 2 years, more or less, *The New York Times* publishes an article in which some astronomer claims to have found an object traveling faster that the speed of light. Many of these reports result from a failure to distinguish what is *seen* from what is *observed*-that is, from a failure to account for light travel time. Here's an example: A star is traveling with speed v at an angle θ to the line of sight (Fin.12.6). What is its apparent speed across the sky? (Suppose the light signal from b reaches the earth at a time Δt after the signal from a, and the star has meanwhile advanced a distance Δs across the celestial sphere; by "apparent speed," I mean $\Delta s / \Delta t$.) What angle θ gives the maximum apparent speed? Show that the apparent speed can be much greater than c, even if v itself is less than c.

Homework of Chap.12

Problem 12.25 A car is traveling along the 45° line in S (Fig.12.25), at (ordinary) speed $(2/\sqrt{5})c$.

(a) Find the components u_x and u_y of the (ordinary) velocity.

(b) Find the components η_x and η_y of the proper velocity.

(c) Find the zeroth components of the 4-velocity, η^0.

System $\bar{S}$ is moving in the x direction with (ordinary) speed $\sqrt{2/5}\,c$, relative to S. By using the appropriate transformation laws:

Problem 12.31 Suppose you have a collection of particles, all moving in the x direction, with energies E_1, E_2, E_3,...and momenta p_1, p_2, p_3,...Find the velocity of the **center of momentum** frame, in which the total momentum is zero.

Problem 12.34 A neutral pion of (rest) mass m and (relativistic) momentum $p = \frac{3}{4}mc$ decays into two photons. One of the photons is emitted in the same direction as the original pion, and the other in the opposite direction. Find the (relativistic) energy of each photon.

Homework of Chap.12

Problem 12.39 Define **proper acceleration** in he obvious way:

$$\alpha^{\mu} \equiv \frac{d\eta^{\mu}}{d\tau} = \frac{d^2 x^{\mu}}{d\tau^2}. \qquad (12.75)$$

(a) Find α^0 and $\boldsymbol{\alpha}$ in terms of **u** and **a** (the ordinary acceleration).

(b) Express $\alpha_{\mu}\alpha^{\mu}$ in terms of **u** and **a** .

(c) Show that $\eta^{\mu}\alpha_{\mu}$=0.

(d) Write the Minkowski version of Newton's second law, Eq. 12.68, in terms of α^{μ}. Evaluate the invariant product $K^{\mu}\eta_{\mu}$.

Problem 12.47

(a) Show that $(\mathbf{E}\cdot\mathbf{B})$ is relativistically invariant.

(b) Show that $(E^2 - c^2B^2)$ is relativistically invariant.

(c) Suppose that in one inertial system $\mathbf{B} = \mathbf{0}$ but $\mathbf{E} \neq \mathbf{0}$ (at some point *P*). Is it possible to find another sysem in which the *electric* field is zero at *P*?