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Chapter 1 Vector Analysis

1.1 Vector Algebra: 1.1.1 Vector Operations (I)

Vectors: Quantities have both magnitude and direction, denoted by **boldface** ($\mathbf{A}$, $\mathbf{B}$, and so on).

Scalars: Quantities have magnitude but no direction denoted by ordinary type.

In diagrams, vectors are denoted by arrows: the length of the arrow is proportional to the magnitude of the vector, and the arrowhead indicates its direction.

Minus $\mathbf{A}$ ($-\mathbf{A}$) is a vector with the same magnitude as $\mathbf{A}$ but of opposite direction.

Vectors have magnitude and direction but not location.

Khan Academy: https://www.khanacademy.org/math/linear-algebra

1.1.1 Vector Operations (II)

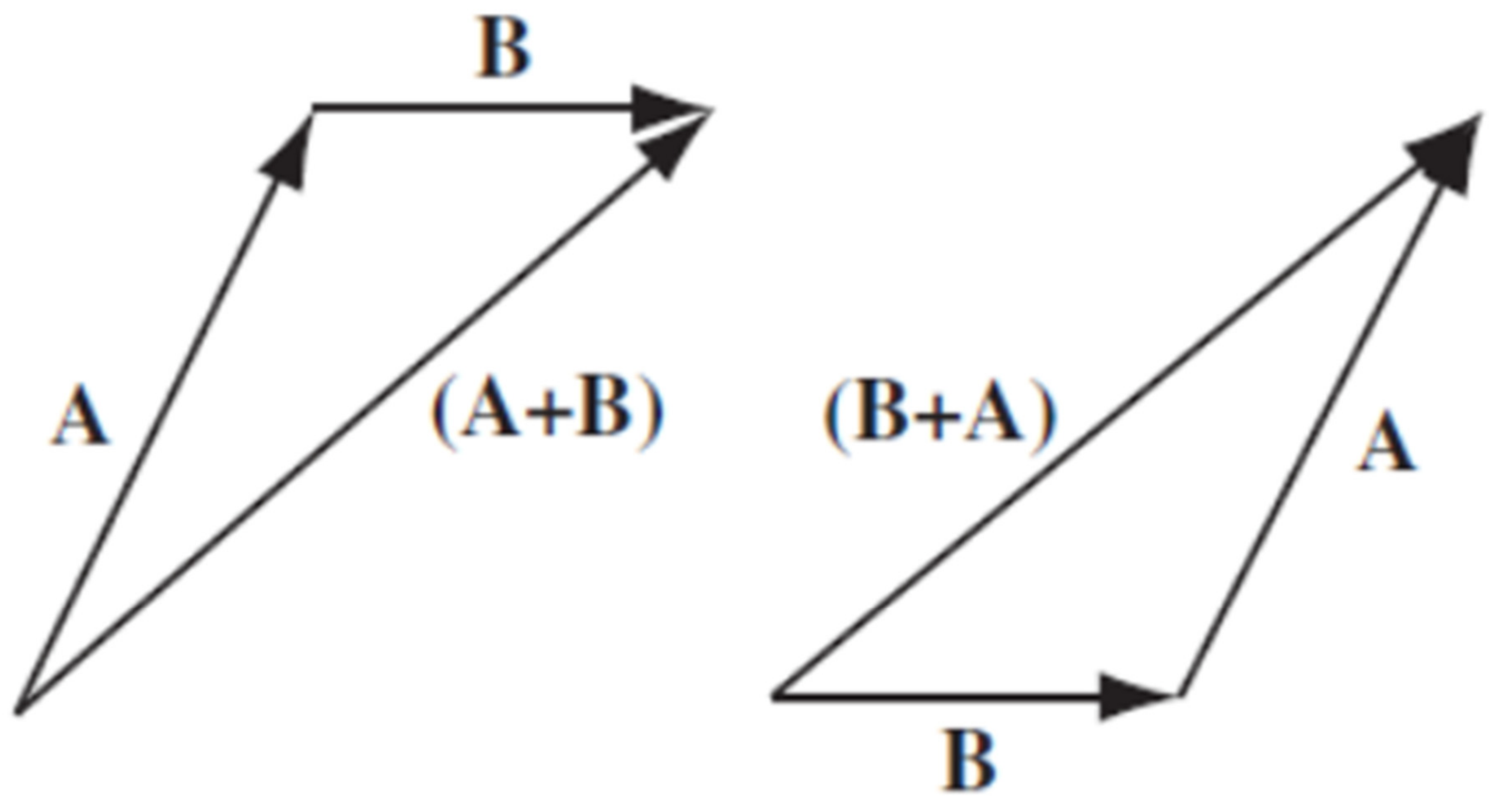
(i) Addition of two vectors:

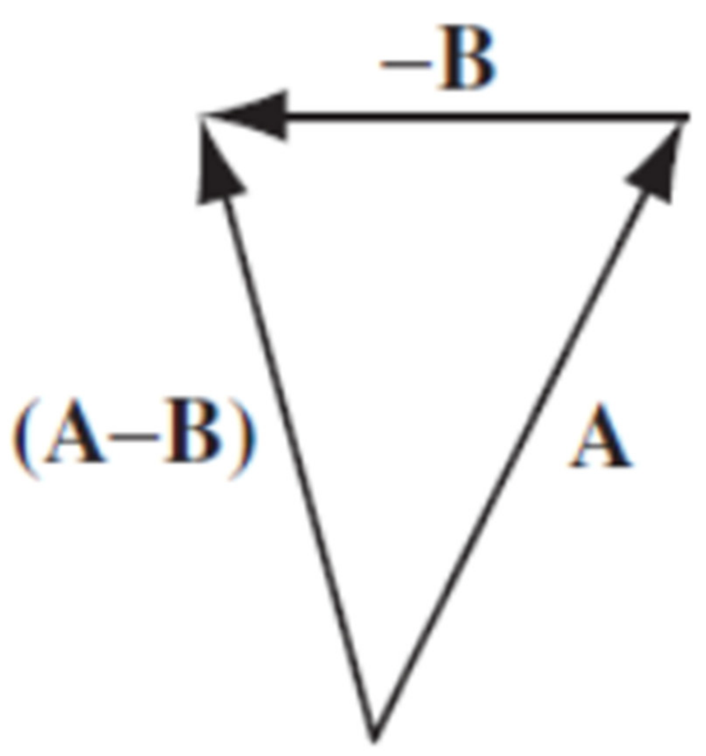
Place the tail of $\mathbf{B}$ at the head of $\mathbf{A}$.

Commutative: $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$

Associative: $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$

$$\mathbf{A}-\mathbf{B}=\mathbf{A}+(-\mathbf{B})$$





1.1.1 Vector Operations (III)

(ii) Multiplication by a scalar:

Multiplies the magnitude but leaves the direction unchanged.

Distributive: $a(\mathbf{A}+\mathbf{B}) = a\mathbf{A} + a\mathbf{B}$

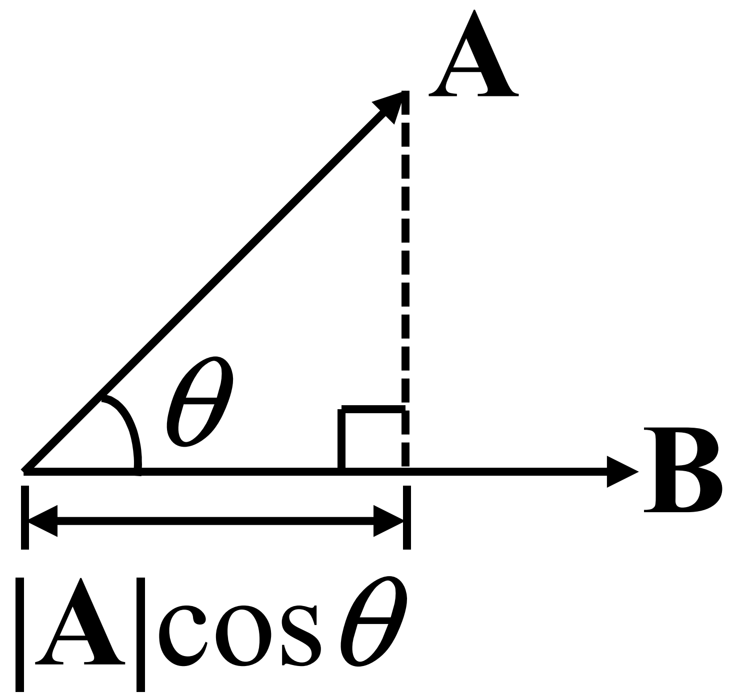
(iii) Dot product of two vectors (scalar product):

The dot product of two vectors is defined by $\mathbf{A}\cdot\mathbf{B} \equiv AB\cos\theta$,

where θ is the angle they form when placed tail-to-tail.

Commutative: $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$

Distributive: $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$



1.1.1 Vector Operations (IV)

(iv) Cross product of two vectors (vector product):

The cross product of two vectors is defined by

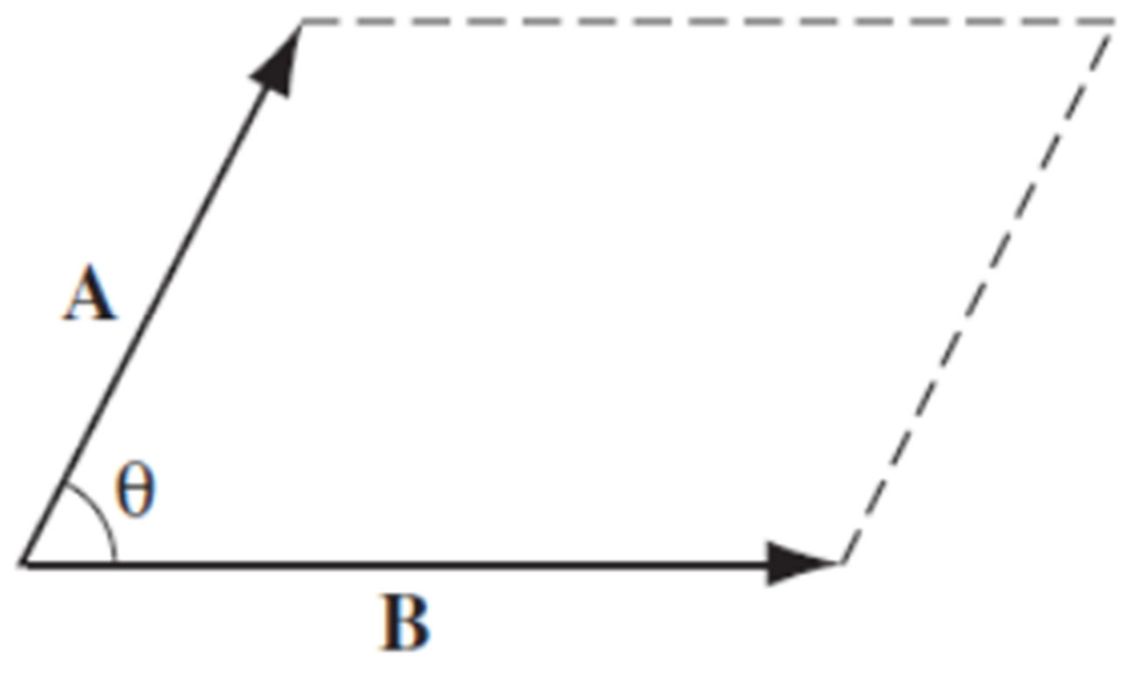
$\mathbf{A}\times\mathbf{B} \equiv AB\sin\theta\,\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector pointing perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$.

(pronounced “n-hat")

A hat is used to designate the unit vector and its direction is determined by the **right-hand** rule.

Distributive: $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$

not commutative: $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$

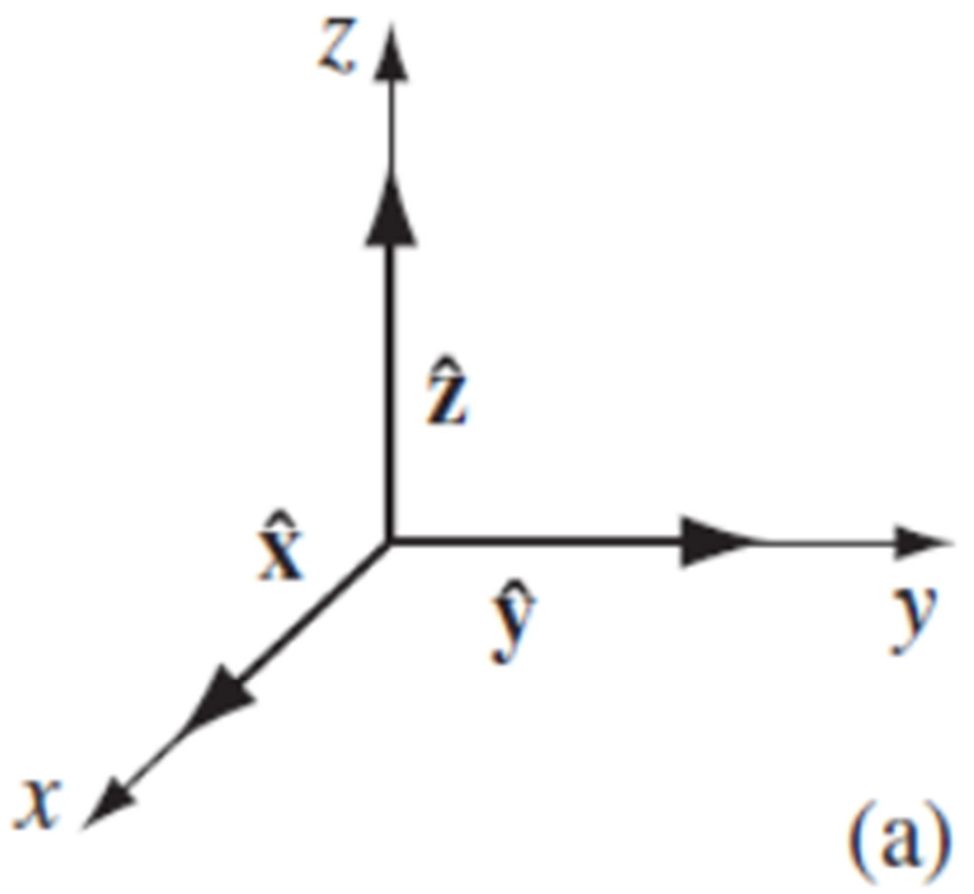


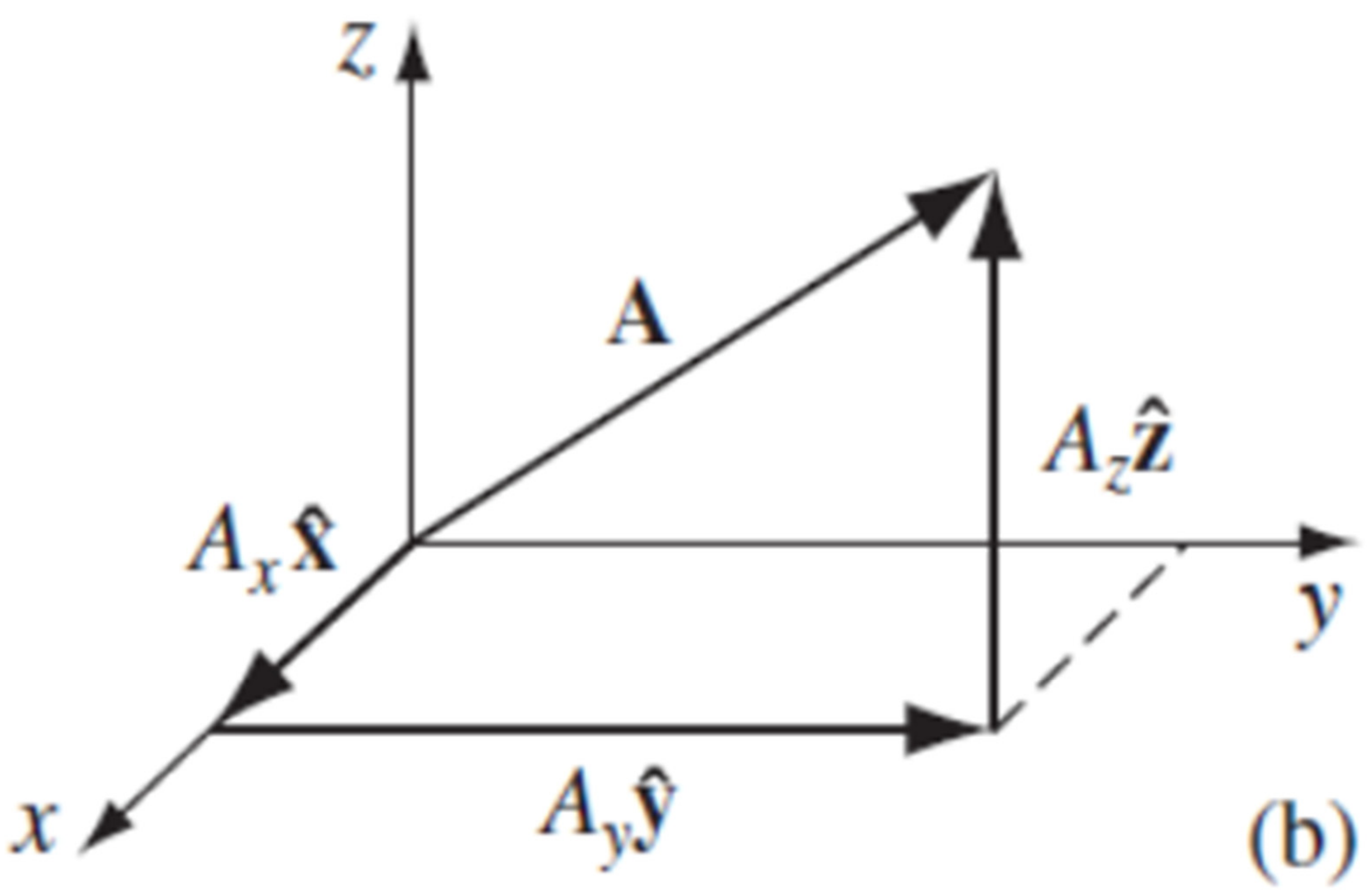
1.1.2 Vector Algebra: Component form (I)

Let $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ be unit vectors parallel to the x, y, and z axes, respectively. An arbitrary vector $\mathbf{A}$ can be expressed in terms of these basis vectors.

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

The numbers A_x, A_y, and A_z are called components.





1.1.2 Vector Algebra: Component form (II)

Reformulate the vector operations as a rule for manipulating components:

(i) To add vectors, add like components.

$$\begin{aligned}\mathbf{A}+\mathbf{B} &= (A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}})+(B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}}) \\ &= (A_x+B_x)\hat{\mathbf{x}}+(A_y+B_y)\hat{\mathbf{y}}+(A_z+B_z)\hat{\mathbf{z}}\end{aligned}$$

(ii) To multiply by a scalar, multiply each component.

$$\begin{aligned}a\mathbf{A} &= a(A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}}) \\ &= aA_x\hat{\mathbf{x}}+aA_y\hat{\mathbf{y}}+aA_z\hat{\mathbf{z}}\end{aligned}$$

1.1.2 Vector Algebra: Component form (III)

(iii) To calculate the dot product, multiply like components, and add.

$$\mathbf{A}\cdot\mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}})\cdot(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$
$$= A_xB_x + A_yB_y + A_zB_z$$

(iv) To calculate the cross product, form the determinant whose first row is $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$, whose second row is $\mathbf{A}$ (in component form), and whose third row is $\mathbf{B}$.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{aligned} &(A_yB_z - A_zB_y)\hat{\mathbf{x}} \\ &+(A_zB_x - A_xB_z)\hat{\mathbf{y}} \\ &+(A_xB_y - A_yB_x)\hat{\mathbf{z}} \end{aligned}$$

1.1.3 Triple Products (I)

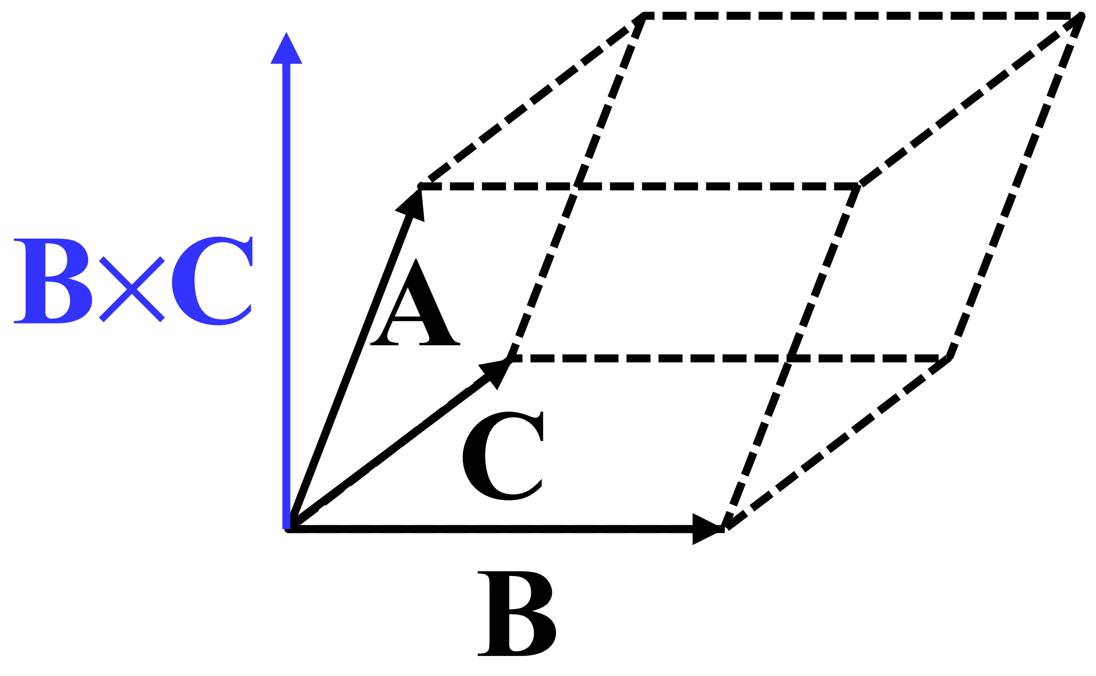
Since the cross product of two vectors is itself a vector, it can be dotted or crossed with a third vector to form a triple product.

(i) **Scalar triple product:** $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. Geometrically, $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$ is the volume of a parallelepiped generated by these three vectors as shown below.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

In component form

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$



Parallelepiped
平行六面體

1.1.3 Triple Products (II)

(ii) **Vector triple product:** $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. The vector triple product can be simplified by the so-called **BAC-CAB** rule.

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Notice that $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}\neq\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=-\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=-\mathbf{A}(\mathbf{B}\cdot\mathbf{C})+\mathbf{B}(\mathbf{A}\cdot\mathbf{C})$$

https://en.wikipedia.org/wiki/Triple_product

Problem 1.6 Under what conditions does

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=\mathbf{A}\times(\mathbf{B}\times\mathbf{C})?$$

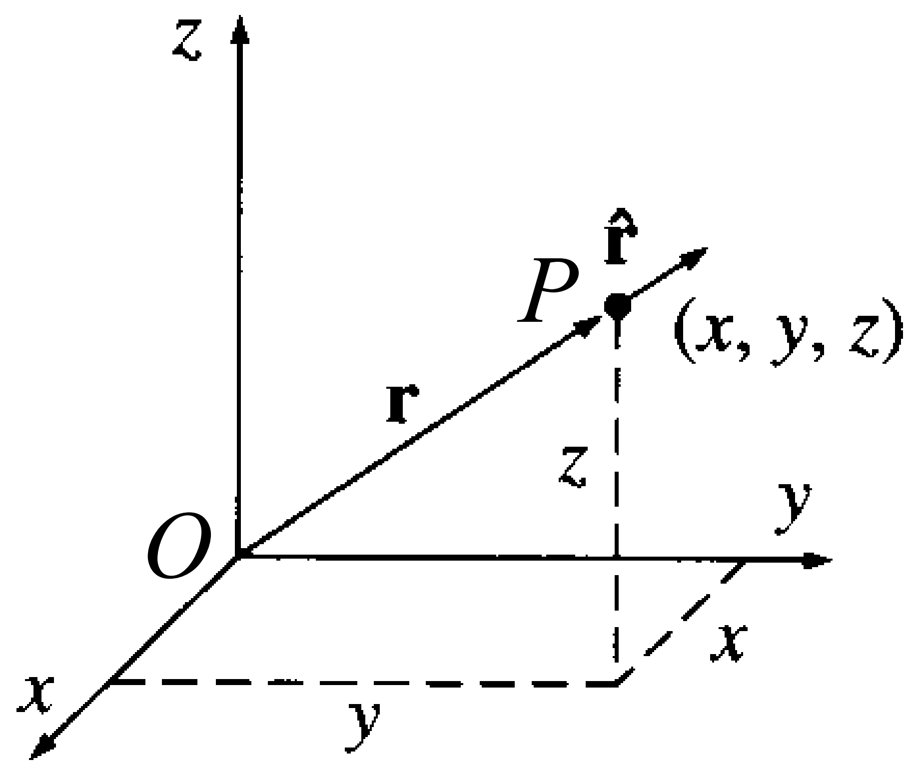
Ans: Either $\mathbf{A}$ is parallel to $\mathbf{C}$,

or $\mathbf{B}$ is perpendicular to $\mathbf{A}$ and $\mathbf{C}$.

1.1.4 Position, Displacement, and Separation Vectors (I)

Position vector: The vector to point *P* from the origin *O*.

$$\mathbf{r} \equiv x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$



Its magnitude (the distance from the origin)

$$r = \sqrt{\mathbf{r}\cdot\mathbf{r}} \equiv \sqrt{x^2 + y^2 + z^2}$$

Its direction unit vector (pointing radially outward)

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

The **infinitesimal displacement vector,** from (x, y, z) to $(x+dx, y+dy, z+dz)$, is

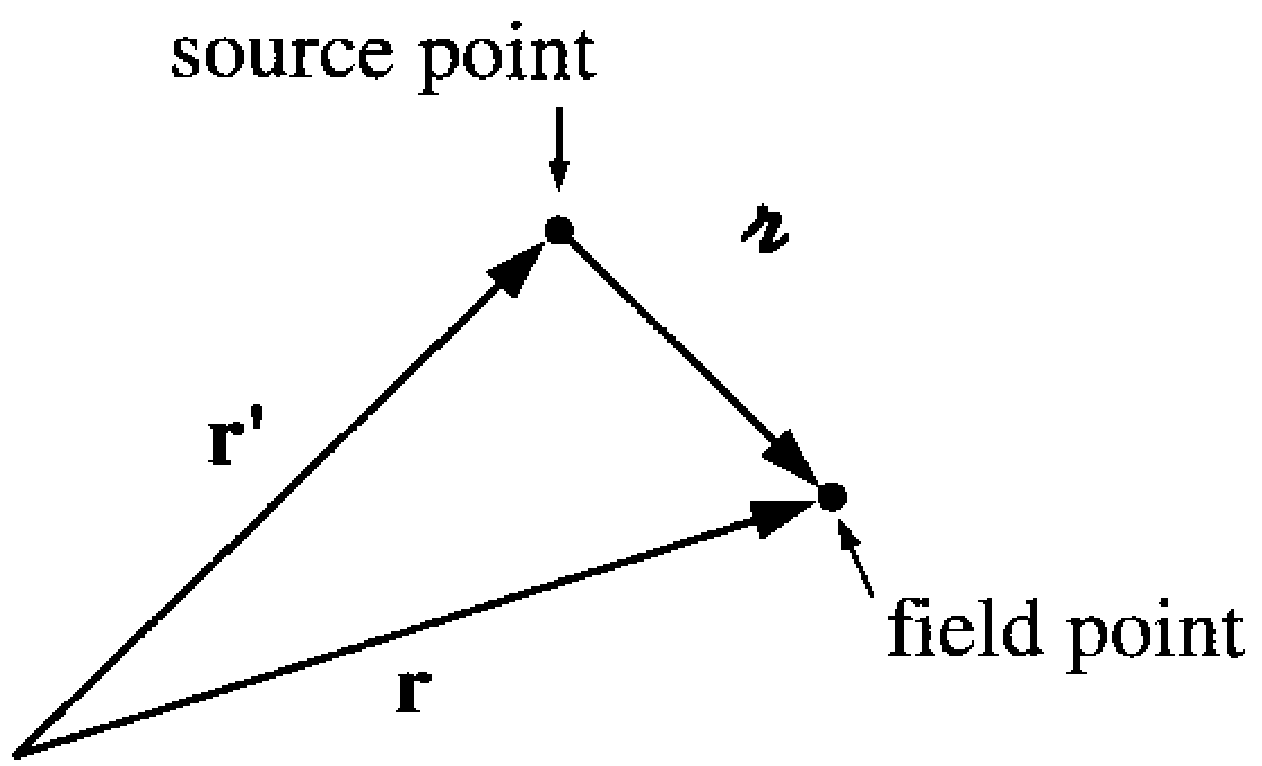
$$d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$$

1.1.4 Position, Displacement, and Separation Vectors (II)

In electrodynamics one frequently encounters problems involving two points:

A source point, $\mathbf{r}'$, where an electric charge is located.
A field point, $\mathbf{r}$, at which you are calculating the electric field.

A short-hand notation for the separation vector from the source point to the field point is

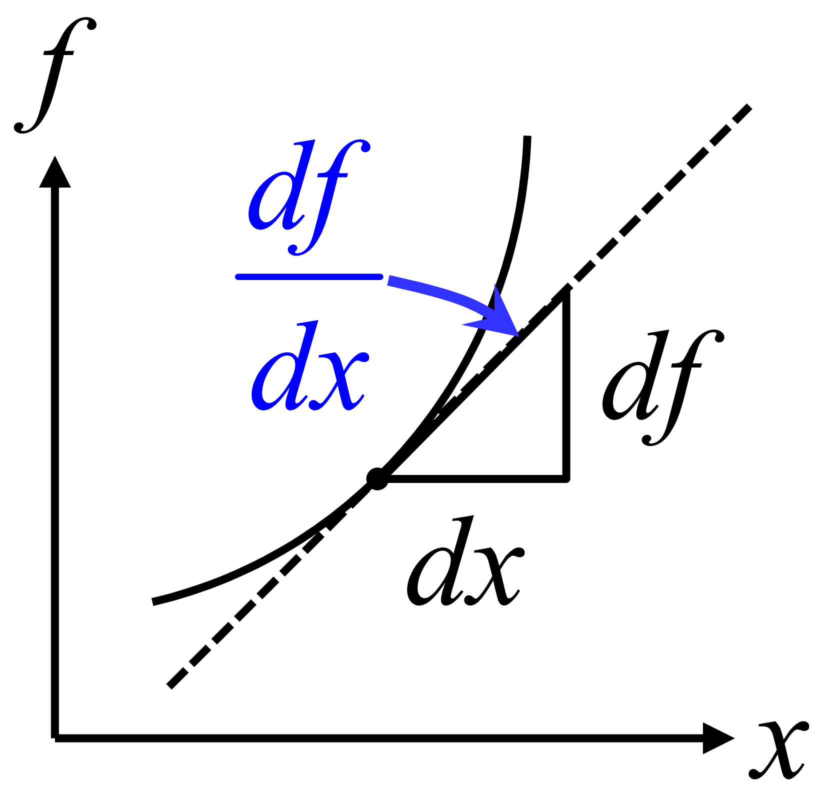


$$\vec{\boldsymbol{\mathscr{r}}} \equiv \mathbf{r} - \mathbf{r}', \text{ magnitude } \mathscr{r} = |\mathbf{r} - \mathbf{r}'|$$

$$\text{unit vector in the direction from } \mathbf{r}' \text{ to } \mathbf{r} \text{ is } \hat{\boldsymbol{\mathscr{r}}} = \frac{\vec{\boldsymbol{\mathscr{r}}}}{\mathscr{r}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

1.2 Differential Calculus
1.2.1 "Ordinary" Derivatives

Suppose we have a function of one variable, $f(x)$. What does the derivative, df/dx, do for us?



Ans: It tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx.

$$df = \left(\frac{df}{dx}\right)dx$$

In words, if we change x by an amount dx, then, f changes by an amount df.

The derivative df/dx is the slope of the graph of f versus x.

1.2.2 Gradient (I)

Khan Academy: Gradient

Suppose we have a function of three variables. What does the derivative mean in this case?

A mountain hill

$H(x, y, z)$

A theorem on partial derivatives states that

$$dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy + \frac{\partial H}{\partial z}dz$$

$$= \left(\frac{\partial H}{\partial x}\hat{\mathbf{x}} + \frac{\partial H}{\partial y}\hat{\mathbf{y}} + \frac{\partial H}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}})$$

$$= (\nabla H) \cdot (d\mathbf{l})$$

The gradient of H is a vector quantity, with three components.

$$\nabla H = \frac{\partial H}{\partial x}\hat{\mathbf{x}} + \frac{\partial H}{\partial y}\hat{\mathbf{y}} + \frac{\partial H}{\partial z}\hat{\mathbf{z}}$$

1.2.2 Gradient (II)

Geometrical interpretation: Like any vector, the gradient has magnitude and direction.

A dot product in abstract form is: $dH = \nabla H \cdot d\mathbf{l} = |\nabla H||d\mathbf{l}|\cos\theta$ where θ is the angle between ∇H and $d\mathbf{l}$.

If we fix the magnitude $|d\mathbf{l}|$ and search around in various directions (that is, vary θ), the maximum change in dH eventually occurs when $\theta = 0$). The gradient ∇H points in the direction of maximum increase of the function H.

Analogous to the derivative of one variable, a vanishing derivative signals a maximum (a summit), a minimum (a valley), or an inflection (a saddle point or a shoulder).

Example 1.3 & Problem 1.13

Example 1.3 Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$

Ans : $\boxed{\nabla r = \frac{\partial r}{\partial x}\hat{\mathbf{x}} + \frac{\partial r}{\partial y}\hat{\mathbf{y}} + \frac{\partial r}{\partial z}\hat{\mathbf{z}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}}$

Problem 1.13 Let $\vec{\mathscr{r}} \equiv (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}$

Show that

(a) $\nabla \mathscr{r}^2 = ?$

$$\boxed{\begin{aligned} \nabla \mathscr{r}^2 &= \nabla[(x - x')^2 + (y - y')^2 + (z - z')^2] \\ &= 2(x - x')\hat{\mathbf{x}} + 2(y - y')\hat{\mathbf{y}} + 2(z - z')\hat{\mathbf{z}} = 2\vec{\mathscr{r}} \end{aligned}}$$

(b) $\nabla(1/\mathscr{r}) = ?$

$$\boxed{\begin{aligned} \nabla \frac{1}{\mathscr{r}} &= \frac{-\nabla \mathscr{r}}{\mathscr{r}^2} = \frac{-\nabla\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}{(x - x')^2 + (y - y')^2 + (z - z')^2} \\ &= -\tfrac{1}{2}[2(x - x')\hat{\mathbf{x}} + 2(y - y')\hat{\mathbf{y}} + 2(z - z')\hat{\mathbf{z}}]/\mathscr{r}^3 \\ &= -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \end{aligned}}$$

1.2.3 The Operator ∇ (I)

The gradient has the formal appearance of a vector, ∇, "multiplying", a scalar H.

$$\nabla H = \underbrace{(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}}_{\textbf{del}})H$$

∇ is a vector operator that acts upon H, not a vector that multiplies H.

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

∇ mimics the behavior of an ordinary vector in virtually every way, if we translate "multiply" by "act upon".

It is a marvelous piece of notational simplification.

1.2.3 The Operator ∇ (II)

An ordinary vector $\mathbf{A}$ can multiply in three ways:

1. Multiply a scalar a : $a\mathbf{A}$
2. Multiply another vector (dot product): $\mathbf{A} \cdot \mathbf{B}$
3. Multiply another vector (cross product): $\mathbf{A} \times \mathbf{B}$

Correspondingly, there are three ways the operator ∇ can act:

1. On a scalar function H: ∇H (**gradient** 梯度)
2. On a vector function (dot product): $\nabla \cdot \mathbf{v}$ (**divergence** 散度)
3. On a vector function (cross product): $\nabla \times \mathbf{v}$ (**curl** 旋度)

1.2.4 The Divergence

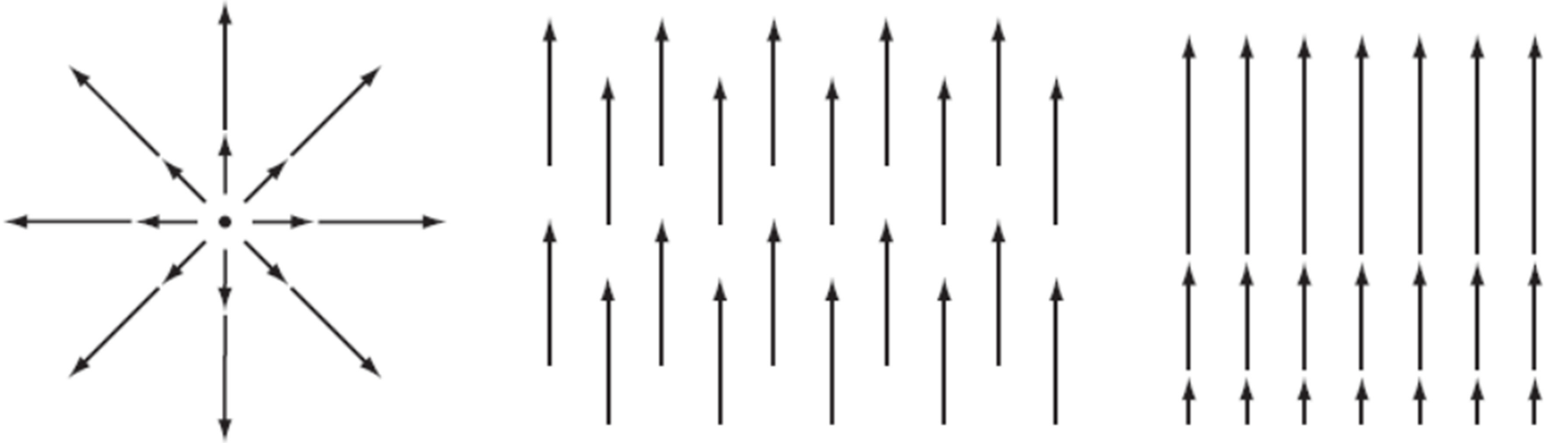
Khan Academy: Divergence

Divergence of a vector $\mathbf{v}$ is:

$$\nabla \cdot \mathbf{v} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}\right)$$

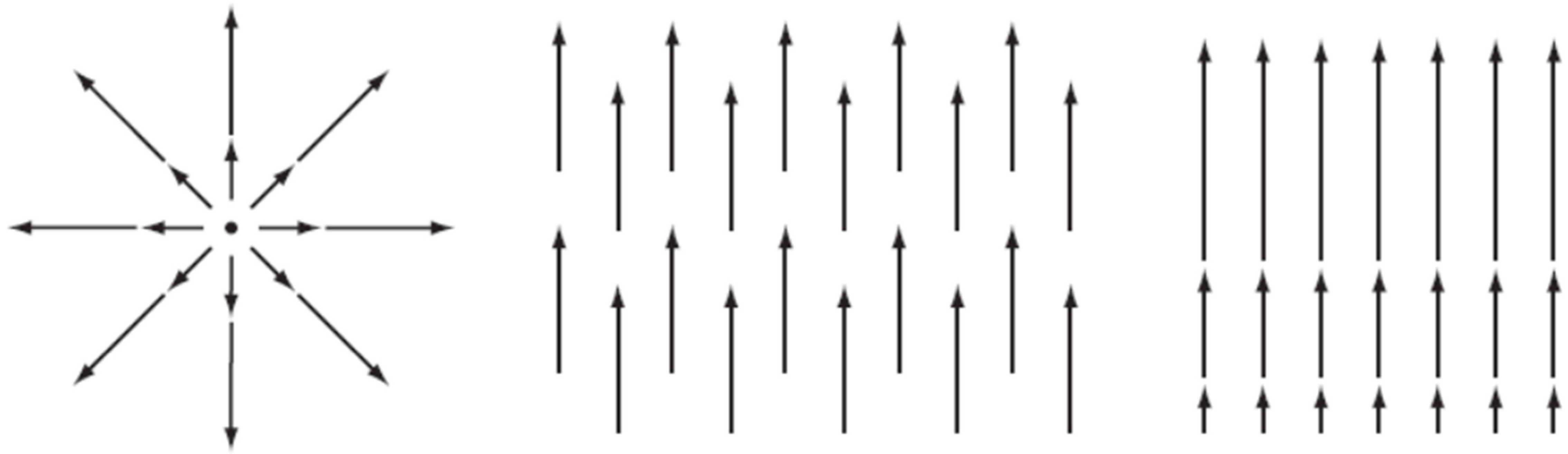
$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$\nabla\cdot\mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ spreads out from the point in question.



(a) positive (b) zero (c) positive

Example 1.4



Example 1.4 Suppose the functions in above three figures are $\mathbf{v}_a = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, $\mathbf{v}_b = \hat{\mathbf{z}}$, $\mathbf{v}_c = z\hat{\mathbf{z}}$. Calculate their divergences.

Ans: $\nabla \cdot \mathbf{v}_a = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 3;$

$$\nabla \cdot \mathbf{v}_b = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial 1}{\partial z} = 0;$$

$$\nabla \cdot \mathbf{v}_c = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial z}{\partial z} = 1.$$

Prob. 1.15

(a) $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$

$\nabla \cdot \mathbf{v}_a = ?$

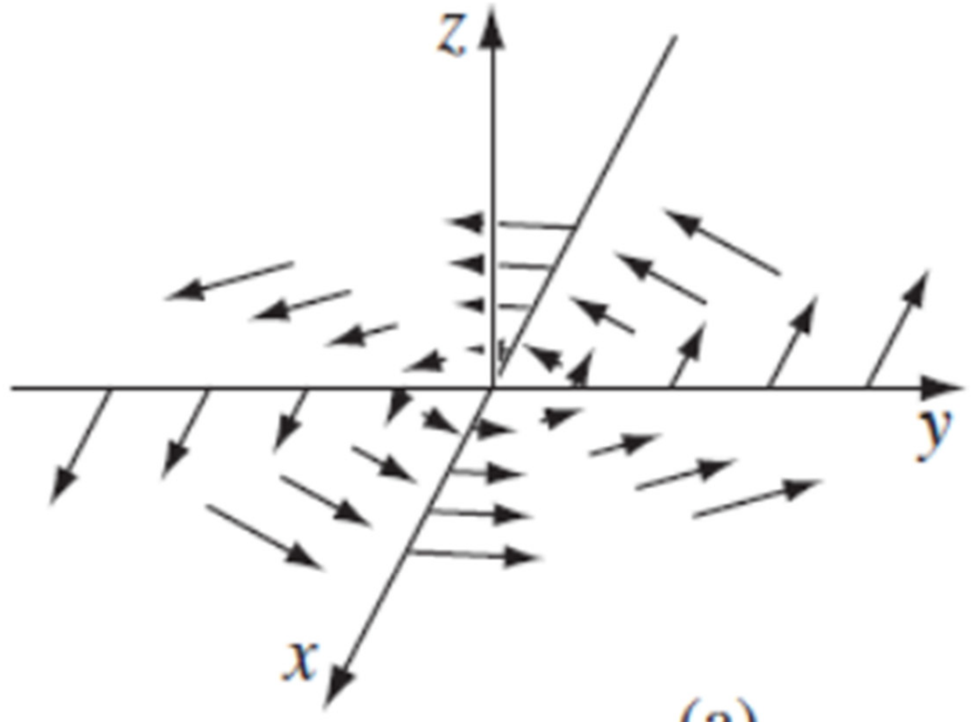
1.2.5 The Curl

Khan Academy: Curl

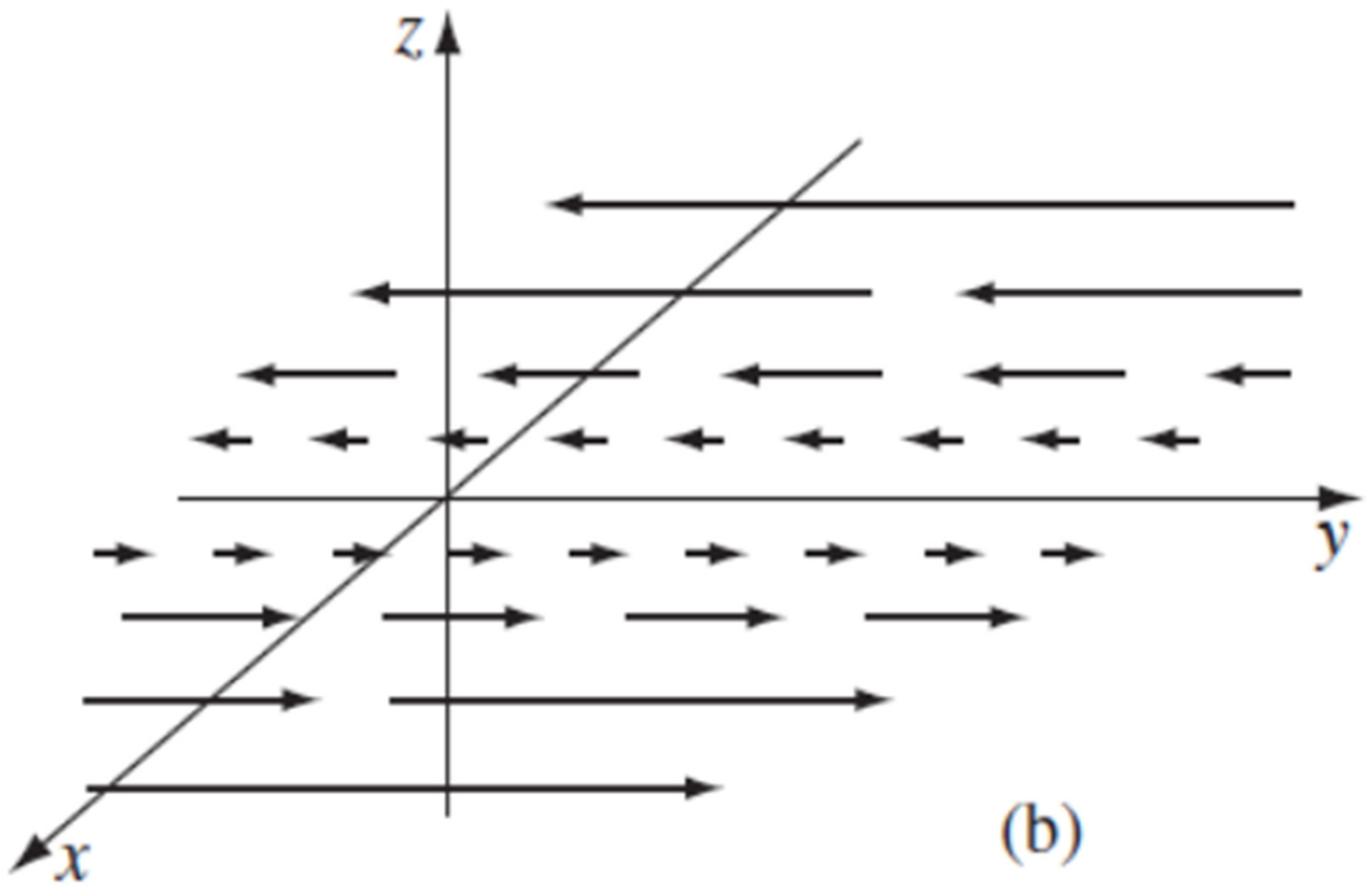
Curl of a vector $\mathbf{v}$ is:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \hat{\mathbf{x}}(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}) + \hat{\mathbf{y}}(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}) + \hat{\mathbf{z}}(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})$$

$\nabla\times\mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ curls around the point in question.

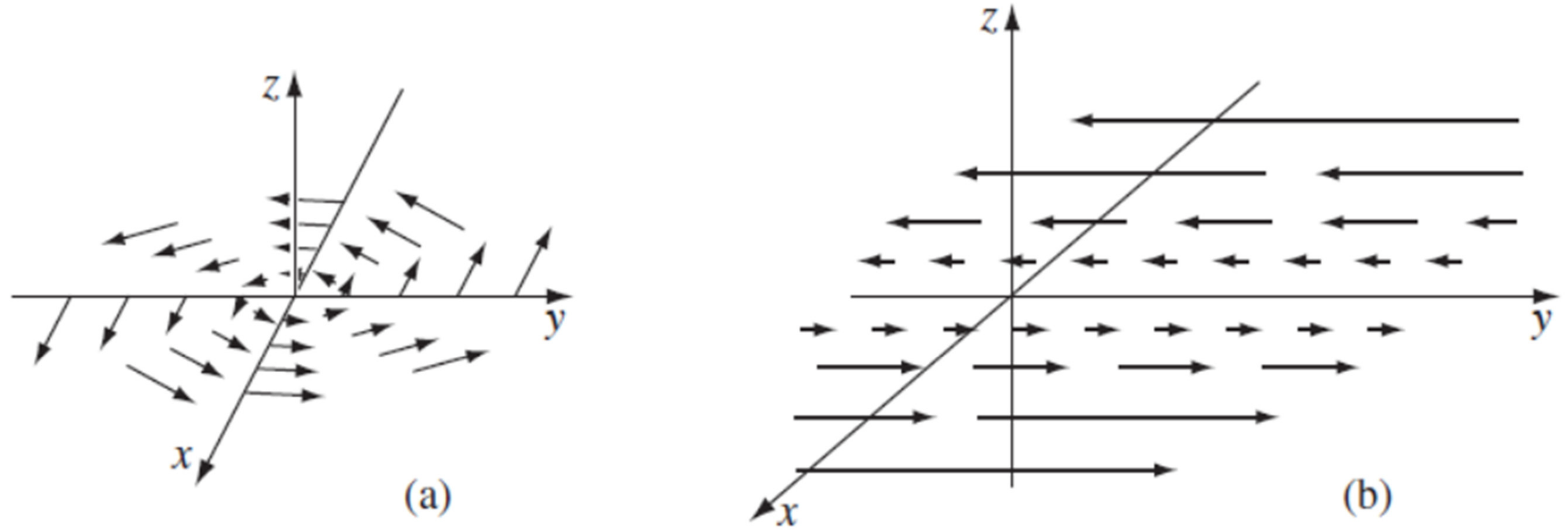


(a)



(b)

Example 1.5



Example 1.5 Suppose the functions in above two figures are $\mathbf{v}_a = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}, \quad \mathbf{v}_b = x\hat{\mathbf{y}}$. Calculate their curls.

Ans: $$\nabla \times \mathbf{v}_a = \hat{\mathbf{x}}\left(\frac{\partial 0}{\partial y} - \frac{\partial x}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial(-y)}{\partial z} - \frac{\partial 0}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) = 2\hat{\mathbf{z}}$$

$$\nabla \times \mathbf{v}_b = \hat{\mathbf{x}}\left(\frac{\partial 0}{\partial y} - \frac{\partial 0}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial 0}{\partial z} - \frac{\partial 0}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial x}{\partial x} - \frac{\partial 0}{\partial y}\right) = \hat{\mathbf{z}}$$

1.2.6 Product Rules (I)

The sum rule:

$$\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx} \qquad \nabla(f+g)=\nabla f+\nabla g$$

$$\nabla\cdot(\mathbf{A}+\mathbf{B})=\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B} \qquad \nabla\times(\mathbf{A}+\mathbf{B})=\nabla\times\mathbf{A}+\nabla\times\mathbf{B}$$

The rule for multiplying by a constant k:

$$\frac{d}{dx}(kf)=k\frac{df}{dx} \qquad \nabla(kf)=k\nabla f$$

$$\nabla\cdot(k\mathbf{A})=k\nabla\cdot\mathbf{A} \qquad \nabla\times(k\mathbf{A})=k\nabla\times\mathbf{A}$$

1.2.6 Product Rules (II)

The product rule: $\begin{cases} \text{scalar}: fg \\ \text{vector}: f\mathbf{A} \end{cases}$

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx} \qquad \nabla(fg) = g\nabla f + f\nabla g$$

$$\nabla \cdot (f\mathbf{A}) = \nabla f \cdot \mathbf{A} + f(\nabla \cdot \mathbf{A}) \qquad \nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} + f(\nabla \times \mathbf{A})$$

$$\begin{cases} \text{scalar}: \mathbf{A}\cdot\mathbf{B} \\ \text{vector}: \mathbf{A}\times\mathbf{B} \end{cases}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$$

Chaps. 8 and 10

1.2.6 Product Rules (III)

The quotient rule:

$$\begin{cases}\text{scalar}: \dfrac{f}{g} \quad \text{vector}: \dfrac{\mathbf{A}}{g}\end{cases}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\dfrac{df}{dx} - f\dfrac{dg}{dx}}{g^2}$$

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

$$\nabla\cdot\left(\frac{\mathbf{A}}{g}\right) = \frac{g(\nabla\cdot\mathbf{A}) - \mathbf{A}\cdot\nabla g}{g^2}$$

$$\nabla\times\left(\frac{\mathbf{A}}{g}\right) = \frac{g(\nabla\times\mathbf{A}) - (\nabla g\times\mathbf{A})}{g^2} = \frac{g(\nabla\times\mathbf{A}) + \mathbf{A}\times\nabla g}{g^2}$$

1.2.7 Second Derivatives (I)

By applying ∇ twice, we can construct five species of second derivatives.

Three first derivatives $\boldsymbol{\nabla} T, \quad \boldsymbol{\nabla} \cdot \mathbf{v}, \quad \boldsymbol{\nabla} \times \mathbf{v}$

(1) Divergence of gradient: $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T)$ $\longleftarrow$ very important

(2) Curl of gradient: $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T)$ $\longleftarrow$ always zero

(3) Gradient of divergence: $\boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{v})$ $\longleftarrow$ Chaps. 8 and 10

(4) Divergence of curl: $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v})$ $\longleftarrow$ always zero

(5) Curl of curl: $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v})$ $\longleftarrow$ reduce to others

1.2.7 Second Derivatives (II)

$$(1)\ \nabla\cdot(\nabla T)=(\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}+\hat{\mathbf{z}}\frac{\partial}{\partial z})\cdot(\hat{\mathbf{x}}\frac{\partial T}{\partial x}+\hat{\mathbf{y}}\frac{\partial T}{\partial y}+\hat{\mathbf{z}}\frac{\partial T}{\partial z})$$

$$=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\nabla^2 T \longleftarrow$$ the **Laplacian** of T

The Laplacian of a vector is similar:

$$(\nabla\cdot\nabla)\mathbf{v}\equiv\nabla^2(\hat{\mathbf{x}}v_x+\hat{\mathbf{y}}v_y+\hat{\mathbf{z}}v_z)=\hat{\mathbf{x}}\nabla^2 v_x+\hat{\mathbf{y}}\nabla^2 v_y+\hat{\mathbf{z}}\nabla^2 v_z$$

$$(2)\ \nabla\times(\nabla T)\neq(\nabla\times\nabla)T$$

The proof hinges on the equality of cross derivatives:

$$\nabla\times(\nabla T)=(\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}+\hat{\mathbf{z}}\frac{\partial}{\partial z})\times(\hat{\mathbf{x}}\frac{\partial T}{\partial x}+\hat{\mathbf{y}}\frac{\partial T}{\partial y}+\hat{\mathbf{z}}\frac{\partial T}{\partial z})=0$$

$$\frac{\partial}{\partial x}(\frac{\partial T}{\partial y})=\frac{\partial}{\partial y}(\frac{\partial T}{\partial x}),\quad \frac{\partial}{\partial y}(\frac{\partial T}{\partial z})=\frac{\partial}{\partial z}(\frac{\partial T}{\partial y}),\quad \frac{\partial}{\partial z}(\frac{\partial T}{\partial x})=\frac{\partial}{\partial x}(\frac{\partial T}{\partial z})$$

1.2.7 Second Derivatives (III)

$$(4)\ \nabla\cdot(\nabla\times\mathbf{v}) = \hat{\mathbf{x}}\frac{\partial}{\partial x}(\hat{\mathbf{x}}(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z})) + \hat{\mathbf{y}}\frac{\partial}{\partial y}(\hat{\mathbf{y}}(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x})) + \hat{\mathbf{z}}\frac{\partial}{\partial z}(\hat{\mathbf{z}}(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}))$$

$$= \frac{\partial}{\partial x}(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y})$$

$= 0 \quad \longleftarrow$ always zero

$(5)\ \nabla\times(\nabla\times\mathbf{v})$ Can we use the following vector identity?

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\nabla\times(\nabla\times\mathbf{v}) = (\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}+\hat{\mathbf{z}}\frac{\partial}{\partial z})\times(\hat{\mathbf{x}}(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}) + \hat{\mathbf{y}}(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}) + \hat{\mathbf{z}}(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}))$$

$$= \ldots = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

TA

We will encounter this derivative when dealing with the vector potential (magnetism).

1.3 Integral Calculus
1.3.1 Line, Surface, and Volume (I)

In electrodynamics, the **line** (or **path**) integrals, **surface** integrals (or **flux**), and **volume** integrals are the most important integrals.

(a) **Line integrals**: a line integral is an expression of the form

$$\int_{\mathbf{a}\,\mathcal{P}}^{\mathbf{b}} \mathbf{v}\cdot d\mathbf{l},$$

where $\mathbf{v}$ is a vector function, $d\mathbf{l}$ is the infinitesimal displacement vector, and the integral is to be carried out along a prescribed path $\mathcal{P}$ from point $\mathbf{a}$ to point $\mathbf{b}$.

Put a circle on the integral, in the path in question forms a closed loop.

$$\oint \mathbf{v}\cdot d\mathbf{l}$$

1.3.1 Line, Surface, and Volume (II)

The value of a line integral depends critically on the particular path taken from $\mathbf{a}$ to $\mathbf{b}$, but there is an important special class of vector functions for which ***the line integral is independent of the path, and is determined entirely by the end points***, e.g.,

$$W = \int_{\mathbf{a}\mathcal{P}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$$

A force that has this property is called **conservative**.

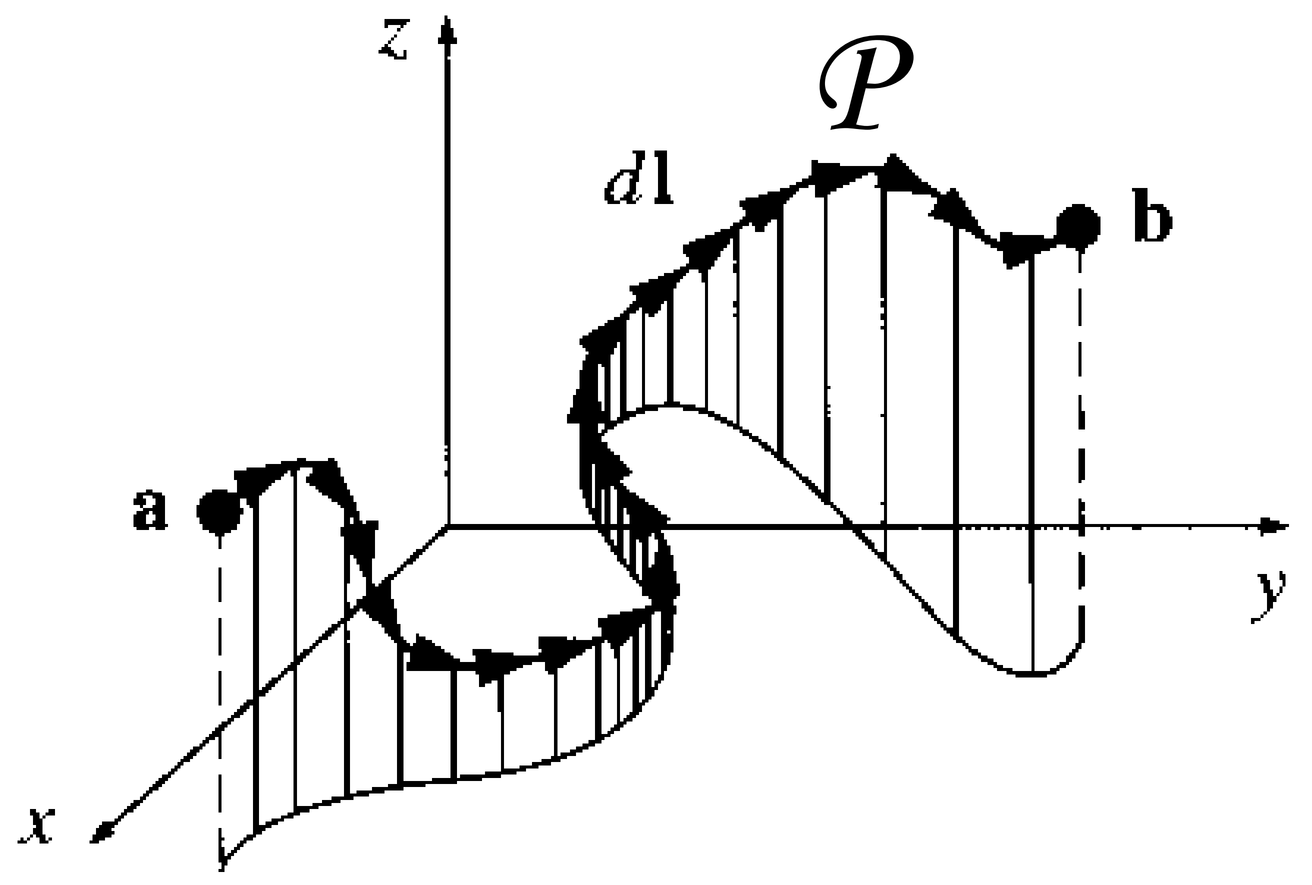
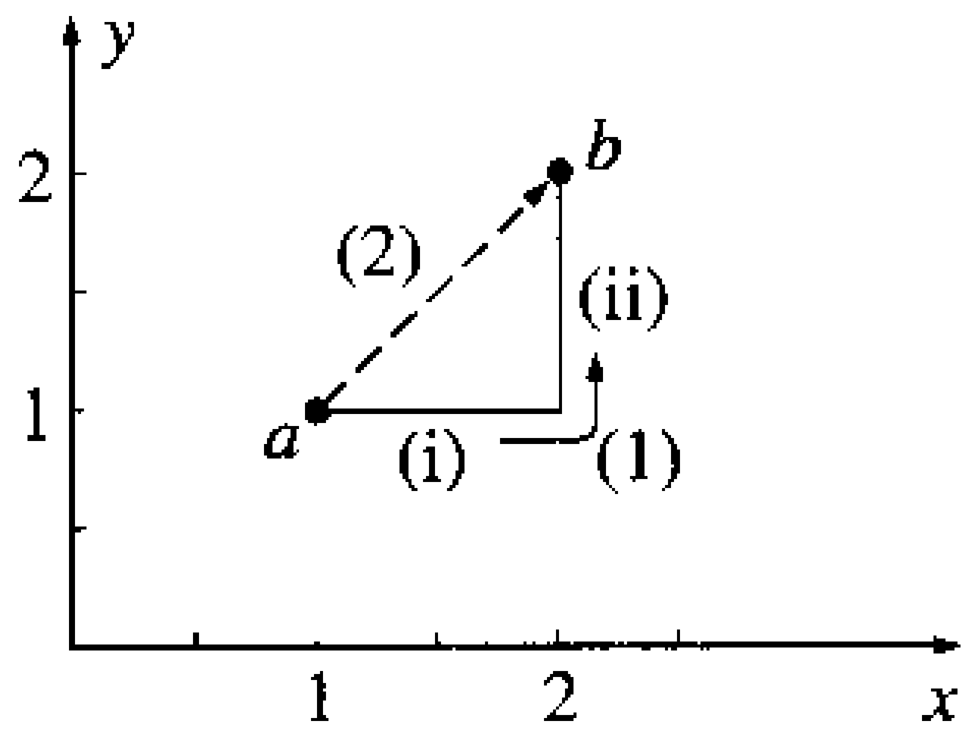


Figure 1.20

Example 1.6 Calculate the line integral of the function $\mathbf{v} = y^2\hat{\mathbf{x}} + 2x(y+1)\hat{\mathbf{y}}$, from the point $\mathbf{a} = (1,1,0)$ to the point $\mathbf{b} = (2,2,0)$, along the paths (1) and (2) in Fig.1.21. What is the loop integral that goes from $\mathbf{a}$ to $\mathbf{b}$ along (1) and returns to $\mathbf{a}$ along (2)?

The strategy here is to get everything in terms of one variable.

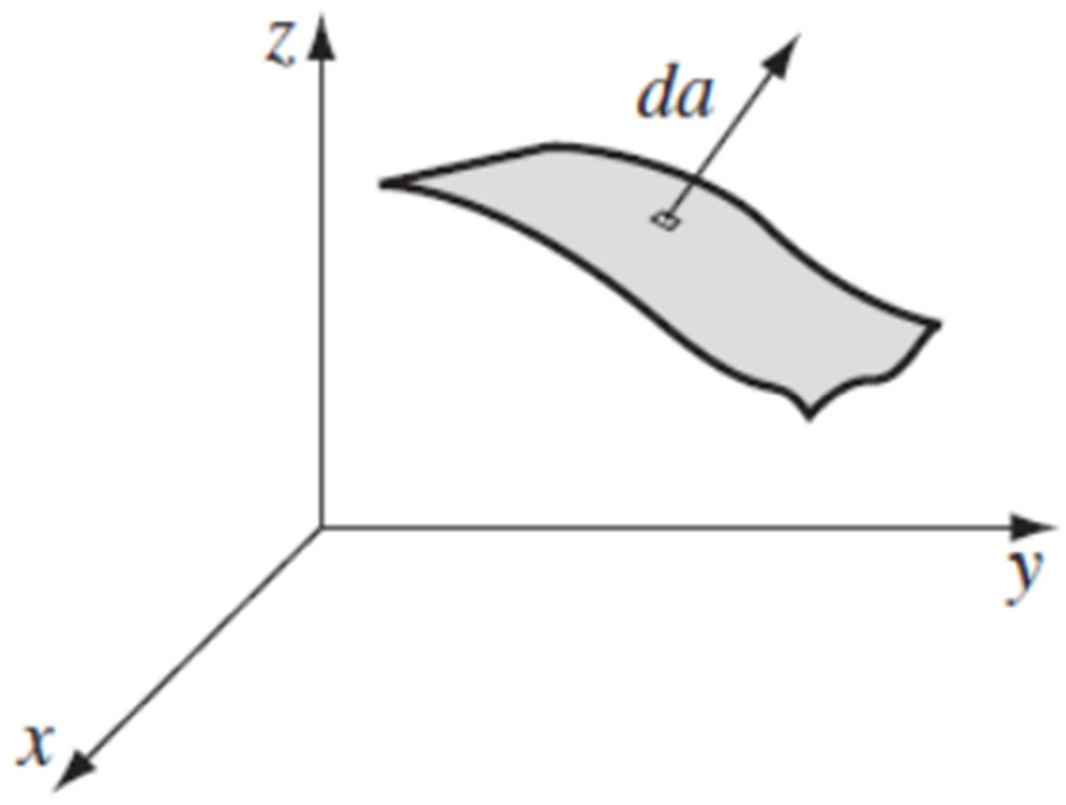


non-conservative
TA

1.3.1 Line, Surface, and Volume (III)

(b) **Surface integrals**: a surface integral is an expression of the form

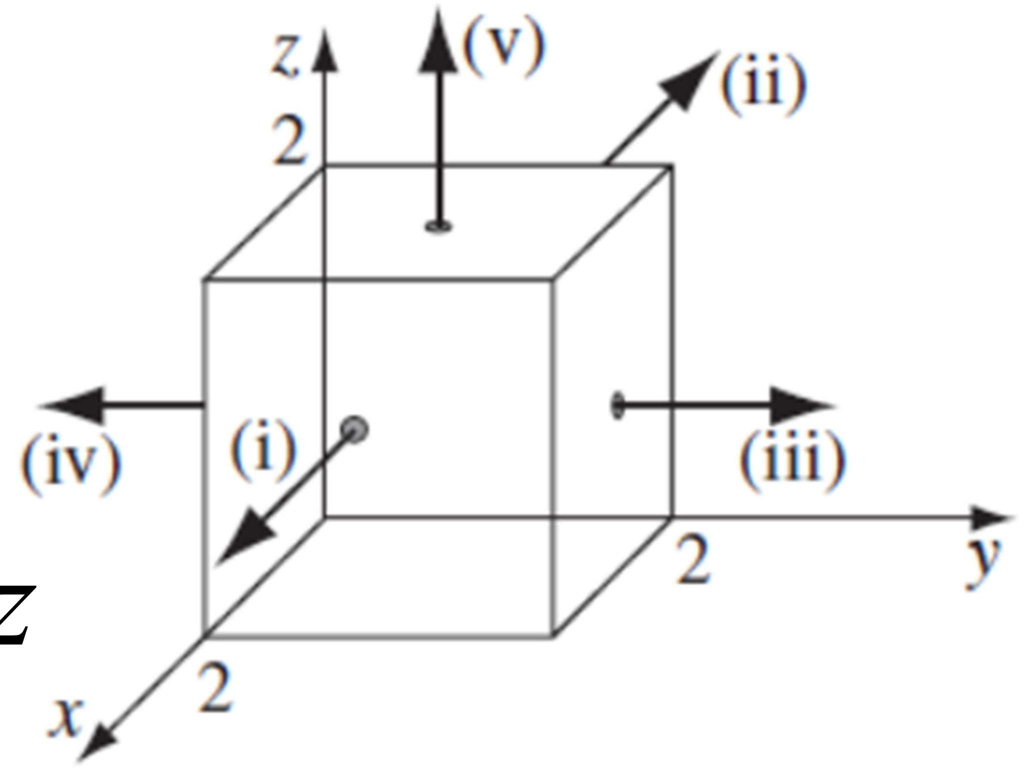
$$\int_S \mathbf{v} \cdot d\mathbf{a},$$



where $\mathbf{v}$ is a vector function, and $d\mathbf{a}$ is the infinitesimal patch of area, with direction perpendicular to the surface.

The value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is independent of the surface, *and is determined entirely by the boundary.*

Example 1.7 Calculate the surface integral of the function $\mathbf{v} = 2xz\hat{\mathbf{x}} + (2+x)\hat{\mathbf{y}} + y(z^2-3)\hat{\mathbf{z}}$ over five sides of the cubical box. Let "upward and outward" be the positive direction, as indicated by the arrow.



Sol: Taking the sides one at a time:

(i) $x = 2,\quad d\mathbf{a} = dydz\hat{\mathbf{x}},\qquad \mathbf{v}\cdot d\mathbf{a} = 2xzdydz = 4zdydz$

$$\int \mathbf{v}\cdot d\mathbf{a} = 4\int_0^2 dy\int_0^2 zdz = 16$$

(v) $z = 2,\quad d\mathbf{a} = dxdy\hat{\mathbf{z}},\qquad \mathbf{v}\cdot d\mathbf{a} = y(z^2-3)dxdy = ydxdy$

$$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^2 dx\int_0^2 ydy = 4$$

1.3.1 Line, Surface, and Volume (IV)

(c) **Volume integrals**: a volume integral is an expression of the form

$$\int_{v} T d\tau,$$

where T is a scalar function, and $d\tau$ is an infinitesimal volume element. In Cartesian coordinates, $d\tau = dxdydz$

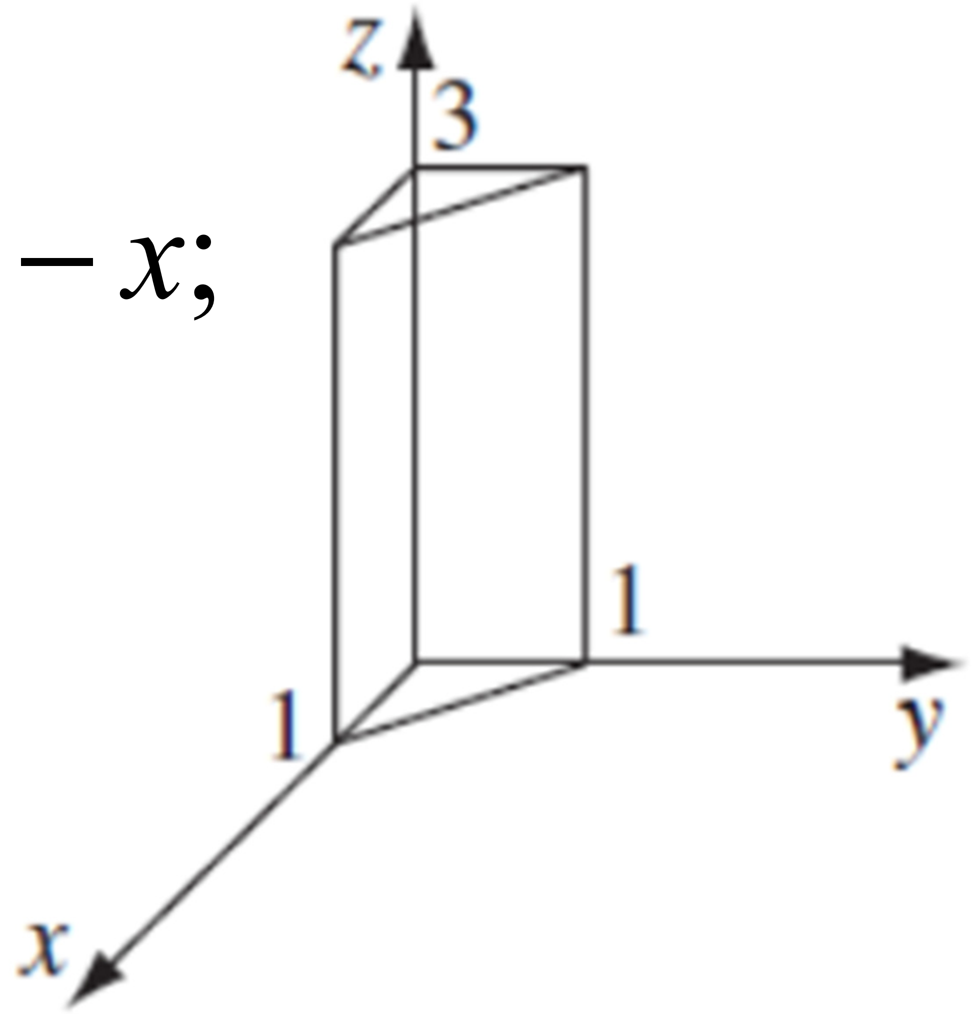
For example, if T is a density of a substance, then the volume integral would give the total mass.

The volume integrals of vector functions:

$$\int \mathbf{v} d\tau = \int (v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}}) d\tau$$

$$= \hat{\mathbf{x}} \int v_x d\tau + \hat{\mathbf{y}} \int v_y d\tau + \hat{\mathbf{z}} \int v_z d\tau$$

Example 1.8 Calculate the volume integral of the function $T = xyz^2$ over the prism in Fig. 1.24.



Sol: Let's do z first (0 to 3); then y from 0 to $1-x$; finally x from 0 to 1.

$$\iiint xyz^2\,dx\,dy\,dz = \int_0^3 z^2 dz \left\{\int_0^1 x\left(\int_0^{1-x} y\,dy\right)dx\right\}$$

$$= 9\left\{\int_0^1 x\left(\frac{1}{2}(1-x)^2\right)dx\right\}$$

$$= 9\left(\frac{1}{2}\right)\left(\frac{1}{12}\right) = \frac{3}{8}$$

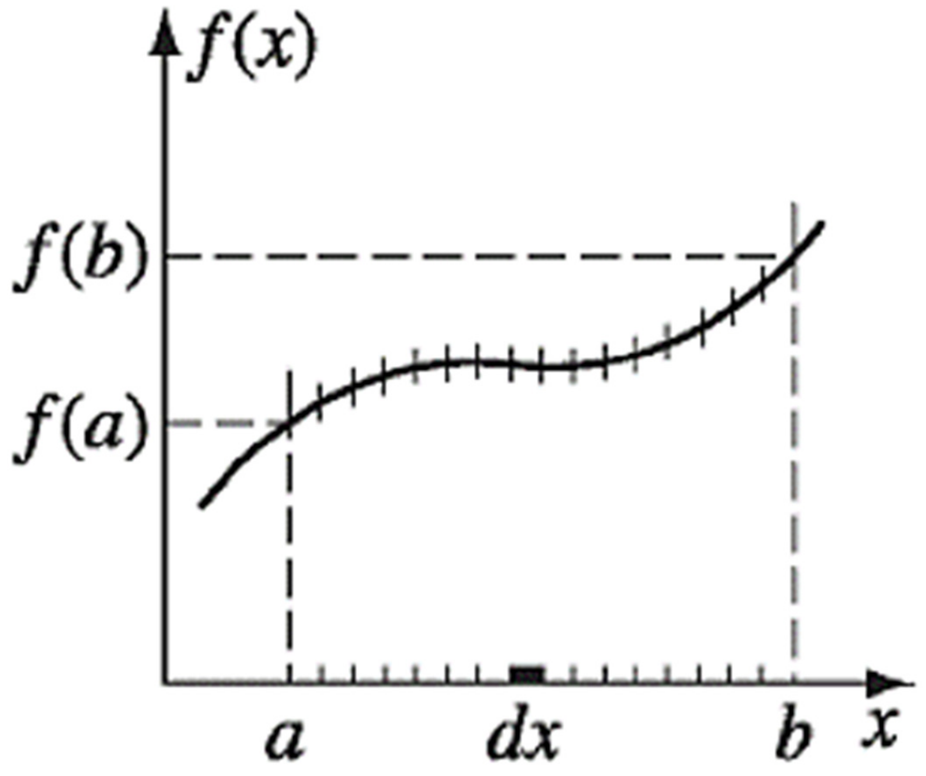
1.3.2 The Fundamental Theorem of Calculus

Fundamental theorem of calculus:

$$\int_a^b \frac{df}{dx} dx = \int_a^b df = f(b) - f(a)$$

Geometrical Interpretation: two ways to determine the total change in the function:

1. go step-by-step adding up all the tiny increments as you go.
2. subtract the values at the ends.



The integral of a derivative over an interval is given by the value of the function at the end points (boundary).

1.3.3 The Fundamental Theorem for Gradients

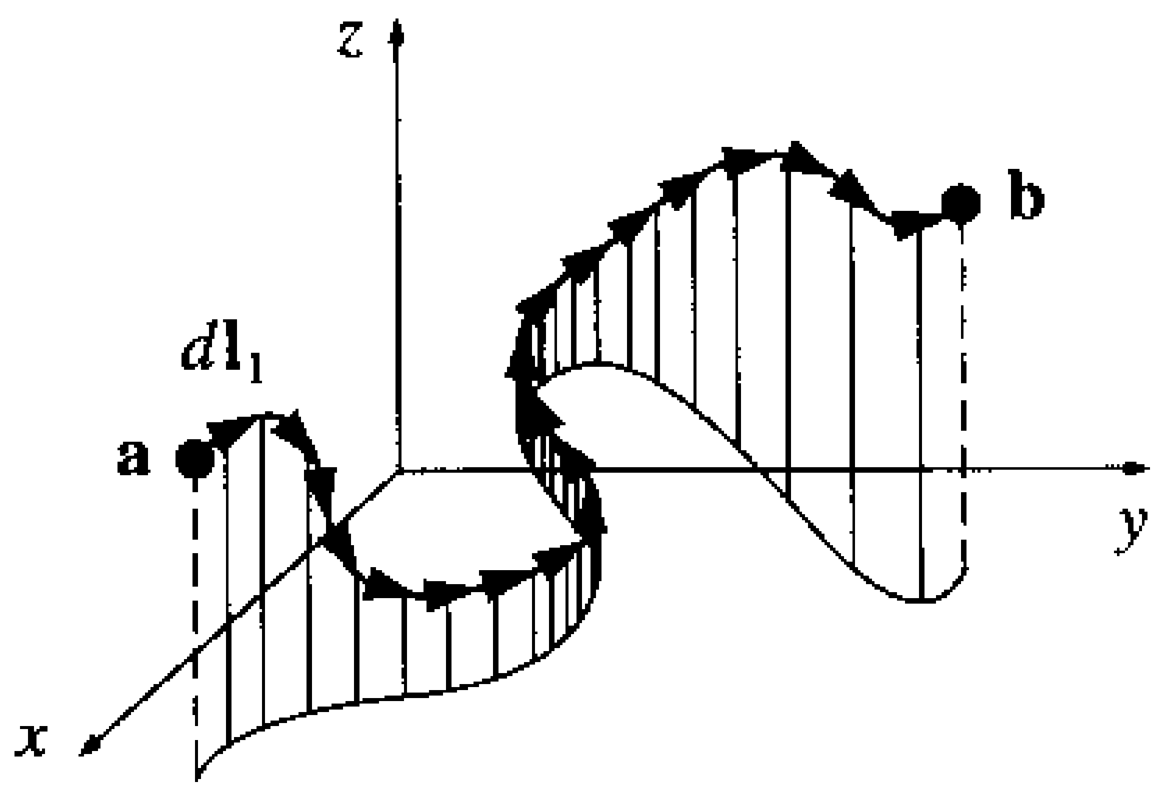
A scalar function of three variables $T(x, y, z)$ changes by a small amount.

$$dT = (\nabla T) \cdot d\mathbf{l}_1$$

The total change in T in going from $\mathbf{a}$ to $\mathbf{b}$ along the path selected is:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$

Fundamental theorem for gradient.



Geometrical Interpretation: Measure the high of a skyscraper.
1. Measure the high of each floor and add them all up.
2. Place an altimeter at the top and the bottom, subtract the readings at the ends.

1.3.3 The Fundamental Theorem for Gradients (II)

$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$ the right side of this equation makes no reference to the path---only to the end points.

Thus gradients have special property that their line integrals are path independent.

Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l}$ is independent of path taken from $\mathbf{a}$ to $\mathbf{b}$.

Corollary 2: $\oint (\nabla T)\cdot d\mathbf{l} = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b})–T(\mathbf{a}) = 0$.

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A conservative force may be associated with a scalar potential energy function, whereas a non-conservative force cannot.

Potential Energy and Conservative Forces

Potential energy defined in terms of work done by the associated conservative force.

$$U_B - U_A = -\int_A^B \mathbf{F}_c \cdot d\mathbf{s}$$

*Conservative forces tend to ***minimize*** the potential energy within any system: If allowed to, an apple falls to the ground and a spring returns to its natural length.

Non-conservative force does not imply it is dissipative, for example, magnetic force, and also does not mean it will decrease the potential energy, such as hand force.

Distinction Between
Conservative and Non-conservative Forces

The distinction between conservative and non-conservative forces is best stated as follows:

A conservative force may be associated with a scalar potential energy function, whereas a non-conservative force cannot.

$$U_B - U_A = -\int_A^B \mathbf{F}_c \cdot d\mathbf{s}$$

$$\mathbf{F}_c = -\nabla U$$

Conservative Force and Potential Energy Function

How can we find a conservative force if the associated potential energy function is given?

A conservative force can be derived from a scalar potential energy function.

$$\mathbf{F}_c = -\nabla U$$

The negative sign indicates that the force points in the direction of ***decreasing*** potential energy.

$$\text{Gravity} \quad U_g = mgy; \qquad F_y = -\frac{dU_g}{dy} = -mg$$

$$\text{Spring} \quad U_{sp} = \frac{1}{2}kx^2; \qquad F_x = -\frac{dU_{sp}}{dx} = -kx$$

1.3.4 The Fundamental Theorem for Divergences

The fundamental theorem for divergences states that:

$$\int_{v} (\nabla \cdot \mathbf{v}) d\tau = \oint_{S} \mathbf{v} \cdot d\mathbf{a}$$

The integration of a derivative (in this case the divergence) over a region (in this case a volume) is equal to the value of the function at the boundary (in this case the surface that bounds the volume)

This theorem has at least three special names: **Gauss's theorem, Green's theorem,** or the **divergence theorem.**

Geometrical Interpretation: Measure the total amount of fluid passing out through the surface, per unit time.
1. Count up all the faucets, recording how much each put out.
2. Go around the boundary, measuring the flow at each point, and add it all up.

Feynman:
Gauss' theorem
Stokes' theorem

Griffiths:
Gauss's theorem
Stokes' theorem

Jackson:
Gauss's theorem
Stokes's theorem

Supplementary

Gauss's divergence theorem
(Transformation between volume integrals and surface integrals)

$$\int_v (\nabla \cdot \mathbf{v}) d\tau = \oint_S \mathbf{v} \cdot \hat{\mathbf{n}} da$$

Rough proof:

$\mathbf{v} = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}}$ and $\hat{\mathbf{n}} = \cos\alpha \hat{\mathbf{x}} + \cos\beta \hat{\mathbf{y}} + \cos\gamma \hat{\mathbf{z}}$

where α, β, and γ are the angles between $\hat{\mathbf{n}}$ and x-, y- and z-axis, respectively.

$$\int_v (\nabla \cdot \mathbf{v}) d\tau = \iiint_v \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dxdydz$$

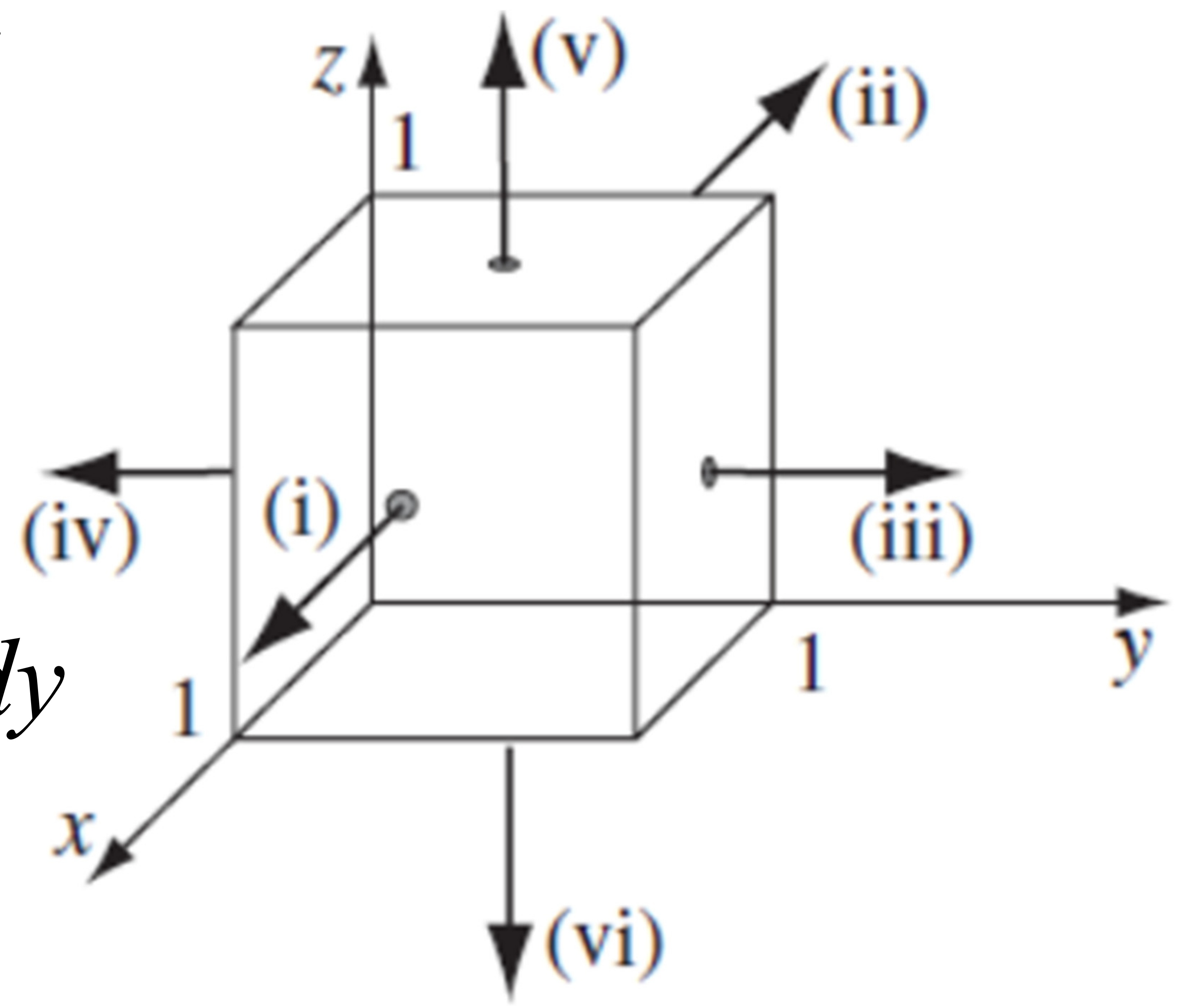
$$= \iint_S (v_x dydz + v_y dzdx + v_z dxdy)$$

$$= \iint_S (v_x \cos\alpha + v_y \cos\beta + v_z \cos\gamma) da = \iint_S \mathbf{v} \cdot \hat{\mathbf{n}} da$$

Rigorous proof can be found in: Erwin Kreyszig, *Advanced Engineering Mathematics* (John Wiley and Sons, New York, 1993), 7th ed. Chap. 9, pp. 546-547.

Example 1.10 Check the divergence theorem using the function $\mathbf{v} = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + (2yz)\hat{\mathbf{z}}$

and the unit cube situated at the origin.



$$\text{Sol: In this case } \nabla \cdot \mathbf{v} = 2(x+y)$$

$$\int_v 2(x+y)\,dxdydz = 2\int_0^1 dz \int_0^1 \int_0^1 (x+y)\,dxdy$$

$$= 2\int_0^1 \int_0^1 (\tfrac{1}{2} + y)\,dy = 2\int_0^1 (\tfrac{1}{2} + y)\,dy = 2$$

$$\therefore \int_v \nabla \cdot \mathbf{v}\, d\tau = 2$$

To evaluate the surface integral we must consider separately the six sides of the cube. The total flux is…

1.3.5 The Fundamental Theorem for Curls (I)

The fundamental theorem for curls---**Stokes' theorem**---states that:

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}$$

The integration of a derivative (here, the curl) over a region (here, a patch of surface) is equal to the value of the function at the boundary (in this case the perimeter of the patch).

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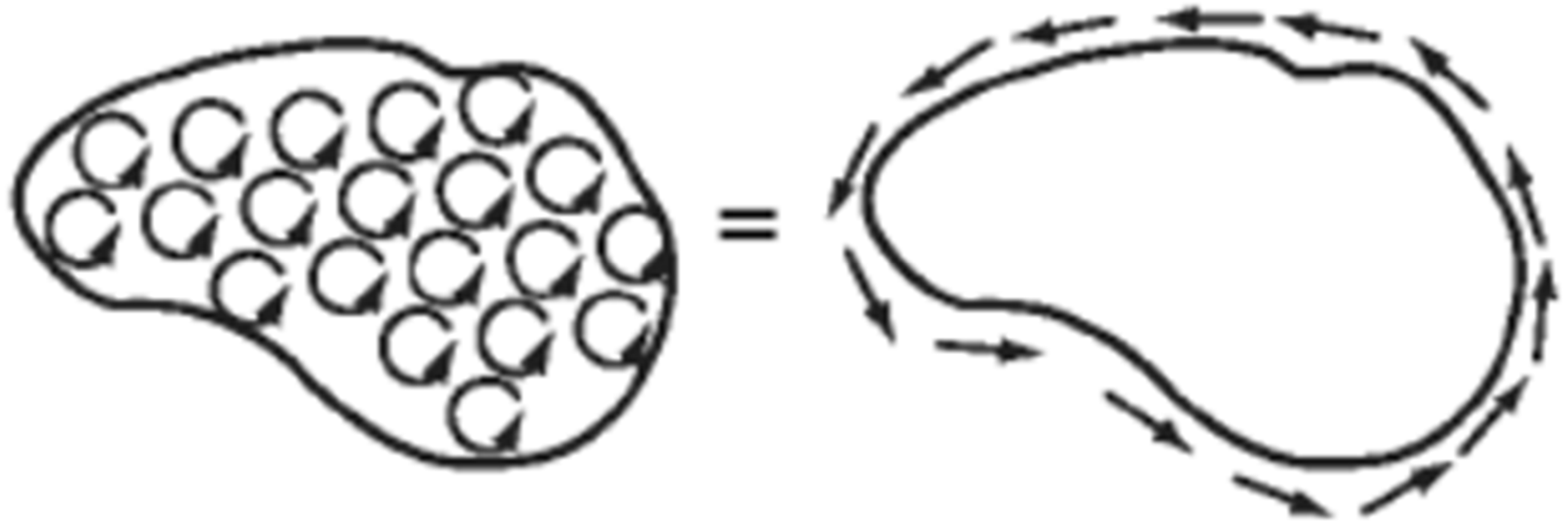
Geometrical Interpretation: Measure the "twist" of the vectors $\mathbf{v}$; a region of high curl is a whirlpool.



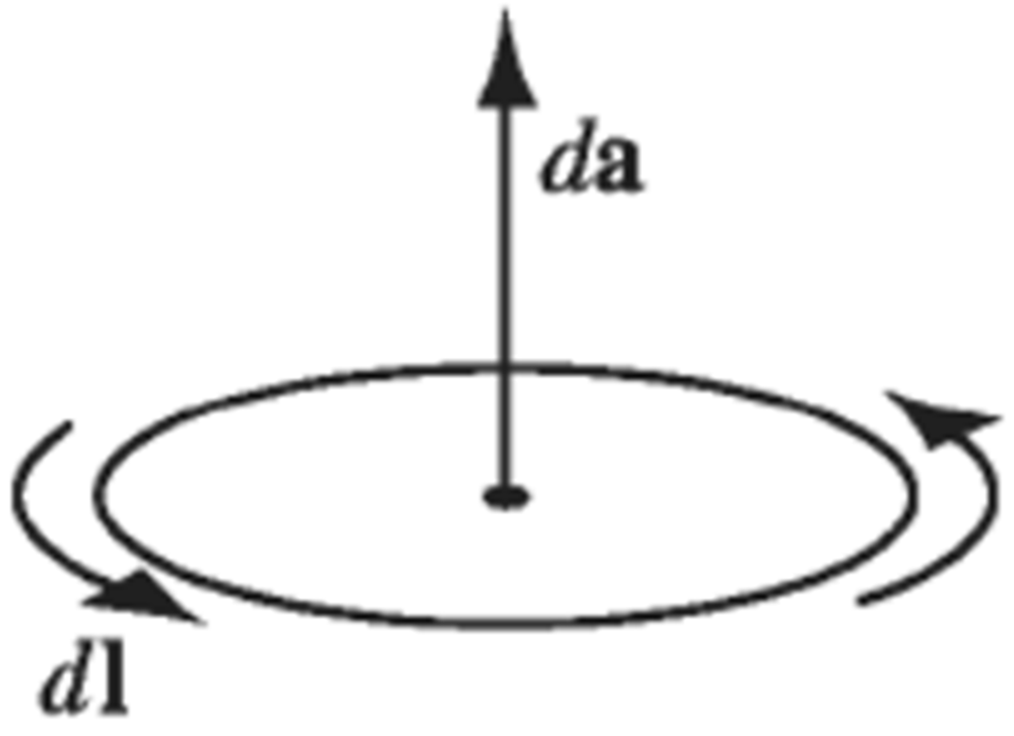
1.3.5 The Fundamental Theorem for Curls (II)

Ambiguity in Stokes' theorem: Concerning the boundary line integral, which way are we supposed to go around (clockwise or counterclockwise)? **The right-hand rule**.

Corollary 1: $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary lines, not on the particular surface used.

Corollary 2: $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line shrinks down to a point.

These corollaries are analogous to those for the gradient theorem.



Supplementary

Stokes' theorem

(Transformation between surface integrals and line integrals)

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}$$

Rigorous proof can be found in:
Erwin Kreyszig, *Advanced Engineering Mathematics* (John Wiley and Sons, New York, 1993), 10th ed. Chap. 10, pp. 464-467.

Comments: graduate level (reference only)

- Green's theorems:

$$\text{Let } \mathbf{v} = f\nabla g \quad \Rightarrow \quad \nabla\cdot\mathbf{v} = \nabla\cdot(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g$$

$$\mathbf{v}\cdot\hat{\mathbf{n}} = f(\hat{\mathbf{n}}\cdot\nabla g)$$

$$\text{Green's first formula: } \int_v (f\nabla^2 g + \nabla f\cdot\nabla g)d\tau = \oint_S f\frac{\partial g}{\partial n}da$$

$$\text{Green's second formula: } \int_v (f\nabla^2 g - g\nabla^2 f)d\tau = \oint_S (f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n})da$$

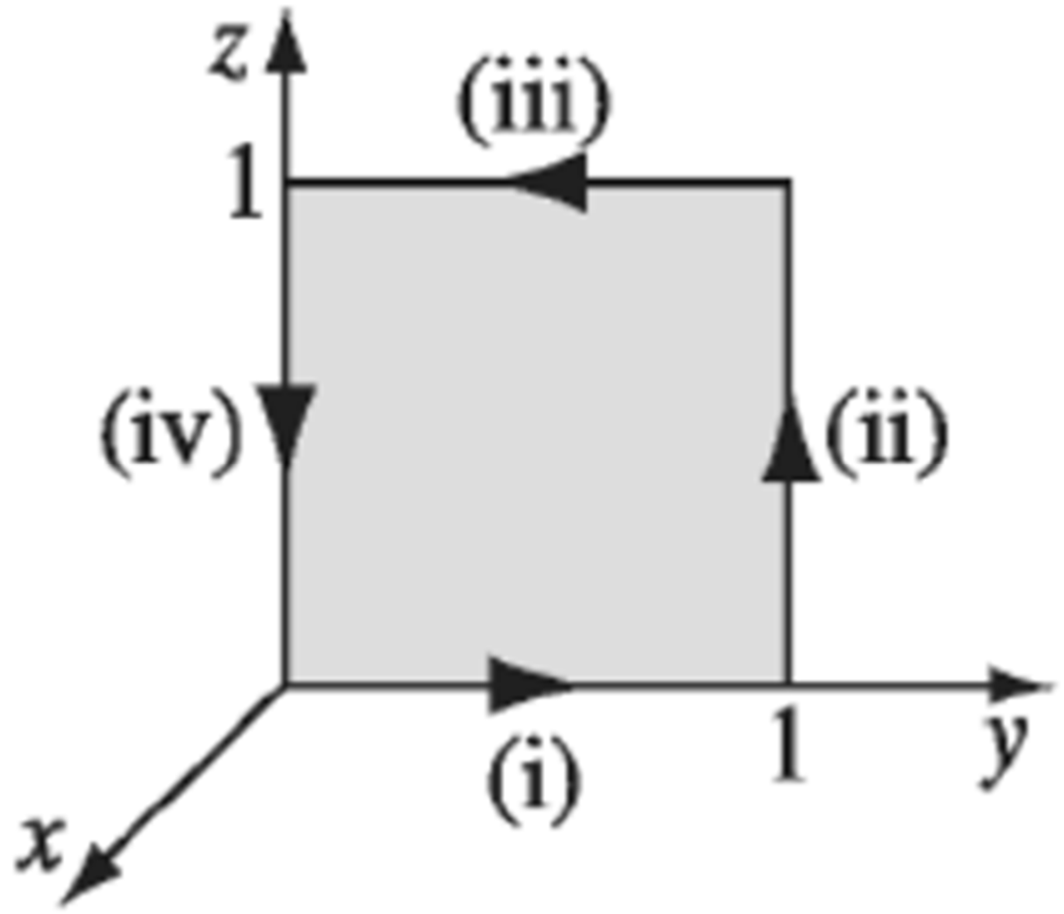
- Green's theorem in the plane as a special case of Stokes' theorem

Let $\mathbf{v}$ be a vector function in the xy-plane.

$$(\nabla\times\mathbf{v})\cdot\hat{\mathbf{n}} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \quad \Rightarrow \quad \iint_S (\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})da = \oint_P (v_x dx + v_y dy)$$

Example 1.11 Suppose $\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}$

Check Stokes' theorem for the square surface shown below.



Sol: $\nabla \times \mathbf{v} = (4z^2 - 2x)\hat{\mathbf{x}} + 2z\hat{\mathbf{z}}; \quad d\mathbf{a} = dydz\hat{\mathbf{x}}$

$$\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 4z^2 dydz = \frac{4}{3}$$

The line integral of the four segments

(i) $x = 0,\ z = 0,\ \mathbf{v} \cdot d\mathbf{l} = 3y^2 dy,\ \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 3y^2 dy = 1,$

(ii) $x = 0,\ \mathrm{y} = 1,\ \mathbf{v} \cdot d\mathbf{l} = 4z^2 dz,\ \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 4z^2 dz = \frac{4}{3},$

(iii) $x = 0,\ z = 1,\ \mathbf{v} \cdot d\mathbf{l} = 3y^2 dy,\ \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 3y^2 dy = -1,$

(iv) $x = 0,\ \mathrm{y} = 0,\ \mathbf{v} \cdot d\mathbf{l} = 0,\ \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 0 dz = 0.$

$$\oint \mathbf{v} \cdot d\mathbf{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}.$$

1.3.6 Integration by Parts

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

Integrating both sides and invoking the fundamental theorem

Left $\int_a^b \frac{d}{dx}(fg)dx = fg\Big|_a^b$

Right $\int_a^b f\frac{dg}{dx}dx + \int_a^b g\frac{df}{dx}dx$

$$\nabla\cdot(f\mathbf{A}) = \nabla f\cdot\mathbf{A} + f(\nabla\cdot\mathbf{A})$$

Integrate it over a volume and invoking the divergence theorem.

Left $\int \nabla\cdot(f\mathbf{A})d\tau = \oint (f\mathbf{A})\cdot d\mathbf{a}$

Right $\int (\nabla f\cdot\mathbf{A} + f(\nabla\cdot\mathbf{A}))d\tau$

$= \int(\nabla f\cdot\mathbf{A})d\tau + \int f(\nabla\cdot\mathbf{A})d\tau$

$$\int_a^b f\frac{dg}{dx}dx = -\int_a^b g\frac{df}{dx}dx + fg\Big|_a^b \quad (1.58)$$

$$\int f(\nabla\cdot\mathbf{A})d\tau = -\int(\nabla f\cdot\mathbf{A})d\tau + \oint (f\mathbf{A})\cdot d\mathbf{a} \quad (1.59)$$

Optional

Applications of Stokes' and Divergence Theorems

$(2)\ \text{Curl of gradient}: \nabla\times(\nabla T)$ $\longleftarrow$ always zero

$(4)\ \text{Divergence of curl}: \nabla\cdot(\nabla\times\mathbf{v})$ $\longleftarrow$ always zero

Stokes' theorem $\int_S (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_P \mathbf{v}\cdot d\mathbf{l}$

$$\int_S (\nabla\times\nabla T)\cdot d\mathbf{a} = \oint_P \nabla T\cdot d\mathbf{l} = T(\mathbf{a}) - T(\mathbf{a}) = 0$$

Divergence theorem $\int_v (\nabla\cdot\mathbf{v})d\tau = \oint_S \mathbf{v}\cdot d\mathbf{a}$

$$\int_v (\nabla\cdot(\nabla\times\mathbf{v}))d\tau = \oint_S (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_P \mathbf{v}\cdot d\mathbf{l} = 0$$

$\oint(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line shrinks down to a point.

Homework of Chap. 1 (part I)

Problem 1.5 Prove the **BAC - CAB** rule by writing out both sides in component form.

Problem 1.7 Find the separation vector $\vec{\mathscr{r}}$ from the source point (2,8,7) to the field point (4,6,8). Determine its magnitude ($\mathscr{r}$), and construct the unit vector $\hat{\mathscr{r}}$.

Problem 1.13 Let $\vec{\mathscr{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that

(a) $\nabla(\mathscr{r}^2) = 2\vec{\mathscr{r}}$.

(b) $\nabla(1/\mathscr{r}) = -\hat{\mathscr{r}}/\mathscr{r}^2$.

(c) What is the *general* formula for $\nabla(\mathscr{r}^n)$?

Problem 1.16 Sketch the vector function

$$\mathrm{v} = \frac{\hat{\mathbf{r}}}{r^2},$$

and compute its divergence. The answer may surprise you. . . can you explain it?

Problem 1.33 Test the divergence theorem for the function $\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$. Take as your volume the cube shown in Fig. 1.30, with sides of length 2.

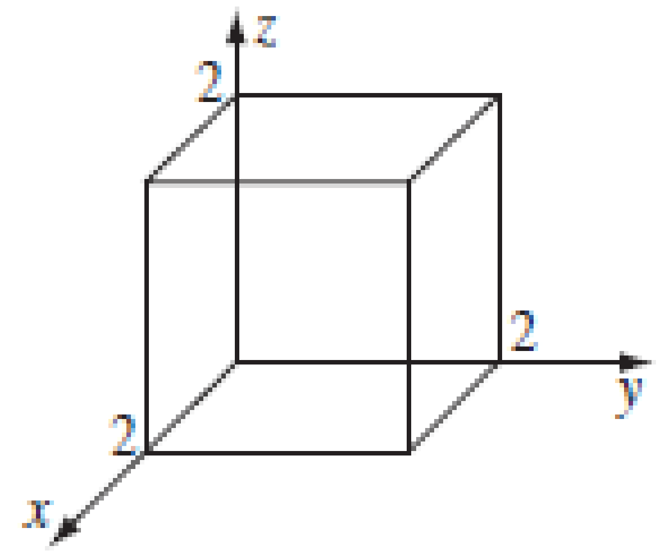


FIGURE 1.30

1.4 Curvilinear Coordinates
1.4.1 Spherical Polar Coordinates (I)

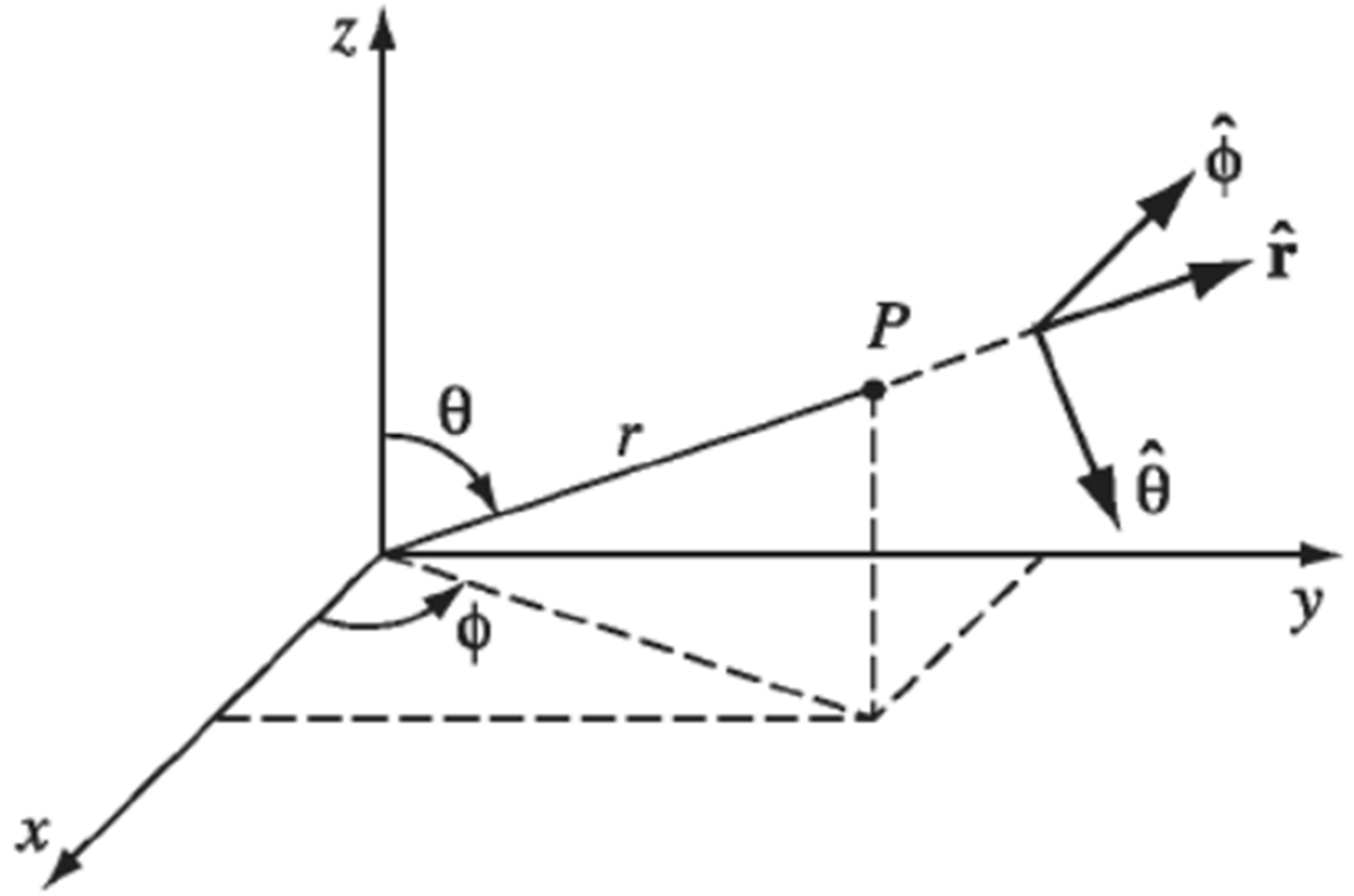
The spherical (polar) coordinates (r, θ, ϕ) of a point P are defined below;

r : the distance from the origin (the magnitude of the position vector).

θ : the angle down from the z-axis (the polar angle).

ϕ : the angle around from the x-axis (the azimuthal angle).

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$



Murray R Spiegel, Vector Analysis (McGraw-Hill, New York, 1989), 6th ed. Chap. 7.

1.4.1 Spherical Polar Coordinates (II)

The direction of the coordinates: the unit vector $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$

They constitute an orthogonal (mutually perpendicular) basis set (just like $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$).

So any vector $\mathbf{A}$ can be expressed in terms of them:

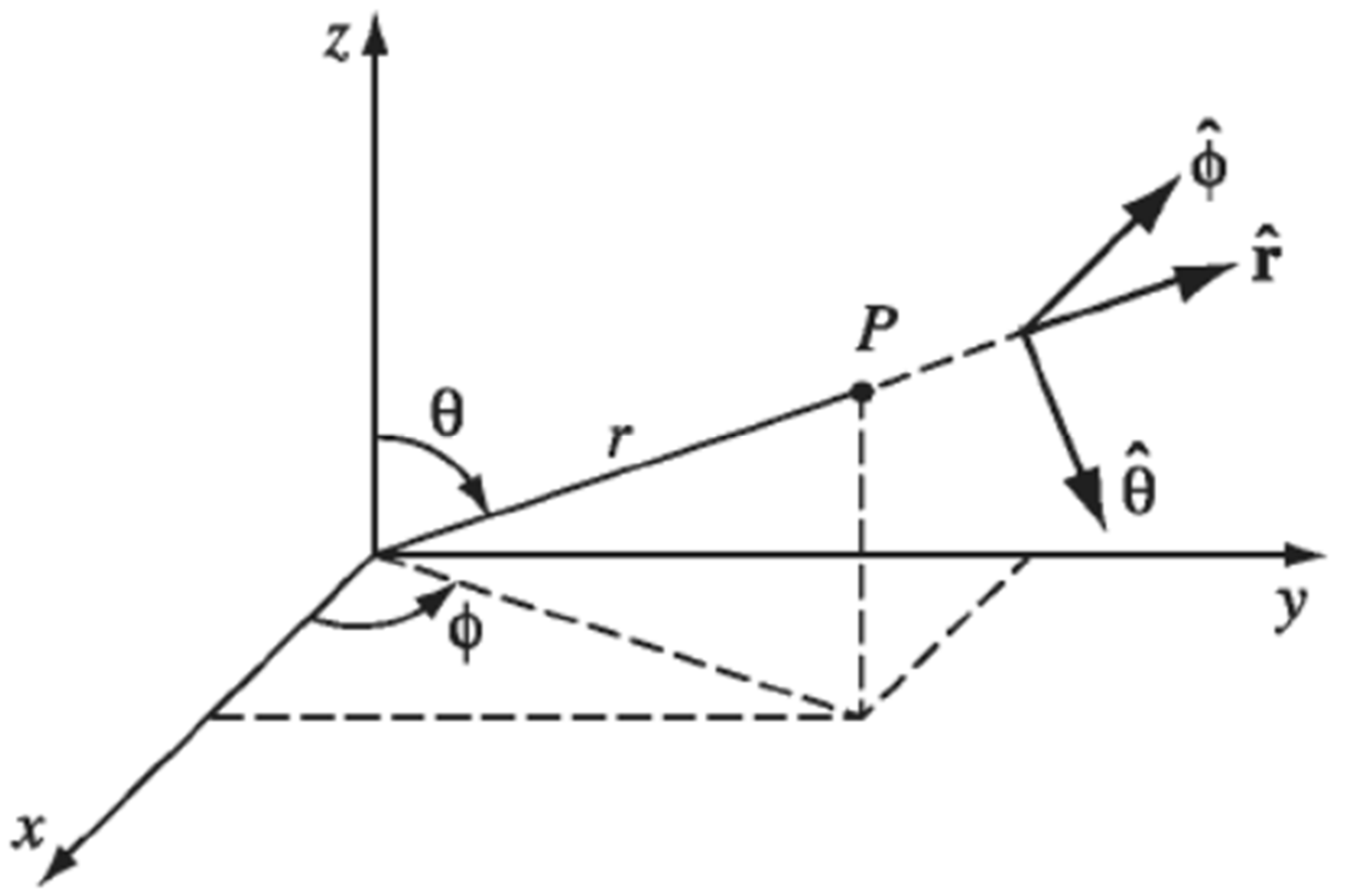
$$\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}$$

In terms of Cartesian unit vector

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$



(Or you can see Appendix A for more details.)

1.4.1 Spherical Polar Coordinates (III)

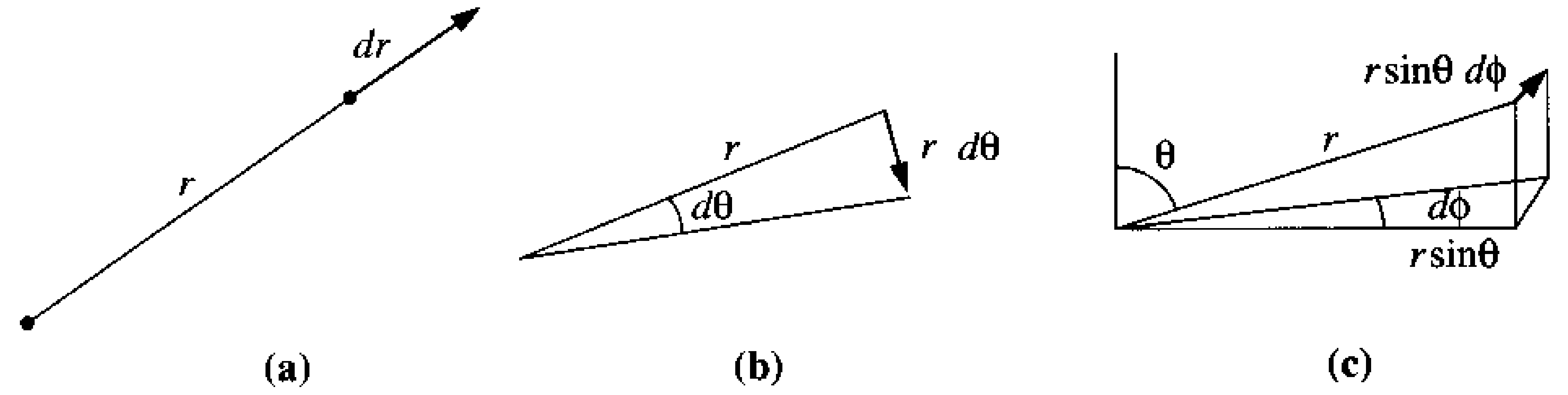
Warning: $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ are associated with particular point *P*, and they change direction as *P* moves around.
For example, $\hat{\mathbf{r}}$ always points radially outward, but “radially outward” can be the x direction, the y direction, or any other direction, depending on where you are.

Notice: Since the unit vectors are function of position, we must handle the differential and integral with care.

1. Differentiate a vector that is expressed in spherical coordinates.

2. Do not take the unit vectors outside an integral.

1.4.1 Spherical Polar Coordinates (IV)



The general infinitesimal displacement:

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

The infinitesimal surface element $d\mathbf{a}$ for the surface of a sphere.

$$d\mathbf{a} = (dl_\theta)(dl_\phi)\hat{\mathbf{r}} = r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$$

The infinitesimal volume element $d\tau$

$$d\tau = (dl_r)(dl_\theta)(dl_\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi$$

1.4.1 Spherical Polar Coordinates (V)

The vector derivatives in spherical coordinates:

Gradient :

$$\nabla T = \frac{\partial T}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Divergence :

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}.$$

Curl :

$$\nabla \times \boldsymbol{v} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}}$$

$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right)\hat{\boldsymbol{\phi}}.$$

Laplacian :

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}.$$

1.4.2 Cylindrical Coordinates (I)

The cylindrical coordinates (s, ϕ, z) of a point P are defined below: $x = s\cos\phi, \quad y = s\sin\phi, \quad z = z$

s: the distance from the z axis.

ϕ: the same meaning as in spherical coordinates.

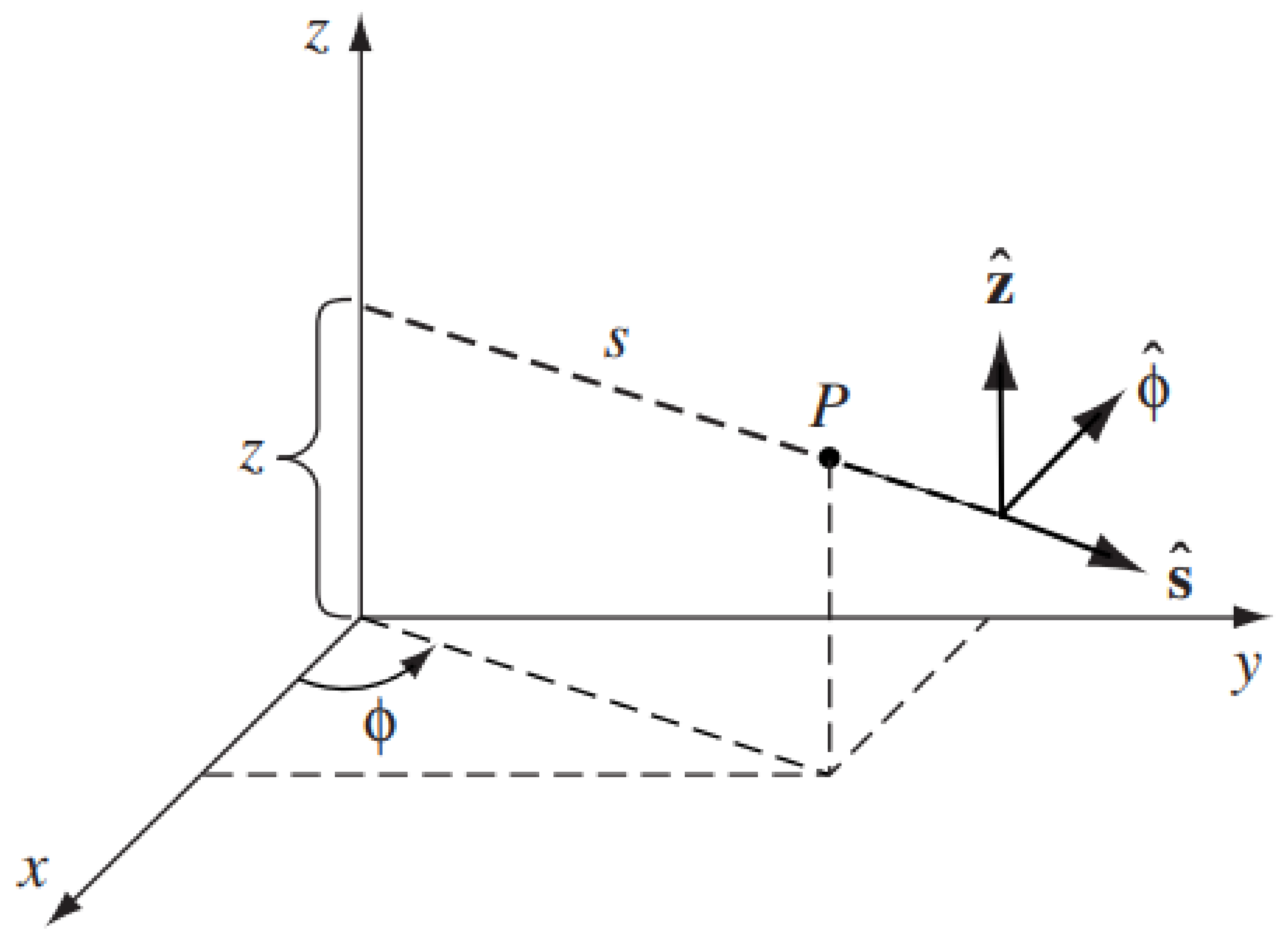
z: the same as Cartesian.

The unit vectors are

$$\hat{\mathbf{s}} = \cos\phi\hat{x} + \sin\phi\hat{y},$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{x} + \cos\phi\hat{y},$$

$$\hat{\mathbf{z}} = \hat{z}.$$



The infinitesimal displacement:

$$d\mathbf{l} = ds\hat{\mathbf{s}} + s\,d\phi\hat{\boldsymbol{\phi}} + dz\hat{\mathbf{z}}$$

1.4.2 Cylindrical Coordinates (II)

The vector derivatives in cylindrical coordinates:

Gradient :

$$\nabla T = \frac{\partial T}{\partial s}\hat{\boldsymbol{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\boldsymbol{z}}.$$

Divergence :

$$\nabla \cdot \boldsymbol{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}.$$

Curl :

$$\nabla \times \boldsymbol{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\boldsymbol{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\boldsymbol{z}}.$$

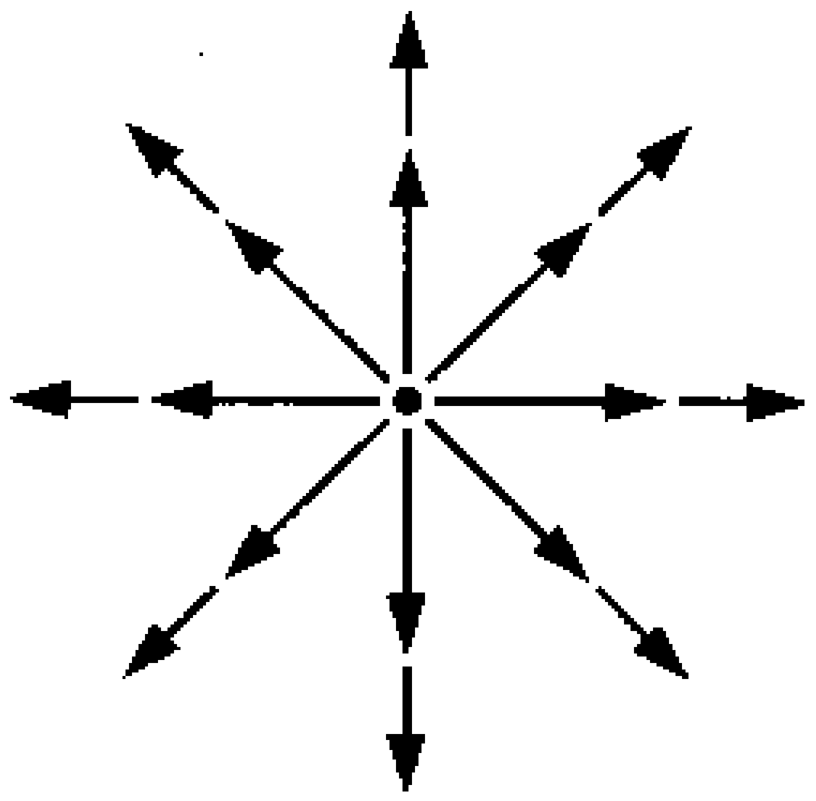
Laplacian :

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}.$$

1.5 The Dirac Delta Function

1.5.1 The Divergence of $\hat{\mathbf{r}}/r^2$

Consider a vector function $\mathbf{v} = \hat{\mathbf{r}}/r^2$



The divergence of this vector function is:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{1}{r^2}) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

The surface integral of this function is:

$$\oint \mathbf{v} \cdot d\mathbf{a} = \int_0^{\pi}\int_0^{2\pi}(\frac{1}{r^2} r^2 \sin\theta) d\theta d\phi$$

$$= \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = 4\pi \neq \int_v (\nabla \cdot \mathbf{v}) d\tau$$

The divergence theorem is false?

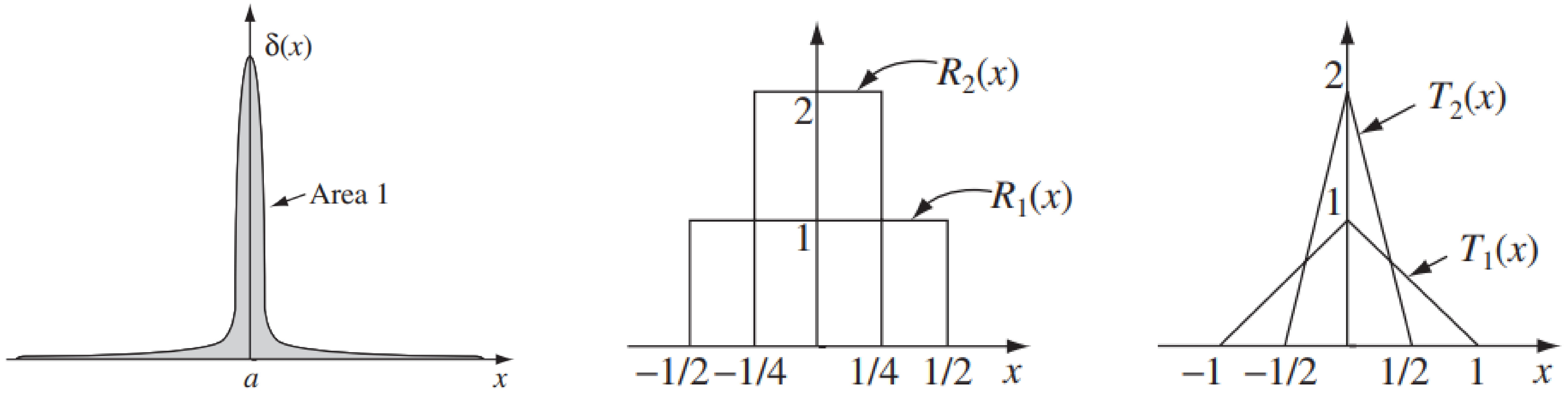
No ➔ The **Dirac delta function**

1.5.2 The One-Dimensional Dirac Delta Function

The 1-D Dirac delta function can be pictured as
an infinitely high, infinitesimally narrow “spike”, with area just 1.

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad \text{with} \quad \int_{-\infty}^{+\infty} \delta(x)dx = 1$$

Technically, $\delta(x)$ is not a function at all, since its value is not finite at $x = 0$. Such function is called the **generalized function**, or **distribution**.



1.5.2 The One-Dimensional Dirac Delta Function (II)

If $f(x)$ is some "ordinary" function (let's say that it is continuous), then the product $f(x)\delta(x)$ is zero everywhere except at $x = 0$. It follows that $f(x)\delta(x) = f(0)\delta(x)$. In particular,

$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0)\int_{-\infty}^{+\infty} \delta(x)dx = f(0)$$

We can shift the spike from $x = 0$ to some other point $x = a$.

$$\delta(x-a) = \begin{cases} 0 & \text{if } x \neq a \\ \infty & \text{if } x = a \end{cases} \quad \text{with} \quad \int_{-\infty}^{+\infty} \delta(x-a)dx = 1$$

A generalized integration equation:

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a)\int_{-\infty}^{+\infty} \delta(x)dx = f(a)$$

1.5.2 The One-Dimensional Dirac Delta Function (III)

Although $\delta(x)$ is not a legitimate function, integrals over $\delta(x)$ are perfectly acceptable.

It is best to think of the delta function as something that is *always intended for use under an integral sign*.

In particular, two expressions involving delta function are considered **equal** if:

$$\int_{-\infty}^{+\infty} f(x) D_1(x) dx = \int_{-\infty}^{+\infty} f(x) D_2(x) dx$$

$$\text{for all ("ordinary") function of } f(x).$$

Example 1.14 Evaluate the integral (a) $\int_0^3 x^3 \delta(x-2) dx$ (b) $\int_0^3 x^3 \delta(x-4) dx$

Example 1.15 Show that $$\delta(kx)=\frac{1}{|k|}\delta(x)$$

where k is any (nonzero) constant.

Sol: Consider the integral for an arbitrary test function $f(x)$,

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx$$

Let $y \equiv kx$, so that $x \equiv y/k$, $dx \equiv 1/k\, dy$

$$k=\begin{cases} positive: \text{the integration runs from } -\infty \text{ to } \infty \\ negative: \text{the integration runs from } \infty \text{ to } -\infty \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \pm\frac{1}{k}\int_{-\infty}^{\infty} f(y/k)\delta(y)dy = \frac{1}{|k|}f(0)$$

So $\delta(kx)$ serves the same purpose as $\frac{1}{|k|}\delta(x)$ and $\delta(-x) = \delta(x)$.

Prob. 1.45

$$\text{(a)}\quad x\frac{d}{dx}(\delta(x)) = -\delta(x)$$

(b) Let $\theta(x)$ be the step function :

$$\theta(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Show that $d\theta/dx = \delta(x)$

1.5.3 The three-Dimensional Dirac Delta Function

The generalized 3D delta function

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

where $\mathbf{r}$ is the position vector. It is zero everywhere except at $(0,0,0)$, where it blows up.

Its volume integral is:

$$\int_{\text{all space}} \delta^3(\mathbf{r})d\tau = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x)\delta(y)\delta(z)dxdydz = 1$$

As in the 1-D case, the integral with delta function picks out the value of the function at the location of the spike.

$$\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})d\tau = f(\mathbf{a})$$

1.5.3 The three-Dimensional Dirac Delta Function (II)

We found that the divergence of $\hat{\mathbf{r}}/r^2$ is zero everywhere except at the origin, and yet its integral over any volume containing the origin is a constant of 4π. The Dirac delta function can be defined as:

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r})$$

More generally,

$$\nabla \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$$

where $\vec{\mathscr{r}}$ is the separation vector $\vec{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$. Note that the differentiation here is with respect to $\mathbf{r}$, while $\mathbf{r}'$ is held constant.

$$\nabla^2\left(\frac{1}{\mathscr{r}}\right) = \nabla \cdot \left(\nabla\left(\frac{1}{\mathscr{r}}\right)\right) = \nabla \cdot \left(-\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = -4\pi\delta^3(\vec{\mathscr{r}})$$

1.6 The Theory of Vector Fields
1.6.1 The Helmholtz Theorem

To what extent is a vector function $\mathbf{F}$ determined by its divergence and curl?

The divergence of $\mathbf{F}$ is a specified scalar function D,

$$\nabla \cdot \mathbf{F} = D$$

and the curl of $\mathbf{F}$ is a specified vector function $\mathbf{C}$,

$$\nabla \times \mathbf{F} = \mathbf{C} \qquad (\text{i.e., } \nabla \cdot (\nabla \times \mathbf{F}) = \nabla \cdot \mathbf{C} = 0)$$

Can you determine the function $\mathbf{F}$?

Helmholtz theorem guarantees that the field $\mathbf{F}$ is uniquely determined by the divergence and curl with appropriate **boundary conditions**. (For more details, see Appendix B of Griffiths)

1.6.2 Potentials (simple example)

If the curl of a vector field ($\mathbf{F}$) vanishes (everywhere), then $\mathbf{F}$ can be written as the gradient of a scalar potential (V):

$$\nabla \times \mathbf{F} = 0 \quad \Rightarrow \quad \mathbf{F} = -\nabla V$$

conventional

If the divergence of a vector field ($\mathbf{F}$) vanishes (everywhere), then $\mathbf{F}$ can be expressed as the curl of a vector potential ($\mathbf{A}$):

$$\nabla \cdot \mathbf{F} = 0 \quad \Rightarrow \quad \mathbf{F} = \nabla \times \mathbf{A}$$

Homework of Chap. 1 (part II)

Problem 1.38 Express the unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ (that is, derive Eq. 1.64). Check your answers several ways ($\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \overset{?}{=} 1$, $\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} \overset{?}{=} 0$, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \overset{?}{=} \hat{\boldsymbol{\phi}}, \ldots$). Also work out the inverse formulas, giving $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ (and θ, ϕ).

Problem 1.40 Compute the divergence of the function

$$\mathbf{v} = (r \cos \theta)\, \hat{\mathbf{r}} + (r \sin \theta) \hat{\boldsymbol{\theta}} + (r \sin \theta \cos \phi) \hat{\boldsymbol{\phi}}.$$

Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R, resting on the xy plane and centered at the origin (Fig. 1.40).

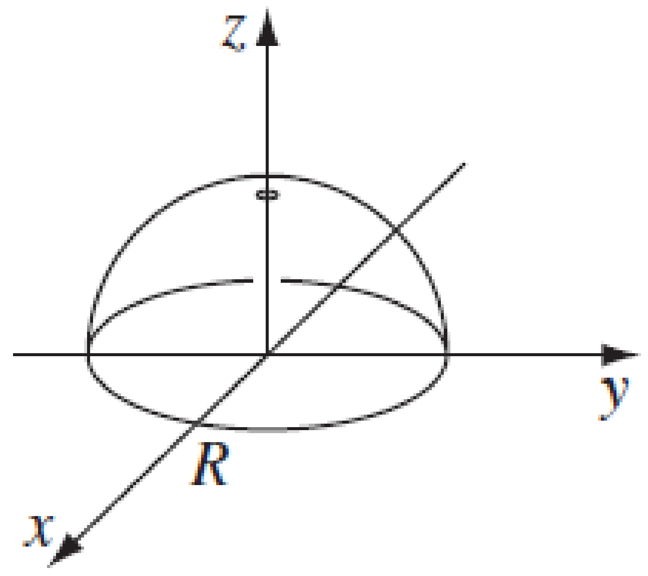


FIGURE 1.40

Problem 1.43

(a) Find the divergence of the function

$$\mathbf{v} = s\,(2 + \sin^2 \phi)\, \hat{\mathbf{s}} + s \sin \phi \cos \phi\, \hat{\boldsymbol{\phi}} + 3z\, \hat{\mathbf{z}}.$$

(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig. 1.43.

(c) Find the curl of $\mathbf{v}$.

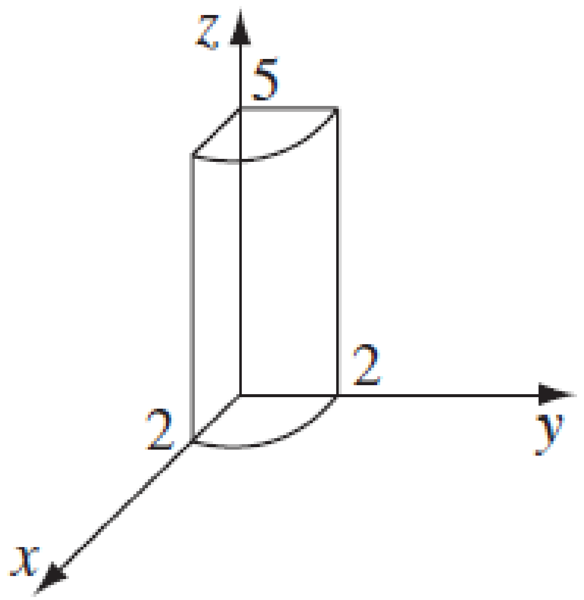


FIGURE 1.43

Homework of Chap. 1 (part III)

Problem 1.46

(a) Show that

$$x\frac{d}{dx}(\delta(x)) = -\delta(x).$$

[*Hint*: Use integration by parts.]

(b) Let $\theta(x)$ be the **step function**:

$$\theta(x) \equiv \begin{Bmatrix} 1, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{Bmatrix}. \qquad (1.95)$$

Show that $d\theta/d\mathrm{x} = \delta(x)$.

Problem 1.49 Evaluate the integral

$$J = \int_v e^{-r}\left(\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}\right)d\tau$$

(where V is a sphere of radius R, centered at the origin) by two different methods, as in Ex. 1.16.