ALGEBRA II

Final Report, 2011.06.08

In the problems below, you should show up the reason, otherwise you have no point.

1. Determine the following polynomials over the given field are irreducible or not.
   (1)(4 points) $x^6 + x^5 + x^3 + x + 1$ over $\mathbb{Z}_2$.
   (2)(4 points) $\Sigma_{i=1}^{n} y^{p(n-i)} x^{pi} + 1$ over $F(y)$, where F is a field with characteristic p for some prime p and y is transcendental over F.
   (3)(4 points) $x^4 - 4x^2 + 5$ over $\mathbb{Q}(i)$.
   (4)(4 points) $x^4 - 4x^3 + 3x^2 + 2x - 2 + \sqrt{3}$ over $\mathbb{Q}(\sqrt{3})$.
   (5)(4 points) $x^4 + 2x^2 + x + 3$ over $\mathbb{Q}$.

2. (20 points) Let u_1, u_2, u_3 be all roots of $x^3 - 2$. Find all the intermediate fields between $\mathbb{Q}(u_1, u_2, u_3)$ and $\mathbb{Q}$.

3. (10 points) Let K be an extension field of F. Suppose $u \in K$ is algebraic of odd degree over F. Show that u^2 is algebraic of odd degree over F and $F(u) = F(u^2)$.

4. (10 points) Show that S_n is generated by $\sigma = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \end{pmatrix}$.

5. (15 points) Find a field F and an irreducible polynomial $f(x)$ in $F[x]$ such that a is a root of multiplicity $k > 1$ of $f(x)$, where $a \in K$ for some extension field K of F. (The definition of multiplicity is in the textbook, Herstein page 209)

6. (15 points) Let K be an extension field of F and $u, v \in K$ are algebraic over F. Suppose $[F(u) : F] = p$ and $[F(v) : F] = q$, where $gcd(p, q) = 1$. Show that $[F(u, v) : F] = pq$.

7. (10 points) Let F be a field with $charF = p \neq 0$. Show that all roots of $x^{p^n} - x \in F[x]$ form a field.

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