Chapter 11: Radiation

11.1 Dipole Radiation 11.1.1 What is Radiation?

A charge at rest does not generate electromagnetic wave; nor does a steady current. It takes *accelerating charges*, and/or *changing currents*.
The purpose of this chapter is to show you how such configurations produce electromagnetic wave.

How charges radiate? Consider Jefimenko's equations.

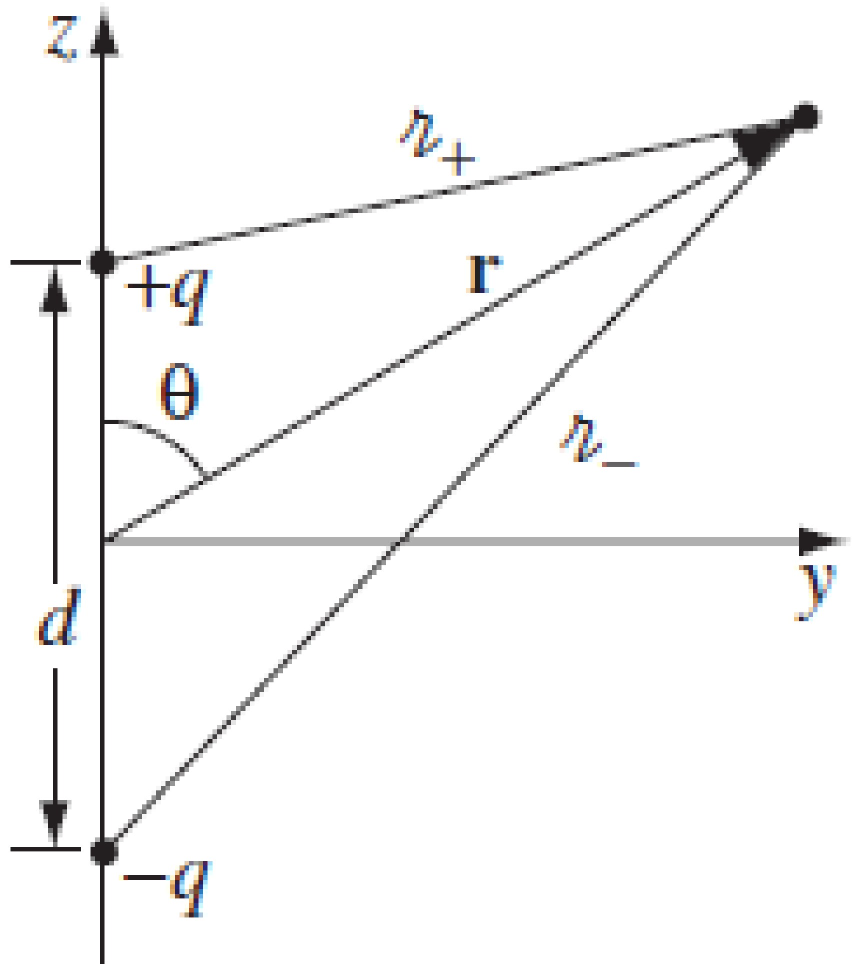
$$\begin{cases} \mathbf{E} = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int [\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2} + \frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}} - \frac{\dot{\mathbf{J}}}{c^2\mathscr{r}}] d\tau' \\ \mathbf{B} = \dfrac{\mu_0}{4\pi}\displaystyle\int [\frac{\mathbf{J}}{\mathscr{r}^2} + \frac{1}{c\mathscr{r}}\dot{\mathbf{J}}] \times \hat{\mathscr{r}} d\tau' \end{cases}$$

We won't use these two equations. Instead, we start from finding the vector and scalar potentials first.

$\dot{\rho}$ and $\dot{\mathbf{J}}$ are responsible for electromagnetic radiation (i.e., EM field at large distance).

11.1.2 Electric Dipole Radiation

Picture two tiny metal spheres separated by a distance d and connected by a fine wire. At time t the charge on the upper sphere is $+q(t)$, and the charge on the lower sphere is $-q(t)$. Suppose that $q(t) = q_0 \cos(\omega t)$



The result is an oscillating electric dipole:

$$\mathbf{p}(t) = q(t)d\hat{\mathbf{z}} = q_0 d\cos(\omega t)\hat{\mathbf{z}} = p_0\cos(\omega t)\hat{\mathbf{z}}, \quad \text{where } p_0 \equiv q_0 d.$$

The retarded potential is:

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{\frac{q_0\cos[\omega(t-\mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0\cos[\omega(t-\mathscr{r}_-/c)]}{\mathscr{r}_-}\right\}$$

Electric Dipole Radiation: Approximations

Approximation #1: Make this physical dipole into a perfect dipole. $d \ll r$

Estimate the spearation distances by the law of cosines.

$$\mathscr{r}_{\pm} = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2} \cong r(1 \mp \frac{d}{2r}\cos\theta)$$

$$\frac{1}{\mathscr{r}_{\pm}} \cong \frac{1}{r}(1 \pm \frac{d}{2r}\cos\theta)$$

$$\cos[\omega(t - \mathscr{r}_{\pm}/c)] \cong \cos[\omega(t - \frac{r}{c}) \pm \frac{\omega d}{2c}\cos\theta]$$

$$= \cos[\omega(t - \frac{r}{c})]\cos(\frac{\omega d}{2c}\cos\theta) \mp \sin[\omega(t - \frac{r}{c})]\sin(\frac{\omega d}{2c}\cos\theta)$$

The Retarded Scalar Potential

Approximation #2: The wavelength is much longer than the dipole size. $d \ll \frac{c}{\omega} = \frac{\lambda}{2\pi}$

$$\cos[\omega(t - \mathscr{r}_{\pm}/c)] \cong \cos[\omega(t-\frac{r}{c})]\overbrace{\cos(\frac{\omega d}{2c}\cos\theta)}^{\approx 1} \mp \sin[\omega(t-\frac{r}{c})]\overbrace{\sin(\frac{\omega d}{2c}\cos\theta)}^{\approx \frac{\omega d}{2c}\cos\theta}$$

$$= \cos[\omega(t-\frac{r}{c})] \mp \sin[\omega(t-\frac{r}{c})]\frac{\omega d}{2c}\cos\theta$$

The retarded scalar potential is:

$$V(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\left\{ \begin{array}{l} \left[\cos[\omega(t-\frac{r}{c})] - \sin[\omega(t-\frac{r}{c})]\frac{\omega d}{2c}\cos\theta\right]\frac{1}{r}(1+\frac{d}{2r}\cos\theta) \\ -\left[\cos[\omega(t-\frac{r}{c})] + \sin[\omega(t-\frac{r}{c})]\frac{\omega d}{2c}\cos\theta\right]\frac{1}{r}(1-\frac{d}{2r}\cos\theta) \end{array} \right\}$$

$$\cong \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c})] + \frac{1}{r}\cos[\omega(t-\frac{r}{c})]\right]$$

The Retarded Scalar Potential

Approximation #3: at the radiation zone. $\frac{\omega}{c} \gg \frac{1}{r}$ or $r \gg \lambda$

The retarded scalar potential is:

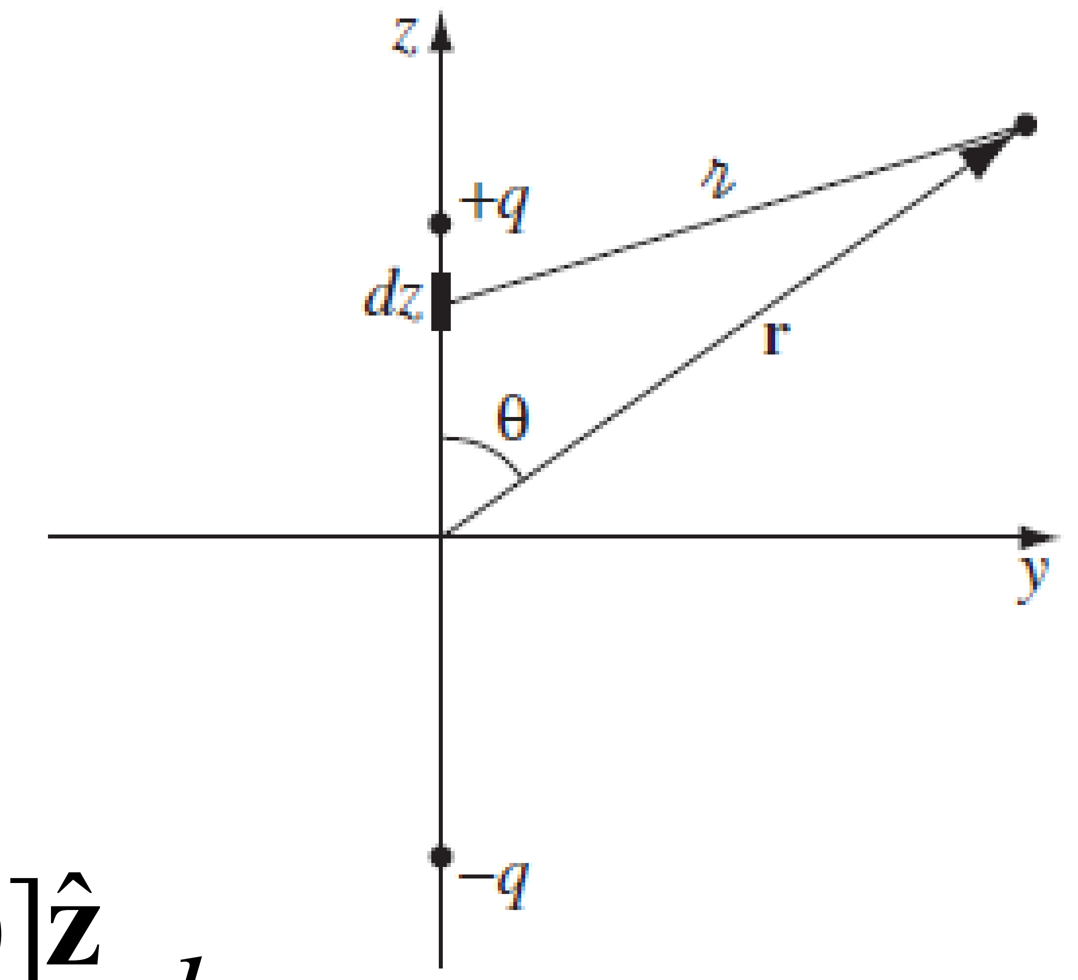
$$V(\mathbf{r},t) \cong \frac{p_0 \cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c})\right]$$

Three approximations

$$d \ll r \qquad d \ll \frac{c}{\omega}(=\frac{\lambda}{2\pi}) \qquad \frac{\omega}{c} \gg \frac{1}{r}$$

$$\Rightarrow d \ll \lambda \ll r$$

The Retarded Vector Potential



The retarded vector potential is determined by the current density.

$$I(t) = \frac{dq}{dt}\hat{\mathbf{z}} = -q_0\omega \sin\omega t\hat{\mathbf{z}}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}} d\tau' = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{-q\omega \sin[\omega(t - \mathscr{r}/c)]\hat{\mathbf{z}}}{\mathscr{r}} dz$$

$$\cong -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t - \frac{r}{c})]\hat{\mathbf{z}} \qquad @\, d \ll \lambda \ll r$$

Retarded potentials:

$$\begin{cases} V(\mathbf{r},t) = -\frac{p_0\omega}{4\pi\varepsilon_0 c}\frac{\cos\theta}{r}\sin[\omega(t - \frac{r}{c})] \\ \mathbf{A}(\mathbf{r},t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t - \frac{r}{c})]\hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} = \nabla \times \mathbf{A} \end{cases}$$

The Electromagnetic Fields and Poynting Vector

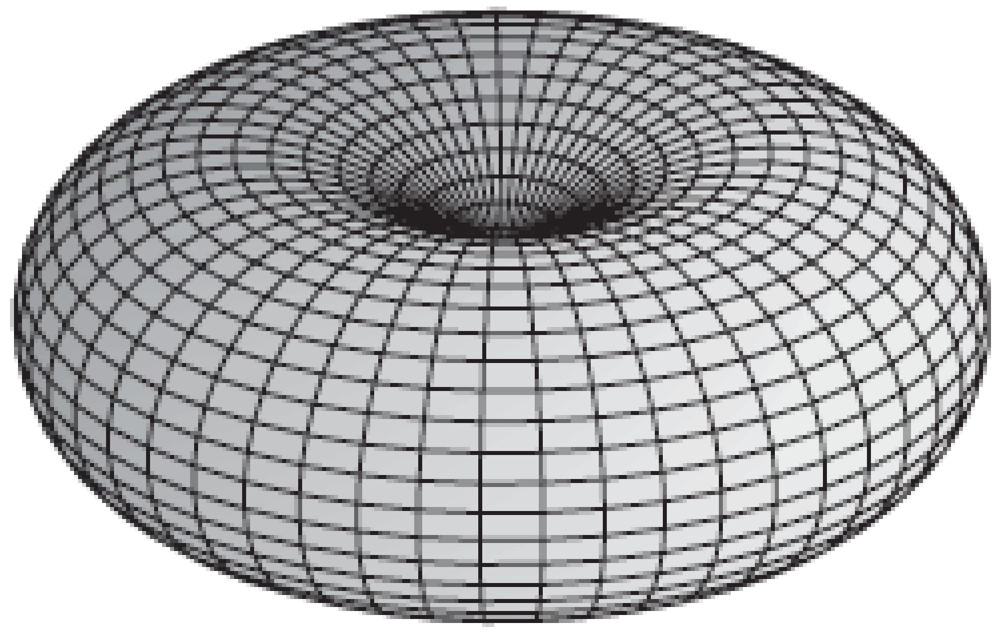
$$\begin{cases} \mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t} = -\dfrac{\mu_0 p_0 \omega^2}{4\pi\varepsilon_0 c}\left(\dfrac{\sin\theta}{r}\right)\cos[\omega(t-\dfrac{r}{c})]\hat{\theta} \\ \mathbf{B} = \nabla \times \mathbf{A} = -\dfrac{\mu_0 p_0 \omega^2}{4\pi c}\left(\dfrac{\sin\theta}{r}\right)\cos[\omega(t-\dfrac{r}{c})]\hat{\boldsymbol{\phi}} \end{cases}$$

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\frac{r}{c})]\right\}^2 \hat{\mathbf{r}}$$

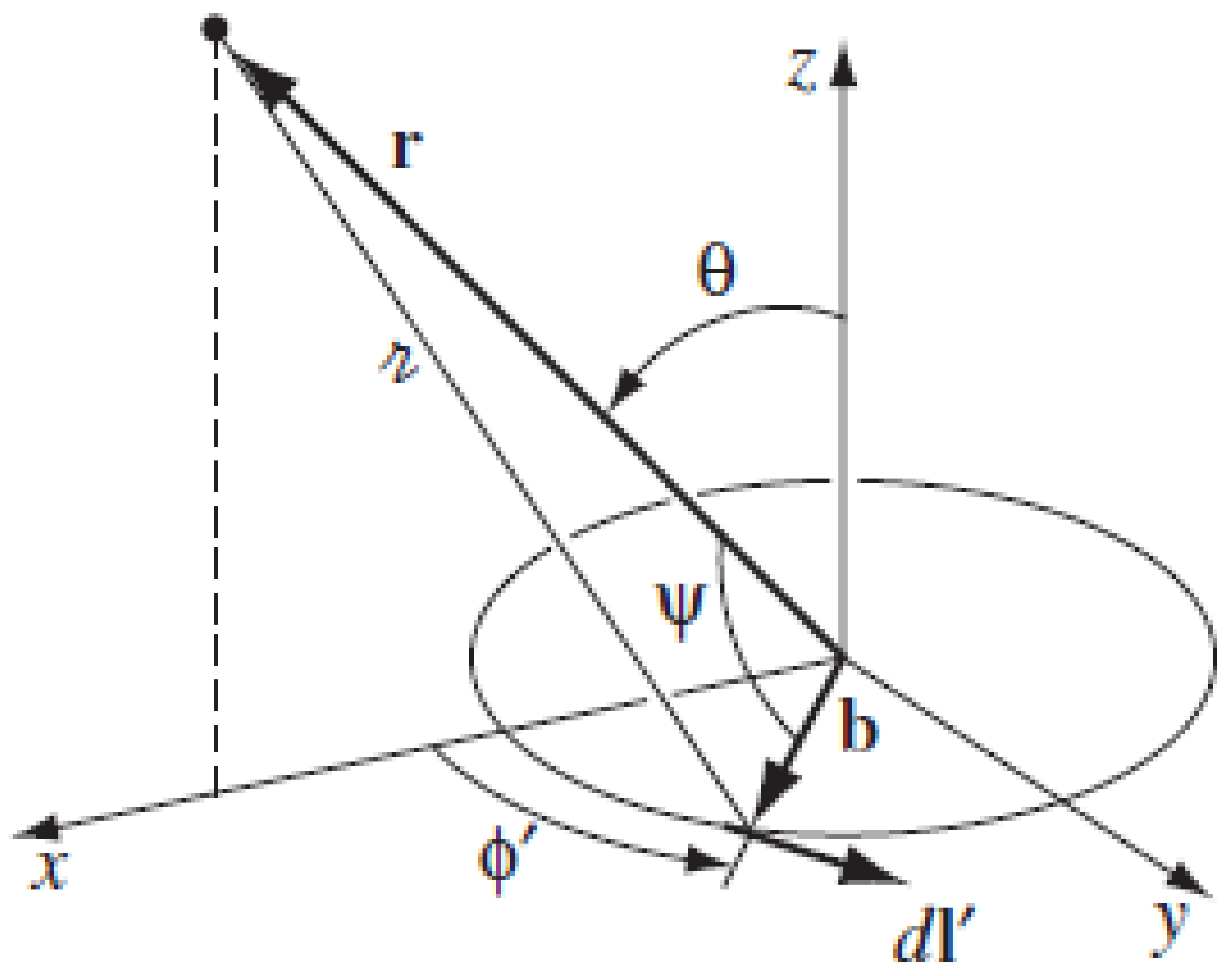
The total power radiated is

$$<P> = \int \langle \mathbf{S} \rangle \cdot d\mathbf{a} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\int \left(\frac{\sin\theta}{r}\right)^2 r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$



11.1.3 Magnetic Dipole Radiation



Suppose we have a loop of radius b, around which we drive an alternating current.

$$I(t) = I_0 \cos \omega t$$

This is a model for an oscillating magnetic dipole,

$$\mathbf{m}(t) = \pi b^2 I(t)\hat{\mathbf{z}} = m_0 \cos \omega t \hat{\mathbf{z}}$$

The loop is uncharged, so the retarded scalar potential is zero. $V = 0$

The retarded vector potential

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}} d\tau' = \frac{\mu_0}{4\pi}\int \frac{I_0 \cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}} d\mathbf{l}'$$

Retarded Vector Potential with Three Approximations

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau' = \frac{\mu_0}{4\pi}\int \frac{I_0\cos[\omega(t-\mathscr{r}/c)]}{\mathscr{r}}d\mathbf{l}'$$

Approximation #1: Make this physical dipole into a perfect dipole. $b \ll r$

Estimate the spearation distances by the law of cosines.

$$\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\cos\psi},$$

where ψ is the angle between the vectors **r** and **b**:

$$rb\cos\psi = \mathbf{r}\cdot\mathbf{b} = rb\sin\theta\cos\phi'$$

$$\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos\phi'} \cong r(1 - \frac{b}{r}\sin\theta\cos\phi')$$

$$\frac{1}{\mathscr{r}} \cong \frac{1}{r}(1 + \frac{b}{r}\sin\theta\cos\phi')$$

Retarded Vector Potential with Three Approximations

$$\cos[\omega(t-\mathscr{r}/c)] = \cos[\omega(t-\frac{r}{c}) + \frac{\omega b}{c}\sin\theta\cos\phi']$$

$$= \cos[\omega(t-\frac{r}{c})]\cos[\frac{\omega b}{c}\sin\theta\cos\phi']$$

$$-\sin[\omega(t-\frac{r}{c})]\sin[\frac{\omega b}{c}\sin\theta\cos\phi']$$

Approximation #2: The size of the dipole is small compared to the wavelength radiated.

$$b \ll \frac{c}{\omega}(=\frac{\lambda}{2\pi})$$

$$\cos[\omega(t-\mathscr{r}/c)] \cong \cos[\omega(t-\frac{r}{c})] - \left(\frac{\omega b}{c}\sin\theta\cos\phi'\right)\sin[\omega(t-\frac{r}{c})]$$

The Retarded Vector Potential

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 I_0 b}{4\pi r}\hat{\mathbf{y}}\int_0^{2\pi}\{\cdots\}\cos\phi' d\phi'$$

$$\{\cdots\} = \cos[\omega(t-\frac{r}{c})] + b\sin\theta\cos\phi'\left(\frac{1}{r}\cos[\omega(t-\frac{r}{c})] - \frac{\omega}{c}\sin[\omega(t-\frac{r}{c})]\right)$$

The second-order term is dropped.

The first term integrates to zero: $\int_0^{2\pi}\cos\phi' d\phi' = 0$

The second term involves the integral of cosine squared. $\int_0^{2\pi}\cos^2\phi' d\phi' = \pi$

Putting this in, and noting that $\mathbf{A}$ points in the $\phi -$ direction.

The Retarded Vector Potential

The vector potential of an oscillating perfect magnetic dipole is:

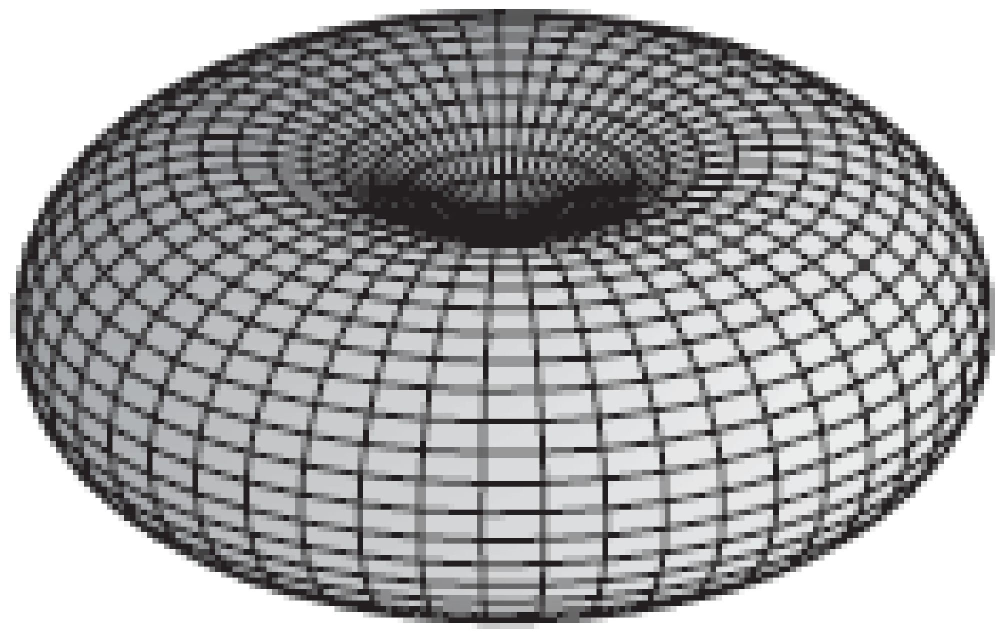
$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0 m_0}{4\pi}\frac{\sin\theta}{r}\left\{\frac{1}{r}\cos[\omega(t-\frac{r}{c})]-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c})]\right\}\hat{\boldsymbol{\phi}}$$

Approximation #3: at the radiation zone. $\frac{c}{\omega}\ll r$

$$\mathbf{A}(\mathbf{r},t)=-\frac{\mu_0 m_0\omega}{4\pi c}\frac{\sin\theta}{r}\sin[\omega(t-\frac{r}{c})]\hat{\boldsymbol{\phi}}$$

$$\begin{cases}\mathbf{E}=-\dfrac{\partial\mathbf{A}}{\partial t}=-\dfrac{\mu_0 m_0\omega^2}{4\pi c}\dfrac{\sin\theta}{r}\cos[\omega(t-\dfrac{r}{c})]\hat{\boldsymbol{\phi}}\\ \mathbf{B}=\nabla\times\mathbf{A}=-\dfrac{\mu_0 m_0\omega^2}{4\pi c^2}\dfrac{\sin\theta}{r}\cos[\omega(t-\dfrac{r}{c})]\hat{\theta}\end{cases}$$

The Electromagnetic Fields and Poynting Vector



$$\begin{cases} \mathbf{E} = -\dfrac{\partial \mathbf{A}}{\partial t} = -\dfrac{\mu_0 m_0 \omega^2}{4\pi c}\dfrac{\sin\theta}{r}\cos[\omega(t-\dfrac{r}{c})]\hat{\boldsymbol{\phi}} \\ \mathbf{B} = \nabla\times\mathbf{A} = -\dfrac{\mu_0 m_0 \omega^2}{4\pi c^2}\dfrac{\sin\theta}{r}\cos[\omega(t-\dfrac{r}{c})]\hat{\theta} \end{cases}$$

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{m_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\frac{r}{c})]\right\}^2\hat{\mathbf{r}}$$

The total power radiated is: $<P> = \int\langle\mathbf{S}\rangle\cdot d\mathbf{a} = \dfrac{\mu_0 m_0^2\omega^4}{12\pi c^3}$

$$\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \frac{m_0^2}{p_0^2 c^2} \ll 1 \quad \text{(Electric dipole radiation dominates)}$$

Only when the system is carefully contrived to exclude any electric contribution will the magnetic dipole radiation reveal itself.

Homework of Chap.11

Problem 11.1 Check that the retarded potentials of an oscillating dipole (Eqs. 11.12 and 11.17) satisfy the Lorenz gauge condition. Do *not* use approximation 3.

Problem 11.2 Equation 11.14 can be expressed in "coordinate-free" form by writing $p_0 \cos\theta = \mathbf{p}_0 \cdot \hat{\mathbf{r}}$. Do so, and likewise for Eqs. 11.17, 11.18. 11.19, and 11.21.

Problem 11.5 Calculate the electric and magnetic fields of an oscillating magnetic dipole *without* using approximation 3. [Do they look familiar? Compare Prob. 9.35.] Find the Poynting vector, and show that the intensity of the radiation is exactly the same as we got using approximation 3.

Problem 11.6 Find the radiation resistance (Prob. 11.3) for the oscillating magnetic dipole in Fig. 11.8. Express your answer in terms of λ and b, and compare the radiation resistance of the *electric* dipole. [*Answer*: $3 \times 10^5 (b/\lambda)^4 \Omega$]