ALGEBRA II

Medium Test, 2011.04.18

1. (10 points) Let R be a commutative ring with unit. An ideal P is said to be a prime ideal if $P \neq R$, and whenever $a, b \in R$ and $ab \in P$ then $a \in P$ or $b \in P$. Show that every maximal ideal M in R is prime.

2. (10 points) Let $R = \{a + bi \mid a, b \in \mathbb{Z}\}$. Show that $M = \{x(2 + i) \mid x \in R\}$ is a maximal ideal of R.

3. (10 points) If R is a Euclidean ring, show that every ideal of R is principal.

4. (10 points) If F is a field, show that the only invertible elements in $F[x]$ are the nonzero elements of F.

5. Check the following polynomials are irreducible in $\mathbb{Q}[x]$ or not.
   (1)(3 points) $x^4 + 3x^2 + 3x - 6$
   (2)(4 points) $x^5 - 5x^3 - 2x^2 + 10$
   (3)(3 points) $x^3 + 3x + 2$

6. (10 points) Let $Aut(\mathbb{Q}[x])$ be the set of all automorphisms from $\mathbb{Q}[x]$ to itself. Show that for all $\varphi \in Aut(\mathbb{Q}[x])$, $\varphi(a) = a$ for every $a \in \mathbb{Q}$.

7. (10 points) Let D be an integral domain, F be the field of quotients of D. If K is any field that contains D, show that there is a field F' with $F' \cong F$ such that $K \supset F' \supset D$.

8. (10 points) Find the inverse of $\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$ and prove it.

9. (10 points) Show that if σ, τ are two disjoint cycles, then $\sigma = \tau\sigma\tau^{-1}$.

10. (10 points) Show that if τ is a k-cycle, then $\sigma\tau\sigma^{-1}$ is also a k-cycle, for any permutation σ.

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