Chapter 10: Potentials and Fields

10.1 The Potential Formulation

10.1.1 Scalar and Vector Potentials

In the electrostatics and magnetostatics,

$$\text{(i)}\ \nabla\cdot\mathbf{E}=\frac{1}{\varepsilon_0}\rho \qquad \text{(iii)}\ \nabla\times\mathbf{E}=0$$

$$\text{(ii)}\ \nabla\cdot\mathbf{B}=0 \qquad \text{(iv)}\ \nabla\times\mathbf{B}=\mu_0\mathbf{J}$$

The electric field and magnetic field can be expressed using potential:

$$\text{(iii)}\ \nabla\times\mathbf{E}=0 \Longrightarrow \mathbf{E}=-\nabla V \qquad -\nabla^2 V=\frac{1}{\varepsilon_0}\rho$$

$$\text{(ii)}\ \nabla\cdot\mathbf{B}=0 \Longrightarrow \mathbf{B}=\nabla\times\mathbf{A} \qquad \nabla\times(\nabla\times\mathbf{A})=\mu_0\mathbf{J}$$

$$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\mathbf{J} \quad \Rightarrow \quad -\nabla^2\mathbf{A}=\mu_0\mathbf{J}$$

If $\nabla\cdot\mathbf{A}=0.$

Scalar and Vector Potentials

In the electrodynamics,

$$\text{(i)}\ \nabla\cdot\mathbf{E}=\frac{1}{\varepsilon_0}\rho \qquad \text{(iii)}\ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\text{(ii)}\ \nabla\cdot\mathbf{B}=0 \qquad \text{(iv)}\ \nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

How do we express the fields in terms of scalar and vector potentials?

$\mathbf{B}$ remains divergenceless, so we can still write, $\mathbf{B}=\nabla\times\mathbf{A}$

Putting this into Faraday's law (iii) yields,

$$\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}(\nabla\times\mathbf{A})=\nabla\times(-\frac{\partial\mathbf{A}}{\partial t}) \quad\Rightarrow\quad \nabla\times(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t})=0$$

$$\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}=-\nabla V$$

Scalar and Vector Potentials

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\text{(i)}\ \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho \quad \Rightarrow \quad -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot A) = \frac{1}{\varepsilon_0}\rho$$

$$\text{(iv)}\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \Rightarrow \quad \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \varepsilon_0 \nabla(\frac{\partial V}{\partial t}) - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

We can further yields.

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\varepsilon_0}\rho$$

$$\left(\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}$$

These two equations contain all the information in Maxwell's equations.

Example 10.1

Find the charge and current distributions that would give rise to the potentials.

$$V = 0,\ \mathbf{A} = \begin{cases} \dfrac{\mu_0 k}{4c}(ct - |x|)^2 \hat{\mathbf{z}} & \text{for } |x| < ct \\ 0 & \text{for } |x| > ct \end{cases}$$

Where k is a constant, and c is the speed of light.

Solution: $\rho = -\varepsilon_0 \dfrac{\partial}{\partial t}(\nabla \cdot \mathbf{A})$

$$\mathbf{J} = -\frac{1}{\mu_0}\left(\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) + \frac{1}{\mu_0}\nabla(\nabla \cdot \mathbf{A})$$

$$\begin{cases} \nabla \cdot \mathbf{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z} = 0 \\ \nabla^2 \mathbf{A} = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right) A_z \hat{\mathbf{z}} = \dfrac{\mu_0 k}{2c}\hat{\mathbf{z}} \\ -\mu_0 \varepsilon_0 \dfrac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \varepsilon_0 \dfrac{\mu_0 k}{4c} 2c^2 \hat{\mathbf{z}} = -\dfrac{\mu_0 k}{2c}\hat{\mathbf{z}} \end{cases}$$

$$\rho = 0$$
$$\mathbf{J} = 0$$

Example 10.1 (ii)

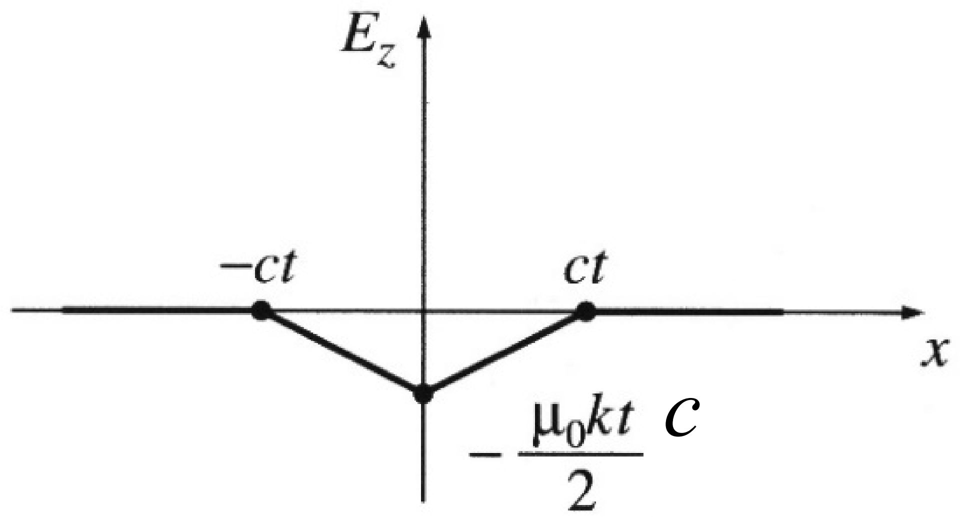
Since the volume charge density and current density are both zero, where are the electric and magnetic fields from?

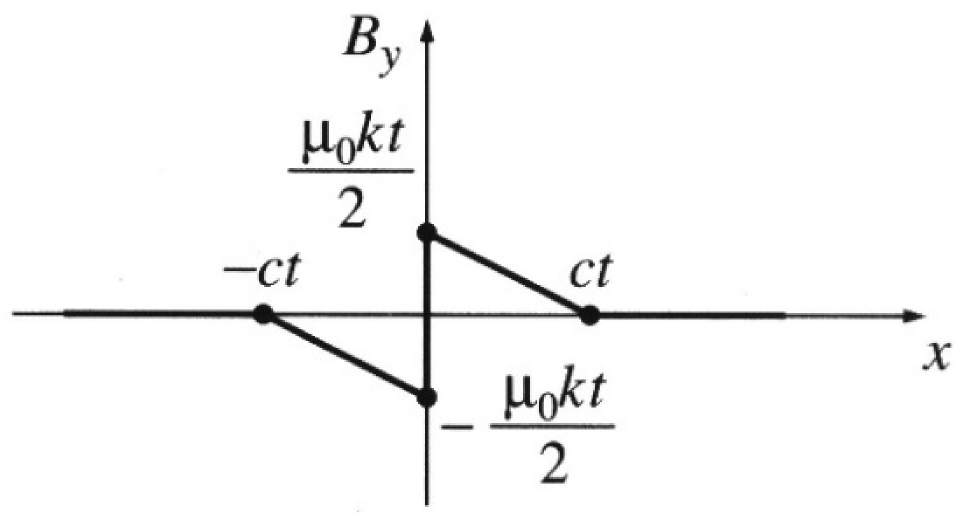
$$\rho = 0 \ \text{ and } \ \mathbf{J} = 0$$

They might originate from surface charge or surface current.

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 k}{2}(ct - |x|)\hat{\mathbf{z}}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 k}{4c}\frac{\partial}{\partial x}(ct - |x|)^2\hat{\mathbf{y}} = \begin{cases} x > 0 & -\dfrac{\mu_0 k}{4c}\dfrac{\partial}{\partial x}(ct - x)^2\hat{\mathbf{y}} = \dfrac{\mu_0 k}{2c}(ct - x)\hat{\mathbf{y}} \\ x < 0 & -\dfrac{\mu_0 k}{4c}\dfrac{\partial}{\partial x}(ct + x)^2\hat{\mathbf{y}} = -\dfrac{\mu_0 k}{2c}(ct + x)\hat{\mathbf{y}} \end{cases}$$





There is a surface current **K** in the *yz* plane.

$$\mathbf{K} = \hat{\mathbf{n}} \times (\mathbf{H}^+ - \mathbf{H}^-)$$

$$= \hat{\mathbf{n}} \times \frac{1}{\mu_0}\frac{\mu_0 k}{c}ct\hat{\mathbf{y}} = kt\hat{\mathbf{z}}$$

How do we know?

10.1.2 Gauge Transformations

We have succeeded in reducing six components ($\mathbf{E}$ and $\mathbf{B}$) down to four (V and $\mathbf{A}$). However, V and $\mathbf{A}$ are not uniquely determined.
We are free to impose extra conditions on V and $\mathbf{A}$, as long as nothing happens to $\mathbf{E}$ and $\mathbf{B}$.

Suppose we have two sets of potential (V, $\mathbf{A}$) and (V', $\mathbf{A}'$), which correspond to the same electric and magnetic fields.

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\alpha} \quad \text{and } V' = V + \beta$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}' \quad \Rightarrow \quad \nabla \times \boldsymbol{\alpha} = 0 \quad \Rightarrow \quad \boldsymbol{\alpha} = \nabla \lambda$$

$$\mathbf{E} = -\nabla V' - \frac{\partial \mathbf{A}'}{\partial t} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} - \left(\nabla \beta + \frac{\partial \boldsymbol{\alpha}}{\partial t} \right)$$

$$\nabla (\beta + \frac{\partial \lambda}{\partial t}) = 0 \quad \Rightarrow \quad (\beta + \frac{\partial \lambda}{\partial t}) = k(t)$$

Gauge Transformations

$$\boldsymbol{\alpha} = \nabla\lambda = \nabla\lambda'$$

$$\beta = -\frac{\partial\lambda}{\partial t} + k(t) = -\frac{\partial\lambda'}{\partial t}$$

$$\Rightarrow \begin{cases} \mathbf{A}' = \mathbf{A} + \nabla\lambda \\ V' = V - \dfrac{\partial\lambda}{\partial t} \end{cases}$$

Conclusion: For any scalar function λ, we can with impunity add $\nabla\lambda$ to $\mathbf{A}$, provided we simultaneously subtract $\partial\lambda/\partial t$ to V.

Such changes in V and $\mathbf{A}$ do not affect $\mathbf{E}$ and $\mathbf{B}$, and are called gauge transformation.

We have the freedom to choose V and $\mathbf{A}$ provided $\mathbf{E}$ and $\mathbf{B}$ do not affect --- gauge freedom.

10.1.3 Coulomb Gauge and Lorentz Gauge

There are many famous gauges in the literature. We will show the two most popular ones.

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\varepsilon_0}\rho$$

$$\left(\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}$$

The Coulomb Gauge: $\nabla \cdot \mathbf{A} = 0$

$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho \text{ (Poisson's equation)}$$

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t)}{\mathscr{r}} d\tau' \text{ (setting } V = 0 \text{ at infinity)}$$

V instantaneously reflects all changes in ρ. Really?

$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ unlike electrostatic case.

The Coulomb Gauge

Advantage: the scalar potential is particularly simple to calculate;

$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho \text{ (Poisson's equation)}$$

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t)}{\mathscr{r}}d\tau' \quad \text{(setting } V=0 \text{ at infinity)}$$

Disadvantage: the vector potential will be very difficult to calculate for the non-static case.

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} + (\mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla(\mu_0\varepsilon_0 \frac{\partial V}{\partial t}))$$

The Coulomb gauge is suitable for the static case.

The Lorentz Gauge

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\varepsilon_0}\rho$$

$$\left(\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}$$

The Lorentz Gauge: $$\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0$$

$$\nabla^2 V - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\varepsilon_0}\rho$$

$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}$$

inhomogeneous wave equations

$$\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \equiv \Box^2$$

$\Box^2$: the d'Alembertian

$$\Box^2 V = -\frac{1}{\varepsilon_0}\rho$$

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

The Lorentz Gauge

Advantage: It treats V and $\mathbf{A}$ on an equal footing and is particularly nice in the context of special relativity. It can be regarded as four-dimensional versions of Poisson's equation.

V and $\mathbf{A}$ satisfy the *inhomogeneous wave equations*, with a "source" term on the right.

$$\Box^2 V = -\frac{1}{\varepsilon_0}\rho$$

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Disadvantage: …

We will use the Lorentz gauge exclusively.

10.2 Continuous Distributions
10.2.1 Retarded Potentials

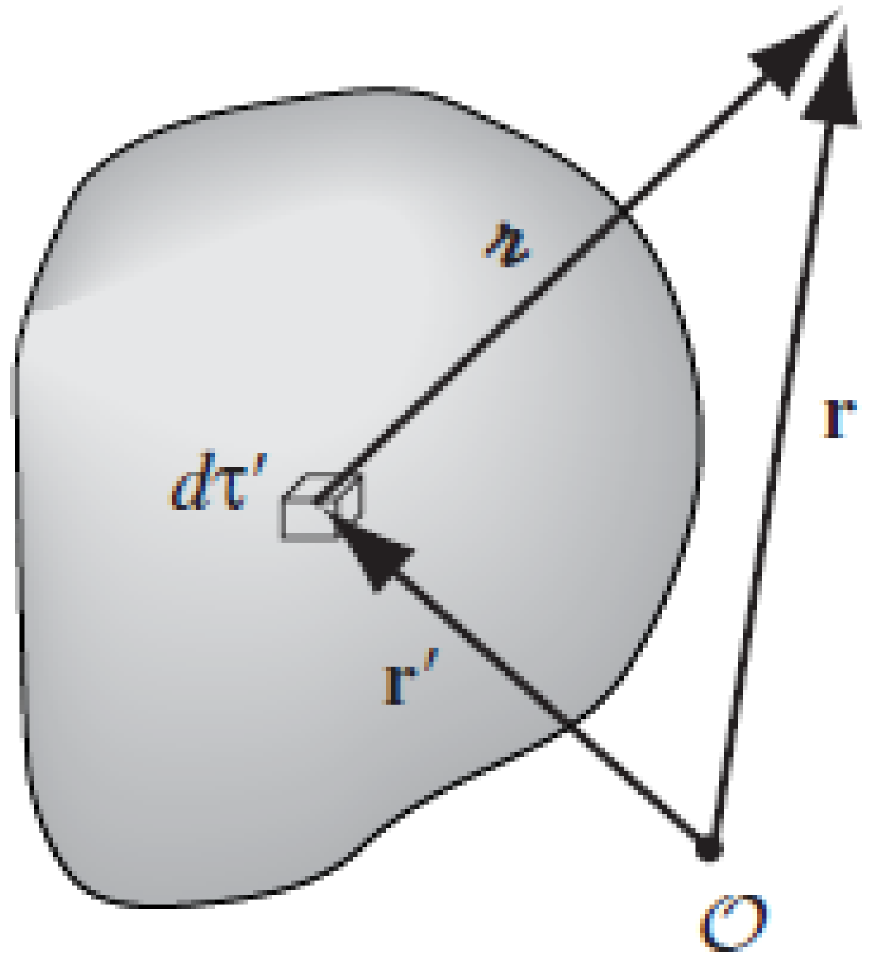
$$\nabla^2 V - \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\varepsilon_0}\rho$$

$$\nabla^2 \mathbf{A} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}$$

static case →

$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho$$

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$



Four copies of Poisson's equation

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} d\tau'$$

Retarded Potentials

In the nonstatic case, it is not the status of the source right now that matters, but rather its condition at some earlier time t_r when the "message" left.

$$t_r \equiv t - \frac{\mathscr{r}}{c} \quad \text{(called the retarded time)}$$

Retarded potentials:

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'$$

Argument: The light we see now left each star at the retarded time corresponding to that star's distance from the earth.

This heuristic argument sounds reasonable, but is it correct? Yes, we will prove it soon.

Retarded Potentials $V(\mathbf{r},t)$ Satisfy the Inhomogeneous Wave Equations

Show that the retarded scalar potentials satisfy the inhomogeneous wave equations.

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau' \qquad \nabla^2 V-\mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2}=-\frac{1}{\varepsilon_0}\rho$$

Sol:

$$\nabla V=\frac{1}{4\pi\varepsilon_0}\int\nabla\left(\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}\right)d\tau'=\frac{1}{4\pi\varepsilon_0}\int\frac{\mathscr{r}(\nabla\rho)-\rho(\nabla\mathscr{r})}{\mathscr{r}^2}d\tau'$$

Using quotient rule: $\nabla\left(\dfrac{f}{g}\right)=\dfrac{g\nabla f-f\nabla g}{g^2}$

$$\nabla\rho=\nabla\rho(\mathbf{r}',t_r)=\frac{d\rho}{dt_r}\nabla t_r=\dot{\rho}\frac{-1}{c}\nabla\mathscr{r} \qquad \nabla\mathscr{r}=\hat{\mathscr{r}}$$

$$\nabla V=\frac{-1}{4\pi\varepsilon_0}\int[\frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}}+\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}]d\tau'$$

$$\nabla t_r=\nabla(t-\frac{\mathscr{r}}{c})=-\frac{1}{c}\nabla\mathscr{r}=-\frac{1}{c}\hat{\mathscr{r}}.$$

Retarded Potentials $V(\mathbf{r},t)$ Satisfy the Inhomogeneous Wave Equations (ii)

$$\nabla\cdot\nabla V=\nabla^2 V=\frac{-1}{4\pi\varepsilon_0}\int\nabla\cdot\left[\frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}}+\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau'$$

$$\nabla\cdot\left[\frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}}+\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}\right]=\frac{1}{c}\nabla\cdot\left(\dot{\rho}\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)+\nabla\cdot\left(\rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$$=\frac{1}{c}\left[\frac{\hat{\mathscr{r}}}{\mathscr{r}}\cdot\nabla\dot{\rho}+\dot{\rho}\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right]+\left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\nabla\rho+\rho\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]$$

$$\nabla\dot{\rho}=\nabla\dot{\rho}(\mathbf{r}',t_r)=\frac{\partial\dot{\rho}}{\partial t_r}\nabla t_r=\ddot{\rho}\frac{-1}{c}\nabla\mathscr{r}=-\frac{\ddot{\rho}}{c}\hat{\mathscr{r}}\quad\text{and}\quad\nabla\rho=\frac{-\dot{\rho}}{c}\hat{\mathscr{r}}$$

$$\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}}=\frac{1}{\mathscr{r}^2}\quad\text{and}\quad\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}=4\pi\delta^3(\vec{\mathscr{r}})$$

$$\nabla\cdot\left[\frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}}+\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}\right]=\frac{1}{c}\left[-\frac{\ddot{\rho}}{c\mathscr{r}}+\frac{\dot{\rho}}{\mathscr{r}^2}\right]+\left[-\frac{1}{\mathscr{r}^2}\frac{\dot{\rho}}{c}+4\pi\rho\delta^3(\vec{\mathscr{r}})\right]$$

$$=-\frac{1}{c^2}\frac{\ddot{\rho}}{\mathscr{r}}+4\pi\rho\delta^3(\vec{\mathscr{r}})$$

Retarded Potentials $V(\mathbf{r},t)$ Satisfy the Inhomogeneous Wave Equations (iii)

$$\nabla^2 V = \frac{-1}{4\pi\varepsilon_0}\int[-\frac{1}{c^2}\frac{\ddot{\rho}}{\mathscr{r}} + 4\pi\rho\delta^3(\vec{\mathscr{r}})]d\tau' = \frac{1}{c^2}\frac{1}{4\pi\varepsilon_0}\int\frac{\ddot{\rho}}{\mathscr{r}}d\tau' - \frac{\rho(\mathbf{r},t)}{\varepsilon_0}$$

$$\frac{1}{4\pi\varepsilon_0}\int\frac{\ddot{\rho}}{\mathscr{r}}d\tau' = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{\mathscr{r}}\frac{\partial^2\rho}{\partial t_r^2}d\tau' = \frac{\partial^2}{\partial t_r^2}\int\frac{\rho}{4\pi\varepsilon_0\mathscr{r}}d\tau' = \frac{\partial^2 V}{\partial t_r^2} = \frac{\partial^2 V}{\partial t^2}$$

$$\nabla^2 V = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{\rho(\mathbf{r},t)}{\varepsilon_0}$$

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho(\mathbf{r},t)}{\varepsilon_0}$$

Retarded Potentials $\mathbf{A}(\mathbf{r},t)$
Satisfy the Inhomogeneous Wave Equations

Show that the retarded *vector* potentials satisfy the inhomogeneous wave equations .

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau' \qquad \nabla^2\mathbf{A}-\mu_0\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\mathbf{J}$$

Sol:

$$\nabla\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}\right)=\frac{\mathscr{r}(\nabla\cdot\mathbf{J})-\mathbf{J}\cdot(\nabla\mathscr{r})}{\mathscr{r}^2} \qquad t_r\equiv t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

Using quotient rule: $\nabla\cdot\left(\frac{\mathbf{A}}{g}\right)=\frac{g(\nabla\cdot\mathbf{A})-\mathbf{A}\cdot(\nabla g)}{g^2}$

Also see Prob. 10.8… Show that the retarded potential satisfy the Lorentz gauge condition.

The Principle of Causality

This proof applies equally well to the advanced potentials.

Advanced potentials:

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t_a)}{\mathscr{r}}d\tau'$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_a)}{\mathscr{r}}d\tau'$$

$$t_a \equiv t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

$$\nabla^2 V - \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\varepsilon_0}\rho$$

$$\nabla^2 \mathbf{A} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}$$

最神聖的信條

The advanced potentials violate the most sacred tenet in all physics: the principle of causality.

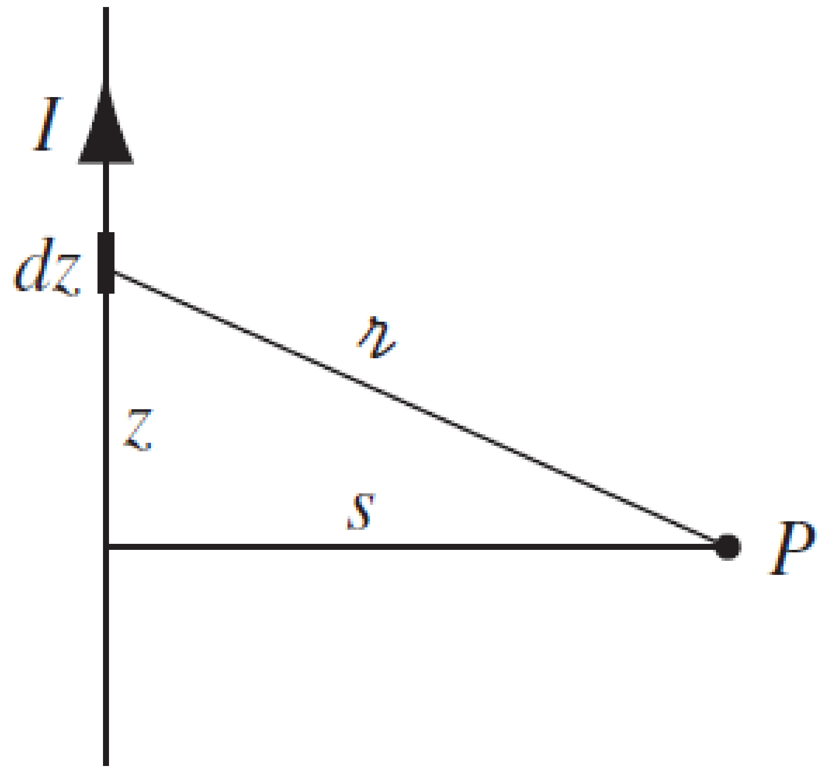
→ No direct physical significance.

Example 10.2

An infinite straight wire carries the current $I(t)=\begin{cases}0 & \text{for } t\le 0\\ I_0 & \text{for } t>0\end{cases}$

Find the resulting electric and magnetic fields.

Sol: The wire is electrically neutral, so the retarded scalar potential is zero.



$$\mathbf{A}(\mathbf{r},t)=\mathbf{A}(s,t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'=\frac{\mu_0}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{\infty}\frac{I(t_r)}{\mathscr{r}}dz$$

For $t < s/c$, the "news" has not yet reached P, and the potential is zero.

For $t > s/c$, only the segment $|z|\le\sqrt{(ct)^2-s^2}$ contributes.

$$\mathbf{A}(s,t) = (\frac{\mu_0 I_0}{4\pi}\hat{\mathbf{z}})\int_{-\sqrt{(ct)^2-s^2}}^{\sqrt{(ct)^2-s^2}} \frac{1}{\sqrt{s^2+z^2}}dz$$

How?

$$= (\frac{\mu_0 I_0}{2\pi}\hat{\mathbf{z}})\ln(\sqrt{s^2+z^2}+z)\Big|_0^{\sqrt{(ct)^2-s^2}}$$

$$= (\frac{\mu_0 I_0}{2\pi}\hat{\mathbf{z}})\ln(\frac{ct+\sqrt{(ct)^2-s^2}}{s})$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2-s^2}}\hat{\mathbf{z}}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\partial A_z}{\partial s}\hat{\boldsymbol{\phi}} = \frac{\mu_0 I_0}{2\pi s}\frac{ct}{\sqrt{(ct)^2-s^2}}\hat{\boldsymbol{\phi}}$$

$$Curl:\ \nabla\times\boldsymbol{v} = [\frac{1}{s}\frac{\partial v_z}{\partial\phi}-\frac{\partial v_\phi}{\partial z}]\hat{s}+[\frac{\partial v_s}{\partial z}-\frac{\partial v_z}{\partial s}]\hat{\phi}+\frac{1}{s}[\frac{\partial}{\partial s}(sv_\phi)-\frac{\partial v_s}{\partial\phi}]\hat{z}$$

Optional

Retarded Fields?

Can we express the electric field and magnetic field using the concept of the retarded potentials? No, but...

Retarded potentials:

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'$$

Retarded fields: (wrong)

$$\mathbf{E}(\mathbf{r},t)\neq\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$$

$$\mathbf{B}(\mathbf{r},t)\neq\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

How to correct this problem?

⇨ Jefimenko’s equations.

Optional

10.2.2 Jefimenko's Equations

Retarded potentials:

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau' \quad \text{and} \quad \mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'$$

$$\mathbf{E}=-\nabla V-\frac{\partial \mathbf{A}}{\partial t} \quad \begin{cases} -\nabla V=\dfrac{1}{4\pi\varepsilon_0}\displaystyle\int[\frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}}+\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}]d\tau' \\ -\dfrac{\partial \mathbf{A}}{\partial t}=-\dfrac{\partial}{\partial t_r}(\dfrac{\mu_0}{4\pi}\displaystyle\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau')\frac{\partial t_r}{\partial t}=-\frac{\mu_0}{4\pi}\int\frac{\dot{\mathbf{J}}}{\mathscr{r}}d\tau' \end{cases}$$

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\int[\frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}}+\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}]d\tau'-\frac{\mu_0}{4\pi}\int\frac{\dot{\mathbf{J}}}{\mathscr{r}}d\tau'$$

$$=\frac{1}{4\pi\varepsilon_0}\int[\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}+\frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}}-\frac{\dot{\mathbf{J}}}{c^2\mathscr{r}}]d\tau'$$

The time-dependent generalization of Coulomb's law.

Optional

Jefimenko's Equations (ii)

Retarded potentials:

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau' \quad \text{and} \quad \mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'$$

$$\mathbf{B}=\nabla\times\mathbf{A}=\frac{\mu_0}{4\pi}\int\nabla\times\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'=\frac{\mu_0}{4\pi}\int[\frac{1}{\mathscr{r}}\nabla\times\mathbf{J}-\mathbf{J}\times\nabla\frac{1}{\mathscr{r}}]d\tau'$$

$$\nabla\times\mathbf{J}=\frac{1}{c}\dot{\mathbf{J}}\times\hat{\mathscr{r}} \quad \text{and} \quad \nabla(\frac{1}{\mathscr{r}})=-\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int[\frac{\mathbf{J}}{\mathscr{r}^2}+\frac{1}{c\mathscr{r}}\dot{\mathbf{J}}]\times\hat{\mathscr{r}}d\tau'$$

The time-dependent generalization of the Biot-Savart law.

These two equations are *of limited utility*, but they provide a satisfying sense of closure to the theory.

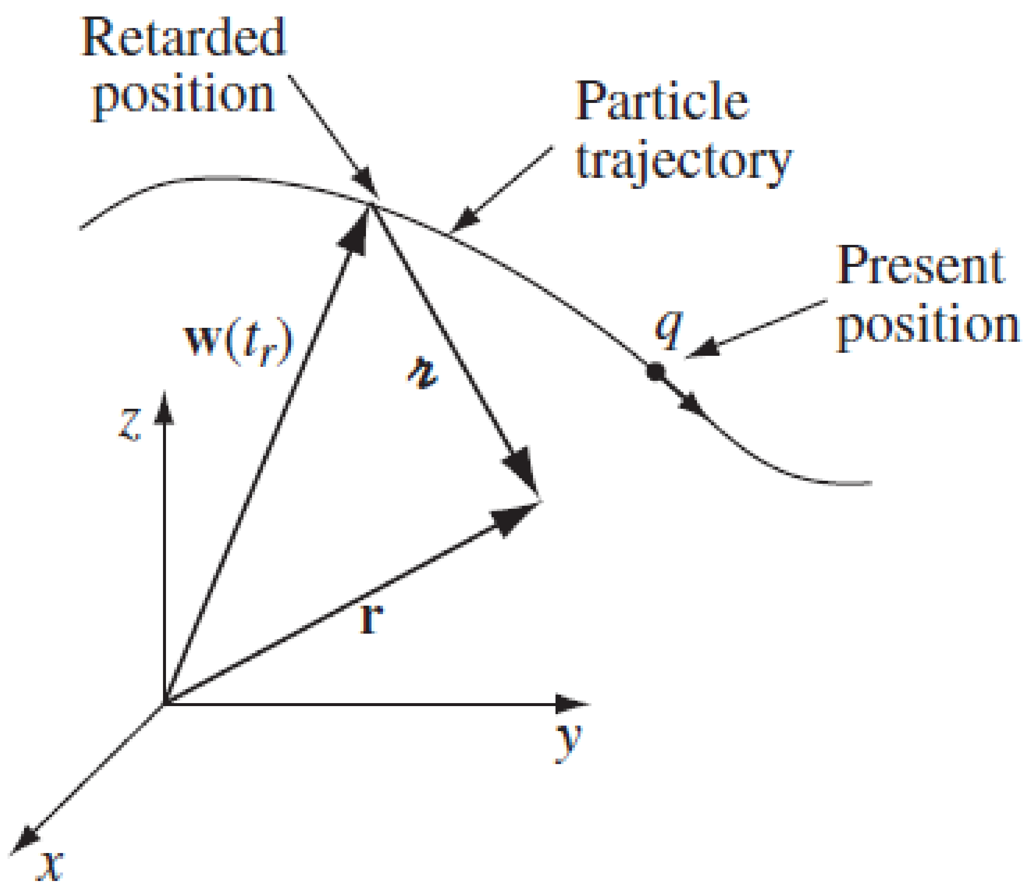
10.3 Point Charges

10.3.1 Lienard-Wiechert Potentials

What are the retarded potentials of a moving point charge q?

Consider a point charge q that is moving on a specified trajectory

$$\mathbf{w}(t) \equiv \text{position of } q \text{ at time } t.$$



The retarded time is: $$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{w}(t_r)|}{c}$$

$$\mathbf{w}(t_r) \text{ the retarded position of the charge.}$$

The separation vector $\vec{\mathscr{r}}$ is the vector from the retarded position to the field point $\mathbf{r}$

$$\vec{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)$$

$\mathbf{r}' = \mathbf{w}(t_r)$,
$\mathbf{r}'$ is function of t_r.

Communication

Is it possible that more than one point on the trajectory are "in communication" with $\mathbf{r}$ at any particular time t?

No, one and only one will contribute.

Suppose there are two such points, with retarded time t_1 and t_2:

$$\mathscr{r}_1 = c(t - t_1) \text{ and } \mathscr{r}_2 = c(t - t_2) \quad \Longrightarrow \quad \mathscr{r}_1 - \mathscr{r}_2 = c(t_1 - t_2)$$

This means the average velocity of the particle in the direction of $\mathbf{r}$ would have to be c. ← violate special relativity.

Only one retarded point contributes to the potentials at any given moment.

Total Charge

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\left|\mathbf{r}-\mathbf{w}(t_r)\right|}d\tau'=\frac{1}{4\pi\varepsilon_0}\frac{1}{\left|\mathbf{r}-\mathbf{w}(t_r)\right|}\underbrace{\int\rho(\mathbf{r}',t_r)d\tau'}_{\neq q}$$

The retardation, $t_r\equiv t-\left|\mathbf{r}-\mathbf{r}'\right|/c$, obliges us to evaluate ρ at different times for different parts of the configuration.

The source in motion lead to a distorted picture of the total charge.

$$\int\rho(\mathbf{r}',t_r)d\tau'=\frac{q}{1-\hat{\boldsymbol{\imath}}\cdot\mathbf{v}/c}$$

No matter how small the charge is.

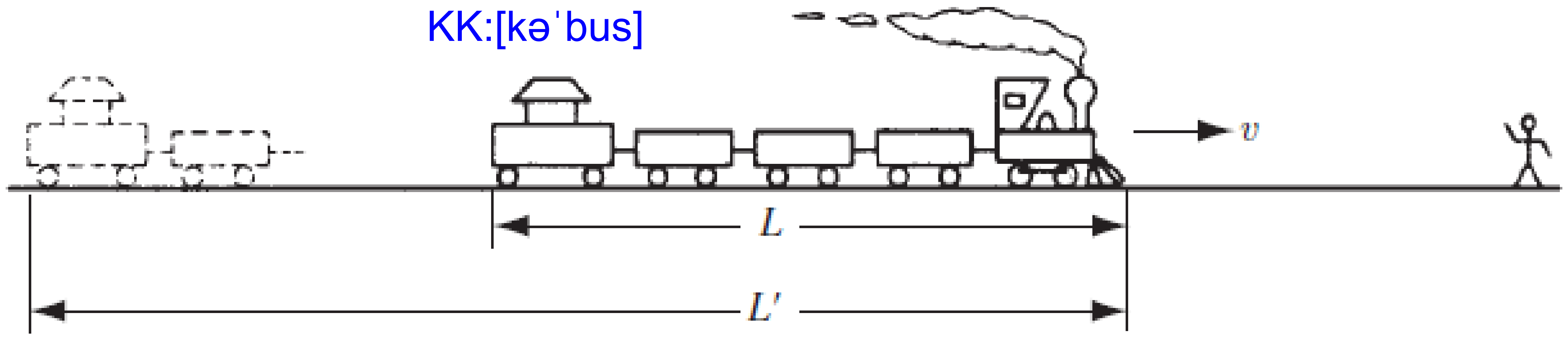
To be proved.

Optional

Total Charge: a Geometrical Effect

A train coming towards you looks a little longer than it really is, because the light you receive from the caboose left earlier than the light you receive simultaneously from the engine.

caboose
KK:[kəˈbus]



$$\frac{L'}{c} = \frac{L'-L}{v} \quad \Rightarrow \quad L' = \frac{L}{1-v/c}$$

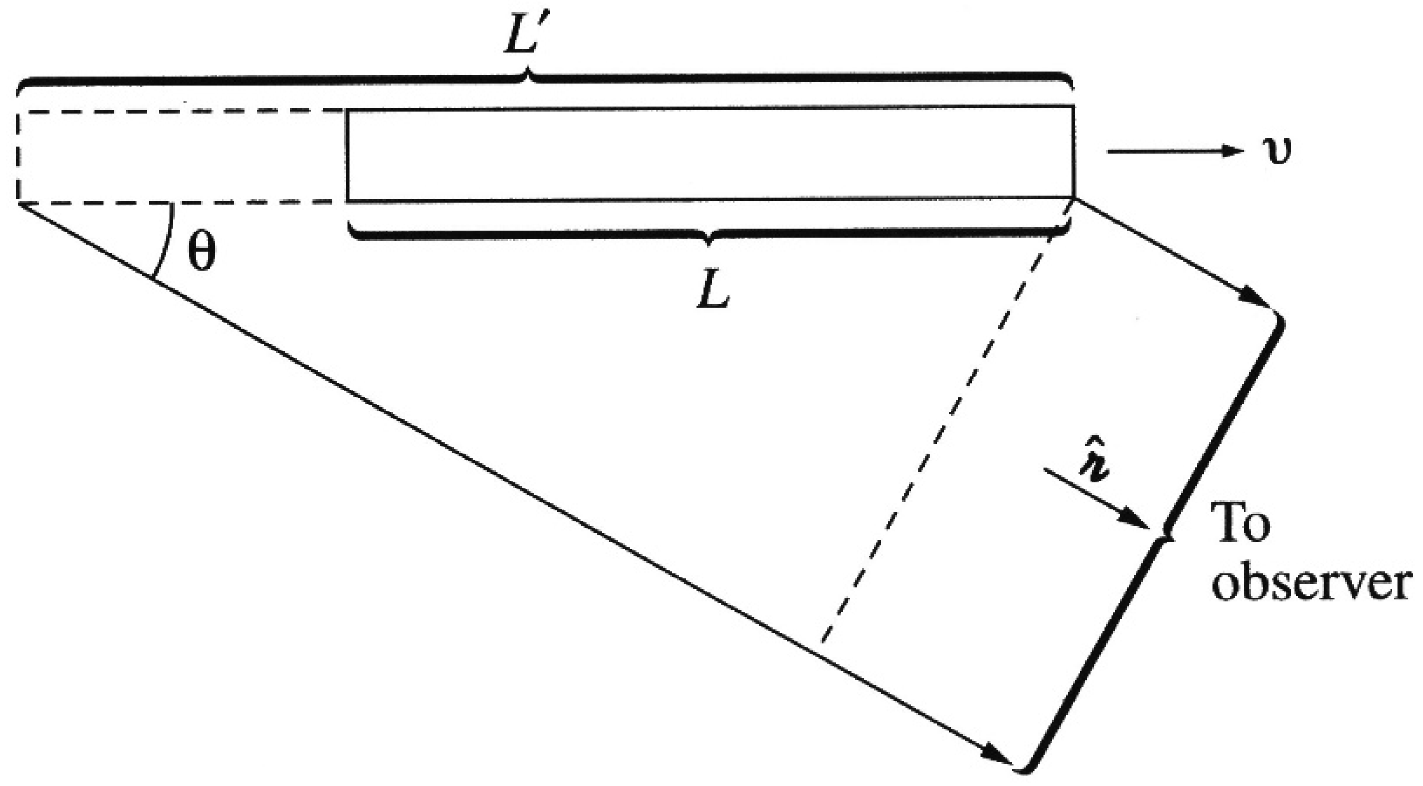
$$L' = \frac{L}{1-v/c}$$ Approaching train appears longer.

$$L' = \frac{L}{1+v/c}$$ A train going away from you looks shorter.

Optional

Total Charge: a Geometrical Effect (ii)

In general, if the train's velocity makes an angle θ with your line of sight, the extra distance light from the caboose must cover is $L'\cos\theta$.



$$\frac{L'\cos\theta}{c}=\frac{L'-L}{v} \quad\Rightarrow\quad L'=\frac{L}{1-v\cos\theta/c}$$

This effect does not distort the dimensions perpendicular to the motion.

The apparent volume τ' of the train is related to the actual volume τ by

$$\tau'=\frac{\tau}{1-\hat{\imath}\cdot\mathbf{v}/c}$$

Lienard-Wiechert Potentials

It follows that

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'=\frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}(1-\hat{\mathscr{r}}\cdot\mathbf{v}/c)},$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\rho(\mathbf{r}',t_r)\mathbf{v}(t_r)}{\mathscr{r}}d\tau'=\frac{\mu_0}{4\pi}\frac{\mathbf{v}(t_r)}{\mathscr{r}}\int\rho(\mathbf{r}',t_r)d\tau'$$

$$=\frac{\mu_0}{4\pi}\frac{q\mathbf{v}}{\mathscr{r}(1-\hat{\mathscr{r}}\cdot\mathbf{v}/c)}=\frac{\mathbf{v}}{c^2}V(\mathbf{r},t)$$

where $\rho(\mathbf{r}',t_r)=q\delta(\mathbf{r}'-\mathbf{r},\ t_r)$.

The famous Lienard-Wiechert potentials for a moving point charge.

$$\begin{cases}V(\mathbf{r},t)=\dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{\mathscr{r}}\dfrac{1}{(1-\hat{\mathscr{r}}\cdot\mathbf{v}/c)}\\[2ex]\mathbf{A}(\mathbf{r},t)=\dfrac{\mathbf{v}}{c^2}V(\mathbf{r},t)\end{cases}$$

Derivation from Wikipedia (i)

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r')}{|\mathbf{r}-\mathbf{r}'|}d\tau',\ \text{ where } t_r'=t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|.$$

For a moving point charge whose trajectory is given as a function of time by $\mathbf{r}_s'(t')$, the charge density is as follows:

$$\rho(\mathbf{r}',t')=q\underline{\delta^3(\mathbf{r}'-\mathbf{r}_s'(t'))}.$$ → Three dimensional Dirac delta function.

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{q\delta^3(\mathbf{r}'-\mathbf{r}_s'(t_r'))}{|\mathbf{r}-\mathbf{r}'|}d\tau'$$

The integral is difficult to evalute in the present form, so we rewrite as:

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\iint\frac{q\delta^3(\mathbf{r}'-\mathbf{r}_s'(t'))}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t_r')dt'd\tau'$$

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$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\iint\frac{q\delta(t'-t_r')}{|\mathbf{r}-\mathbf{r}'|}\delta^3(\mathbf{r}'-\mathbf{r}_s'(t'))d\tau'dt'=\frac{1}{4\pi\varepsilon_0}\int\frac{q\delta(t'-t_r')}{|\mathbf{r}-\mathbf{r}_s'(t')|}dt'$$

Derivation from Wikipedia (ii)

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{q\delta(t'-t_r')}{|\mathbf{r}-\mathbf{r}_s'(t')|}dt'$$

Since the retarded time t_r' is a function of the field point $(\mathbf{r},\ t)$ and the source trajectory $\mathbf{r}_s'(t')$, and hence depends on t'.

$$\delta(f(t'))=\sum_i\frac{\delta(t'-t_i)}{f'(t_i)},$$ where each t_i is a zero of f.

Because there is only one retarded time t_r for any given space-time coordinate $(\mathbf{r},t)$ and source trajectory $\mathbf{r}_s(t')$, the above equation reduces to:

$$\delta(t'-t_r')=\frac{\delta(t'-t_r)}{\frac{\partial}{\partial t'}(t'-t_r')\Big|_{t'=t_r}}=\frac{\delta(t'-t_r)}{\frac{\partial}{\partial t'}(t'-(t-\frac{1}{c}|\mathbf{r}-\mathbf{r}_s(t')|))\Big|_{t'=t_r}}$$

$$=\frac{\delta(t'-t_r)}{1+\frac{1}{c}(\mathbf{r}-\mathbf{r}_s(t'))\cdot(-\mathbf{v}_s)\Big/|\mathbf{r}-\mathbf{r}_s(t')|\Big|_{t'=t_r}}$$

$$=\frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_s\cdot\hat{\mathbf{n}}}$$ where $\boldsymbol{\beta}_s=\frac{\mathbf{v}_s}{c}$ and $\hat{\mathbf{n}}=\frac{(\mathbf{r}-\mathbf{r}_s(t'))}{|\mathbf{r}-\mathbf{r}_s(t')|}\Big|_{t'=t_r}$

Derivation from Wikipedia (iii)

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{q\delta(t'-t_r')}{|\mathbf{r}-\mathbf{r}_s'(t')|}dt'$$

$$\begin{cases} V(\mathbf{r},t)=\dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{q}{(1-\boldsymbol{\beta}_s\cdot\hat{\mathbf{n}})|\mathbf{r}-\mathbf{r}_s(t')|}\right)_{t'=t_r} \\ \mathbf{A}(\mathbf{r},t)=\dfrac{\mu_0 c}{4\pi}\left(\dfrac{q\boldsymbol{\beta}_s}{(1-\boldsymbol{\beta}_s\cdot\hat{\mathbf{n}})|\mathbf{r}-\mathbf{r}_s(t')|}\right)_{t'=t_r} \end{cases}$$

Lienard-Wiechert Potentials

$$\begin{cases} V(\mathbf{r},t)=\dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{\mathscr{r}}\dfrac{1}{(1-\hat{\mathscr{r}}\cdot\mathbf{v}/c)} \\ \mathbf{A}(\mathbf{r},t)=\dfrac{\mathbf{v}}{c^2}V(\mathbf{r},t) \end{cases}$$

Example 10.3

Find the potentials of a point charge moving with constant velocity. Assume the particle passes through the origin at time $t = 0$.

Sol: The trajectory is: $\mathbf{w}(t) = \mathbf{v}t$

First compute the retarded time: $|\mathbf{r} - \mathbf{w}(t_r)| = |\mathbf{r} - \mathbf{v}t_r| = c(t - t_r)$

$$r^2 - 2\mathbf{r}\cdot\mathbf{v}t_r + v^2 t_r^2 = c^2(t^2 - 2tt_r + t_r^2)$$

$$(c^2 - v^2)t_r^2 + 2(\mathbf{r}\cdot\mathbf{v} - c^2 t)t_r + (c^2 t^2 - r^2) = 0$$

$$t_r = \frac{(c^2 t - \mathbf{r}\cdot\mathbf{v}) \pm \sqrt{(\mathbf{r}\cdot\mathbf{v} - c^2 t)^2 - (c^2 - v^2)(c^2 t^2 - r^2)}}{(c^2 - v^2)}$$

Which sign is correct?

Consider $v = 0$ $\quad t_r = t \pm \sqrt{t^2 - (t^2 - r^2/c^2)} = t \pm r/c$

We want the minus sign

Contd.:

$$t_r = \frac{(c^2 t - \mathbf{r}\cdot\mathbf{v}) - \sqrt{(\mathbf{r}\cdot\mathbf{v} - c^2 t)^2 - (c^2 - v^2)(c^2 t^2 - r^2)}}{(c^2 - v^2)}$$

$$\mathscr{r} = c(t - t_r), \text{ and } \hat{\mathscr{r}} = \frac{\mathbf{r} - \mathbf{v}t_r}{c(t - t_r)}$$

$$\mathscr{r} - \vec{\mathscr{r}}\cdot\mathbf{v}/c = c(t - t_r)\left[1 - \frac{\mathbf{v}}{c}\cdot\frac{\mathbf{r} - \mathbf{v}t_r}{c(t - t_r)}\right] = c(t - t_r) - (\frac{\mathbf{v}\cdot\mathbf{r}}{c} - \frac{v^2}{c}t_r)$$

$$= \frac{1}{c}\left[(c^2 t - \mathbf{r}\cdot\mathbf{v}) - (c^2 - v^2)t_r\right]$$

$$= \frac{1}{c}\sqrt{(\mathbf{r}\cdot\mathbf{v} - c^2 t)^2 - (c^2 - v^2)(c^2 t^2 - r^2)}$$

$$\begin{cases} V(\mathbf{r},t) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qc}{\sqrt{(\mathbf{r}\cdot\mathbf{v} - c^2 t)^2 - (c^2 - v^2)(c^2 t^2 - r^2)}} \\ \mathbf{A}(\mathbf{r},t) = \dfrac{\mu_0}{4\pi}\dfrac{qc\mathbf{v}}{\sqrt{(\mathbf{r}\cdot\mathbf{v} - c^2 t)^2 - (c^2 - v^2)(c^2 t^2 - r^2)}} \end{cases}$$

10.3.2 The Fields of a Moving Point Charge

Using the Lienard-Wiechert potentials we can calculate the fields of a moving point charge.

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}(1-\hat{\mathscr{r}}\cdot\mathbf{v}/c)} \quad \text{and} \quad \mathbf{A}(\mathbf{r},t)=\frac{\mathbf{v}}{c^2}V(\mathbf{r},t)$$

Find: $\mathbf{E}=-\nabla V-\dfrac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B}=\nabla\times\mathbf{A}$

The separation vector: $\vec{\mathscr{r}}=\mathbf{r}-\mathbf{r}'=\mathbf{r}-\mathbf{w}(t_r)$ and $\mathbf{v}=\dot{\mathbf{w}}(t_r)$

The retarded time t_r: $\left|\mathbf{r}-\mathbf{w}(t_r)\right|=c(t-t_r)$

t_r is a function of $\mathbf{r}$ and t.

Gradient of the Scalar Potential

$$\nabla V = \frac{1}{4\pi\varepsilon_0}\frac{-q}{(\mathscr{r} - \vec{\mathscr{r}}\cdot\mathbf{v}/c)^2}\nabla(\mathscr{r} - \vec{\mathscr{r}}\cdot\mathbf{v}/c)$$

$$\nabla \mathscr{r} = \nabla c(t - t_r) = -c\nabla t_r$$

See Chap.1 p.23

$$\nabla(\vec{\mathscr{r}}\cdot\mathbf{v}) = \underbrace{(\vec{\mathscr{r}}\cdot\nabla)\mathbf{v}}_{\#1} + \underbrace{(\mathbf{v}\cdot\nabla)\vec{\mathscr{r}}}_{\#2} + \underbrace{\vec{\mathscr{r}}\times(\nabla\times\mathbf{v})}_{\#3} + \underbrace{\mathbf{v}\times(\nabla\times\vec{\mathscr{r}})}_{\#4}$$

$$\#1\quad (\vec{\mathscr{r}}\cdot\nabla)\mathbf{v} = (\mathscr{r}_x\frac{\partial}{\partial x} + \mathscr{r}_y\frac{\partial}{\partial y} + \mathscr{r}_z\frac{\partial}{\partial z})\mathbf{v}$$

$$= (\mathscr{r}_x\frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial x} + \mathscr{r}_y\frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial y} + \mathscr{r}_z\frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial z})$$

$$= \mathbf{a}(\vec{\mathscr{r}}\cdot\nabla t_r)$$

acceleration

1.2.6 Product Rules (II)

Chap.1 p.23

The product rule: $\begin{cases} \text{scalar}: fg \\ \text{vector}: f\mathbf{A} \end{cases}$

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx} \qquad \nabla(fg) = g\nabla f + f\nabla g$$

$$\nabla\cdot(f\mathbf{A}) = \nabla f\cdot\mathbf{A} + f(\nabla\cdot\mathbf{A}) \qquad \nabla\times(f\mathbf{A}) = \nabla f\times\mathbf{A} + f(\nabla\times\mathbf{A})$$

$$\begin{cases} \text{scalar}: \mathbf{A}\cdot\mathbf{B} \\ \text{vector}: \mathbf{A}\times\mathbf{B} \end{cases}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$$

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$$\#2\quad (\mathbf{v}\cdot\nabla)\vec{\boldsymbol{\mathscr{r}}} = (\mathbf{v}\cdot\nabla)\mathbf{r} - (\mathbf{v}\cdot\nabla)\mathbf{w}(t_r) = \mathbf{v} - (v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z})\mathbf{w}(t_r)$$

$$= \mathbf{v} - (v_x \frac{d\mathbf{w}}{dt_r}\frac{\partial t_r}{\partial x} + v_y \frac{d\mathbf{w}}{dt_r}\frac{\partial t_r}{\partial y} + v_z \frac{d\mathbf{w}}{dt_r}\frac{\partial t_r}{\partial z}) = \mathbf{v}(1 - (\mathbf{v}\cdot\nabla t_r))$$

$$\#3\quad \vec{\boldsymbol{\mathscr{r}}}\times(\nabla\times\mathbf{v}) = \vec{\boldsymbol{\mathscr{r}}}\times\left[(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z})\hat{\mathbf{x}} + (\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x})\hat{\mathbf{y}} + (\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})\hat{\mathbf{z}} \right]$$

$$= \vec{\boldsymbol{\mathscr{r}}}\times\left[(\frac{dv_z}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dv_y}{dt_r}\frac{\partial t_r}{\partial z})\hat{\mathbf{x}} + (\frac{dv_x}{dt_r}\frac{\partial t_r}{\partial z} - \frac{dv_z}{dt_r}\frac{\partial t_r}{\partial x})\hat{\mathbf{y}} + (\frac{dv_y}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dv_x}{dt_r}\frac{\partial t_r}{\partial y})\hat{\mathbf{z}} \right]$$

$$= \vec{\boldsymbol{\mathscr{r}}}\times\left(-\mathbf{a}\times\nabla t_r\right)$$

$$\#4\quad \mathbf{v}\times(\nabla\times\vec{\boldsymbol{\mathscr{r}}}) = \mathbf{v}\times\left[\begin{array}{l} (\frac{\partial(z-w_z)}{\partial y} - \frac{\partial(y-w_y)}{\partial z})\hat{\mathbf{x}} + (\frac{\partial(x-w_x)}{\partial z} \\ - \frac{\partial(z-w_z)}{\partial x})\hat{\mathbf{y}} + (\frac{\partial(y-w_y)}{\partial x} - \frac{\partial(x-w_x)}{\partial y})\hat{\mathbf{z}} \end{array} \right]$$

$$= \mathbf{v}\times\left(\mathbf{v}\times\nabla t_r\right)$$

$$\nabla(\vec{\mathscr{r}}\cdot\mathbf{v}) = \underbrace{(\vec{\mathscr{r}}\cdot\nabla)\mathbf{v}}_{\#1} + \underbrace{(\mathbf{v}\cdot\nabla)\vec{\mathscr{r}}}_{\#2} + \underbrace{\vec{\mathscr{r}}\times(\nabla\times\mathbf{v})}_{\#3} + \underbrace{\mathbf{v}\times(\nabla\times\vec{\mathscr{r}})}_{\#4}$$

$$= \mathbf{a}(\vec{\mathscr{r}}\cdot\nabla t_r) + \mathbf{v}(1-(\mathbf{v}\cdot\nabla t_r)) - \vec{\mathscr{r}}\times\left(\mathbf{a}\times\nabla t_r\right) + \mathbf{v}\times\left(\mathbf{v}\times\nabla t_r\right)$$

$$= \mathbf{v} + (\vec{\mathscr{r}}\cdot\mathbf{a} - v^2)\nabla t_r$$

$$\nabla t_r = -\nabla\frac{\mathscr{r}}{c} = -\frac{1}{c}\nabla\mathscr{r} = -\frac{1}{c}\nabla(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}})^{1/2} = -\frac{1}{2c(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}})^{1/2}}\nabla(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}})$$

$$= -\frac{1}{2c(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}})^{1/2}}2\left[\vec{\mathscr{r}}\times(\nabla\times\vec{\mathscr{r}}) + (\vec{\mathscr{r}}\cdot\nabla)\vec{\mathscr{r}}\right]$$

where

$$\begin{cases} \vec{\mathscr{r}}\times(\nabla\times\vec{\mathscr{r}}) = \vec{\mathscr{r}}\times(\mathbf{v}\times\nabla t_r) \\ (\vec{\mathscr{r}}\cdot\nabla)\vec{\mathscr{r}} = (\vec{\mathscr{r}}\cdot\nabla)(\mathbf{r}-\mathbf{w}(t_r)) = \vec{\mathscr{r}} - \mathbf{v}(\vec{\mathscr{r}}\cdot\nabla t_r) \end{cases}$$

$$\nabla t_r = -\frac{1}{c(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}})^{1/2}}\left[\vec{\mathscr{r}}\times(\mathbf{v}\times\nabla t_r) + \vec{\mathscr{r}} - \mathbf{v}(\vec{\mathscr{r}}\cdot\nabla t_r)\right]$$

$$= -\frac{1}{c\mathscr{r}}\left[\vec{\mathscr{r}} - (\vec{\mathscr{r}}\cdot\mathbf{v})\nabla t_r\right] \Rightarrow \nabla t_r = \frac{-\vec{\mathscr{r}}}{c\mathscr{r} - \vec{\mathscr{r}}\cdot\mathbf{v}}$$

$\mathbf{w}(t_r)$ is function of t_r.

P.14, $\nabla t_r = \frac{-1}{c}\nabla\mathscr{r} = -\frac{1}{c}\hat{\mathscr{r}}$ because $\mathbf{r}'$ is independet of t_r.

$$\nabla V = \frac{1}{4\pi\varepsilon_0} \frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}} \cdot \mathbf{v})^3} \left[(\mathscr{r}c - \vec{\mathscr{r}} \cdot \mathbf{v})\mathbf{v} - (c^2 - v^2 + \vec{\mathscr{r}} \cdot \mathbf{a})\vec{\mathscr{r}} \right]$$

Similar calculations

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{4\pi\varepsilon_0} \frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}} \cdot \mathbf{v})^3} \left[\begin{array}{l} (\mathscr{r}c - \vec{\mathscr{r}} \cdot \mathbf{v})(-\mathbf{v} + \vec{\mathscr{r}} \cdot \mathbf{a} / c) \\ + \frac{\mathscr{r}}{c}(c^2 - v^2 + \vec{\mathscr{r}} \cdot \mathbf{a})\mathbf{v} \end{array} \right]$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = \frac{q}{4\pi\varepsilon_0} \frac{\mathscr{r}}{(\vec{\mathscr{r}} \cdot \mathbf{u})^3} \left[(c^2 - v^2)\mathbf{u} + \vec{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a}) \right]$$

where $\mathbf{u} \equiv c\hat{\mathscr{r}} - \mathbf{v}$

Curl of the Vector Potential

$$\nabla\times\mathbf{A}=\frac{1}{c^2}\nabla\times(V\mathbf{v})=\frac{1}{c^2}\left(V(\nabla\times\mathbf{v})-\mathbf{v}\times\nabla V\right)$$

$$=-\frac{1}{c}\frac{q}{4\pi\varepsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\mathbf{u})^3}\vec{\mathscr{r}}\times\left[(c^2-v^2)\mathbf{v}+(\vec{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}-(\vec{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}\right]$$

$$=\frac{1}{c}\frac{q}{4\pi\varepsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\mathbf{u})^3}\vec{\mathscr{r}}\times\left[(c^2-v^2)\mathbf{u}+\vec{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right]=\frac{1}{c}\vec{\mathscr{r}}\times\mathbf{E}$$

where $\vec{\mathscr{r}}\times\mathbf{v}=-\vec{\mathscr{r}}\times\mathbf{u}$.

$$\mathbf{B}=\frac{1}{c}\hat{\mathscr{r}}\times\mathbf{E}$$

The magnetic field of a point charge is always perpendicular to the electric field, and to the vector from the retarded point.

Generalized Coulomb Field

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{\mathscr{r}}{(\vec{\mathscr{r}} \cdot \mathbf{u})^3} \left[\underbrace{(c^2 - v^2)\mathbf{u}}_{\text{velocity field}} + \underbrace{\vec{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})}_{\substack{\text{acceleration field} \\ \text{radiation field}}} \right]$$

$\mathbf{v} = 0 \text{ and } \mathbf{a} = 0$

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{\mathscr{r}}{(c\mathscr{r})^3} (c^3)\hat{\mathscr{r}} = \frac{q}{4\pi\varepsilon_0} \frac{1}{\mathscr{r}^2} \hat{\mathscr{r}}$$

Example 10.4

Calculate the electric and magnetic fields of a point charge moving with constant velocity.

Solution:

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{\mathscr{r}}{(\vec{\mathscr{r}} \cdot \mathbf{u})^3}(c^2 - v^2)\mathbf{u}, \text{ since } \mathbf{a} = 0.$$

$$\mathbf{u} = c\hat{\mathscr{r}} - \mathbf{v}$$

$$\Rightarrow \mathscr{r}\mathbf{u} = c\vec{\mathscr{r}} - \mathscr{r}\mathbf{v} = c(\mathbf{r} - \mathbf{v}t_r) - c(t - t_r)\mathbf{v} = c(\mathbf{r} - \mathbf{v}t);$$

$$\Rightarrow \vec{\mathscr{r}} \cdot \mathbf{u} = c\mathscr{r} - \vec{\mathscr{r}} \cdot \mathbf{v} = Rc\sqrt{1 - v^2 \sin^2\theta / c^2} \text{ (Prob. 10.16)}$$

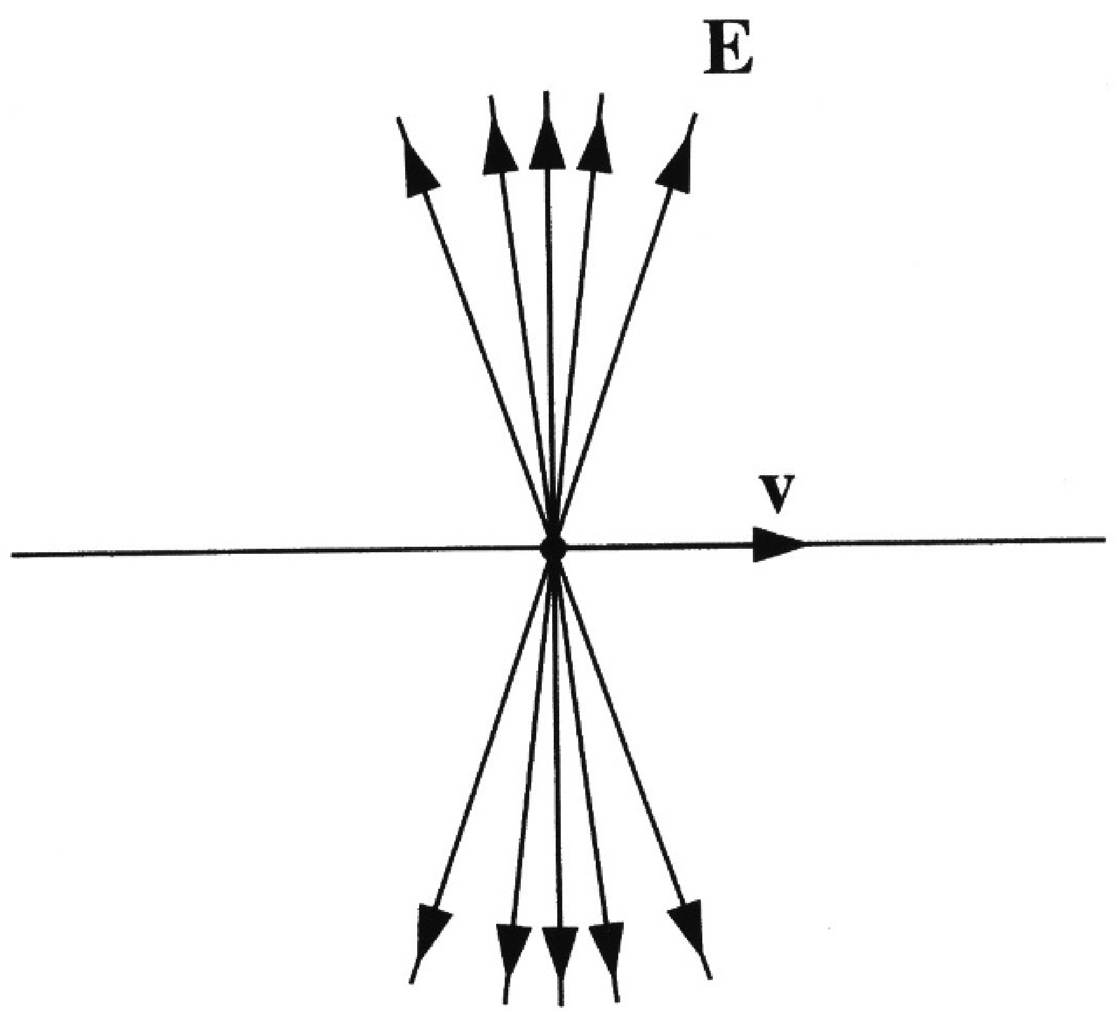
where θ is the angle between $\mathbf{R}$ and $\mathbf{v}$.

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2\theta / c^2)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}, \quad \text{where } \mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$$

Fields of a Moving Point Charge

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2\theta / c^2)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2},$$

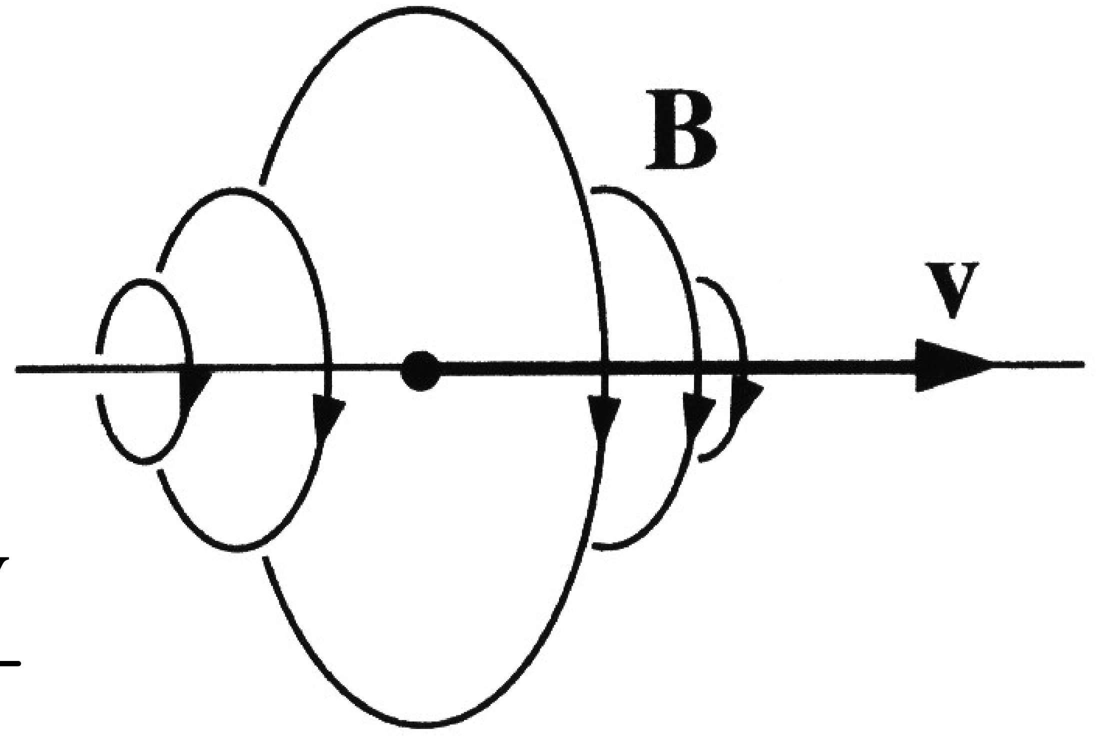
where $\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$



Obtain the same result by using the Lorentz transformation. Chap.12

$$\mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})$$

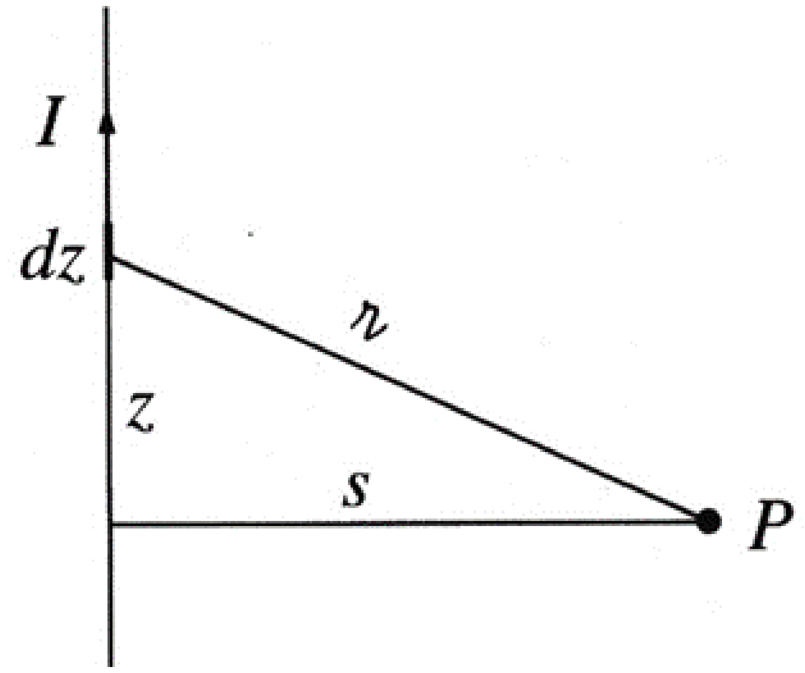
since $\hat{\boldsymbol{\mathscr{r}}} = \frac{\mathbf{r} - \mathbf{v}t_r}{\mathscr{r}} = \frac{(\mathbf{r} - \mathbf{v}t) + (t - t_r)\mathbf{v}}{\mathscr{r}} = \frac{\mathbf{R}}{\mathscr{r}} + \frac{\mathbf{v}}{c}$



Homework of Chap.10

Problem 10.4 Suppose $V = 0$ and $\mathbf{A} = A_0 \sin(kx - \omega t)\hat{\mathbf{y}}$, where A_0, ω, and k are constants. Fine **E** and **B**, and check that they satisfy Maxwell's equations in vacuum. What condition must you impose on ω and k?

Problem 10.11



(a) Suppose the wire in Ex. 10.2 carries a linearly increasing current

$$I(t) = kt,$$

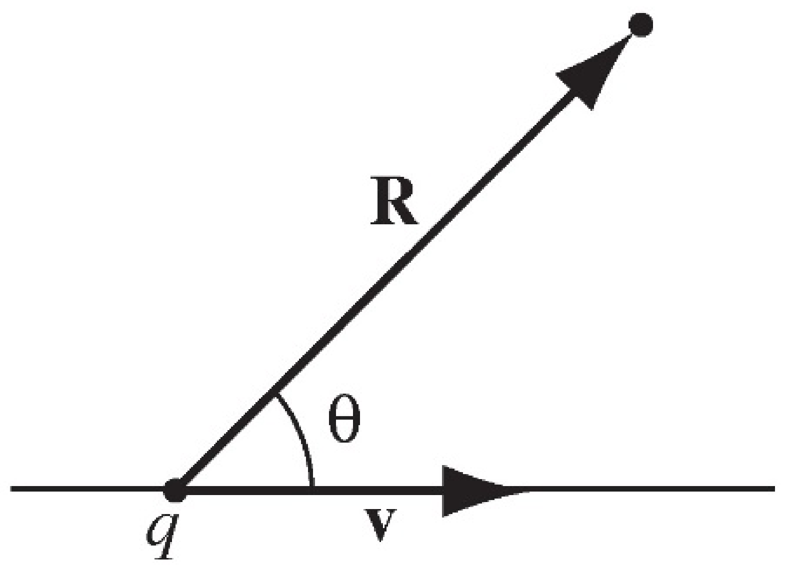
for $t > 0$. Find the electric and magnetic fields generated.

(b) Do the same for the case of a sudden burst of current:

$$I(t) = q_0 \delta(t).$$

Problem 10.16 Show that the scalar potential of a point charge moving with constant velocity (Eq. 10.49) can be written more simply as?

$$V(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \frac{q}{R\sqrt{1 - v^2 \sin^2\theta / c^2}}, \qquad (10.51)$$



where $\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$ is the vector from the present (!) position of the particle to thefield point $\mathbf{r}$, and θ is the angle between **R** and **v**.

Note that for nonrelativistic velocities ($v^2 \ll c^2$),

$$V(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \frac{q}{R}$$

Homework of Chap.10

Problem 10.15 A particle of charge q moves in a circle of radius a at constant angular velocity ω. (Assume that the circle lies in the xy plane, centered at the origin, and at time $t=0$ the charge is at $(a,0)$, on the positive x axis.) Find the Lienard-Wiechert potentials for points on the z axis.

Problem 10.27 Check that the potentials of a point charge moving at constant velocity (Eqs. 10.49 and 10.50) satisfy the Lorenz gauge condition (Eq. 10.12).

Problem 10.28 One particle, of charge q_1, is held at rest at the origin. Another particle, of charge q_2, approaches along the x axis, in hyperbolic motion:

$$x(t) = \sqrt{b^2 + (ct)^2};$$

it reaches the closest point, b, at time $t=0$, and then returns out to infinity.

(a) What is the force F_2 on q_2 (due to q_1) at time t?

(b) What total impulse ($I_2 = \int_{-\infty}^{\infty} F_2 dt$) is delivered to q_2 by q_1?

(c) What is the force F_1 on q_1 (due to q_2) at time t?

(d) What total impulse ($I_1 = \int_{-\infty}^{\infty} F_1 dt$) is delivered to q_1 by q_2? [*Hint*: It might help to review Prob. 10.17 before doing this integral. *Answer*: $I_2 = -I_1 = q_1 q_2 / 4\varepsilon_0 bc$]