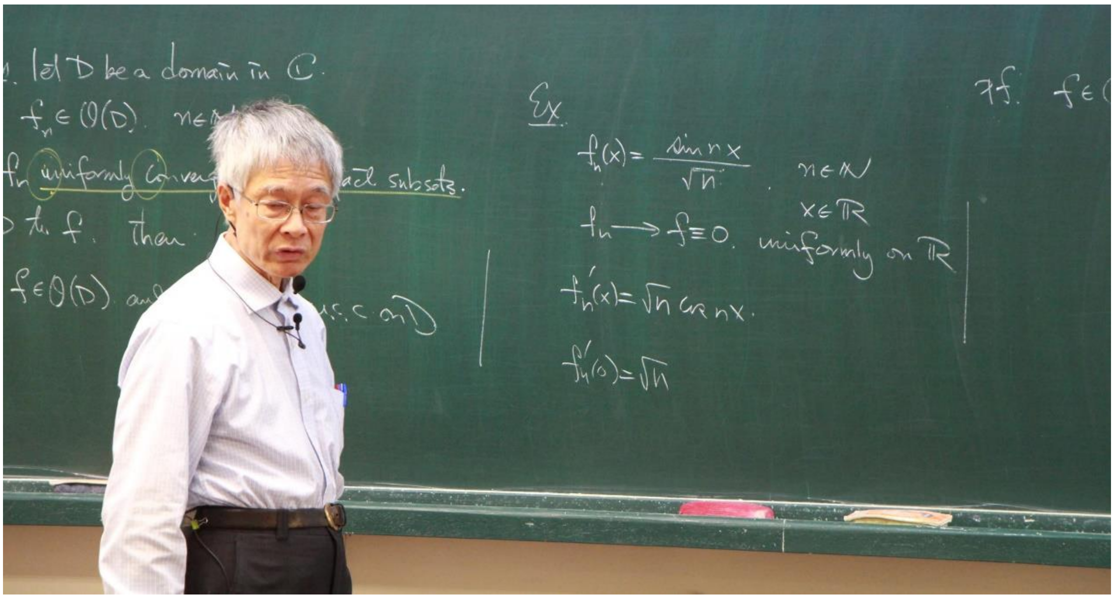
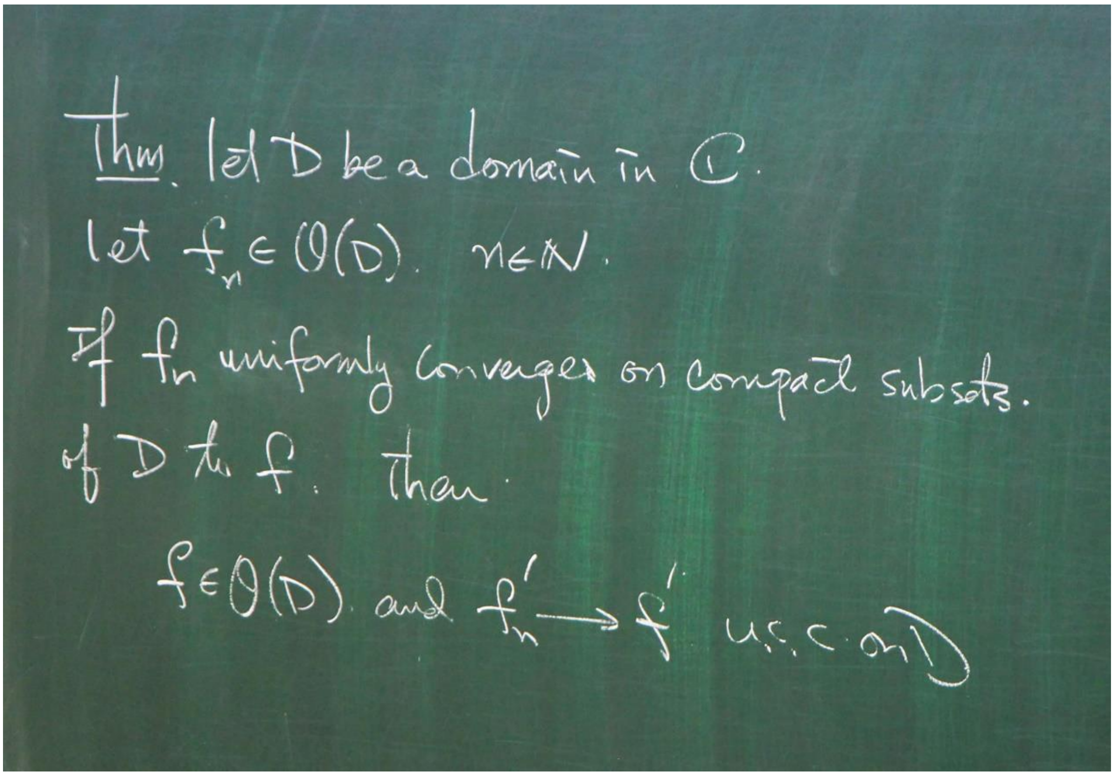
【10920 程守慶教授複變數函數論 / 第 6 堂版書】





Ex.

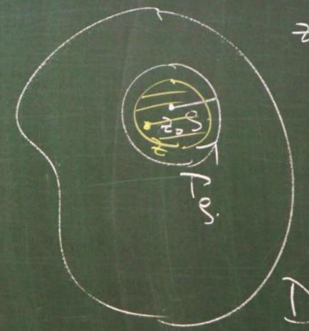
$$f_n(x) = \frac{\sin nx}{\sqrt{n}} \qquad n\in\mathbb{N},\ x\in\mathbb{R}$$

$f_n \to f\equiv 0$ uniformly on $\mathbb{R}$

$$f_n'(x) = \sqrt{n}\cos nx.$$

$$f_n'(0) = \sqrt{n}$$

Pf. $f\in C(D)$

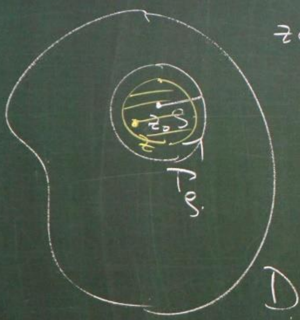


choose

$\overline{B}(z_0;\rho)\subseteq D$

$z\in B(z_0;\rho)$

Pf. $f\in C(D)$ choose $\overline{B}(z_0;\rho)\subseteq D$ $z\in B(z_0;\rho)$



$$f_n(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{f_n(w)}{w-z}\,dw$$

$\lim_{n\to\infty}$

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{f(w)}{w-z}\,dw$$

$$f_n'(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{f_n(w)}{(w-z)^2}\,dw$$

$\xrightarrow{n\to\infty}$ $\downarrow$ uniformly

$$f'(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{f(w)}{(w-z)^2}\,dw$$

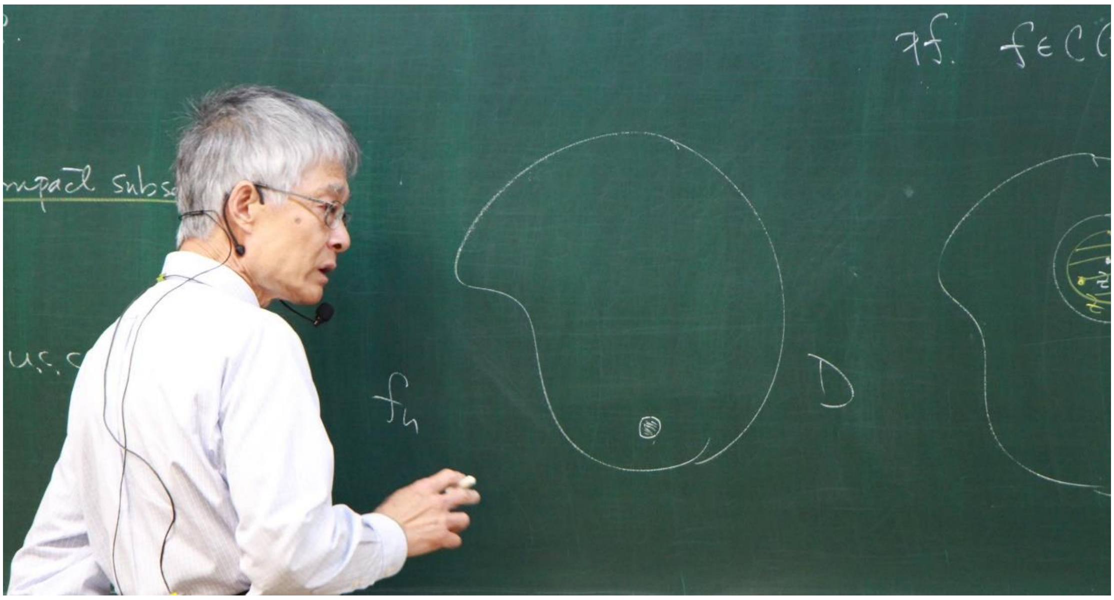
choose $\overline{B}(z_0;\rho)\subseteq D$ $z\in B(z_0;\rho)$

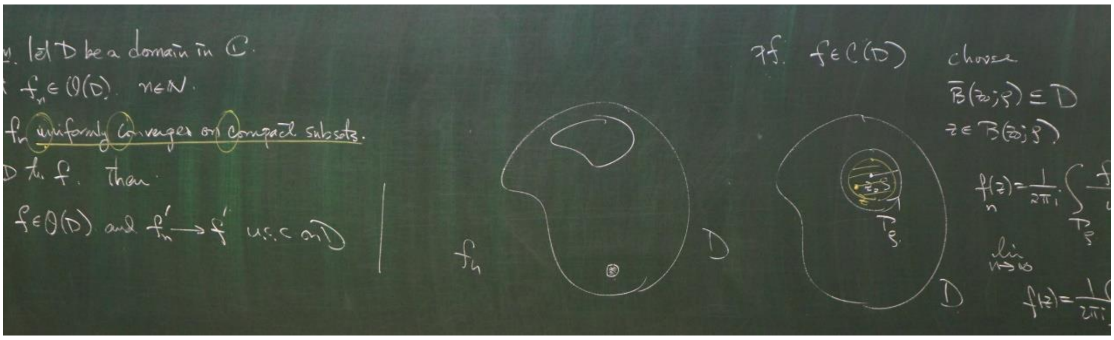
$f_n(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{f_n(w)}{w-z}\,dw$ $\lim_{n\to\infty}$ $f(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{f(w)}{w-z}\,dw$

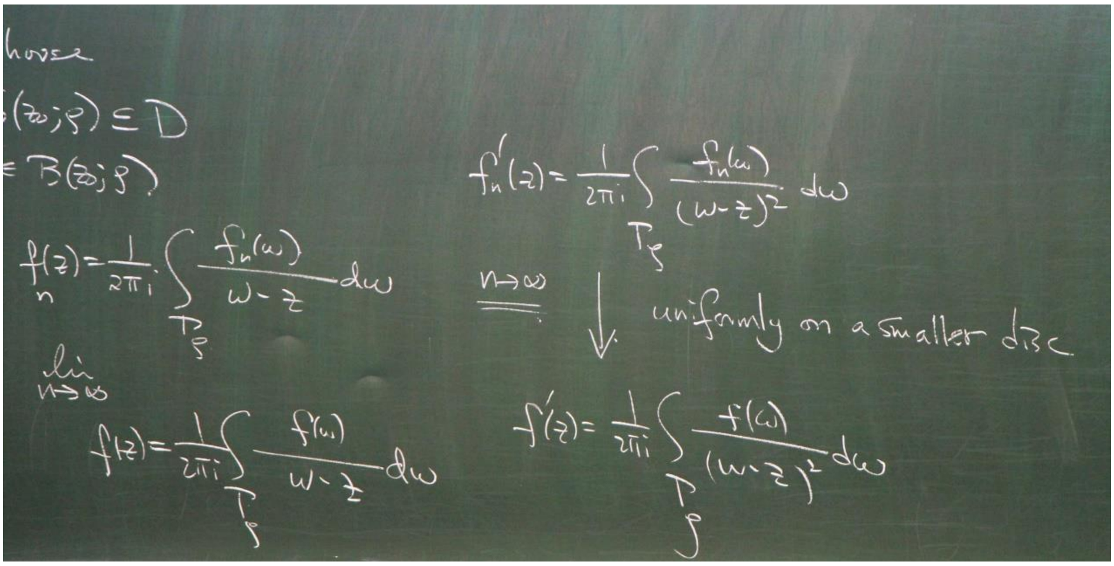
$f_n'(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{f_n(w)}{(w-z)^2}\,dw$

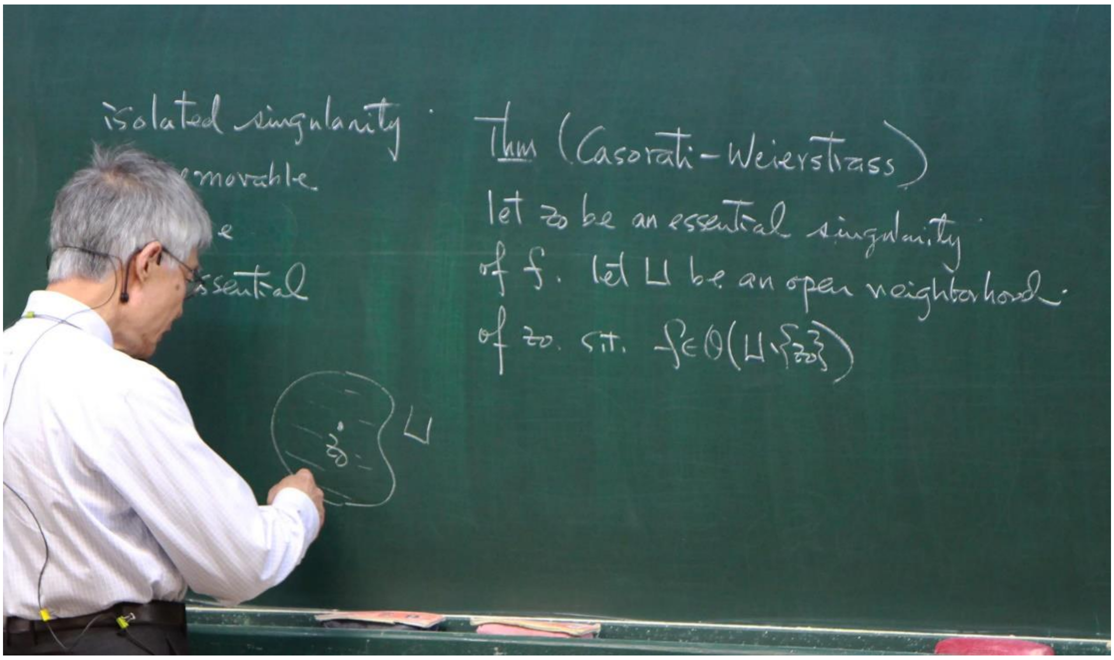
$\xrightarrow{n\to\infty}$ $\downarrow$ uniformly on a smaller disc

$f'(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{f(w)}{(w-z)^2}\,dw$

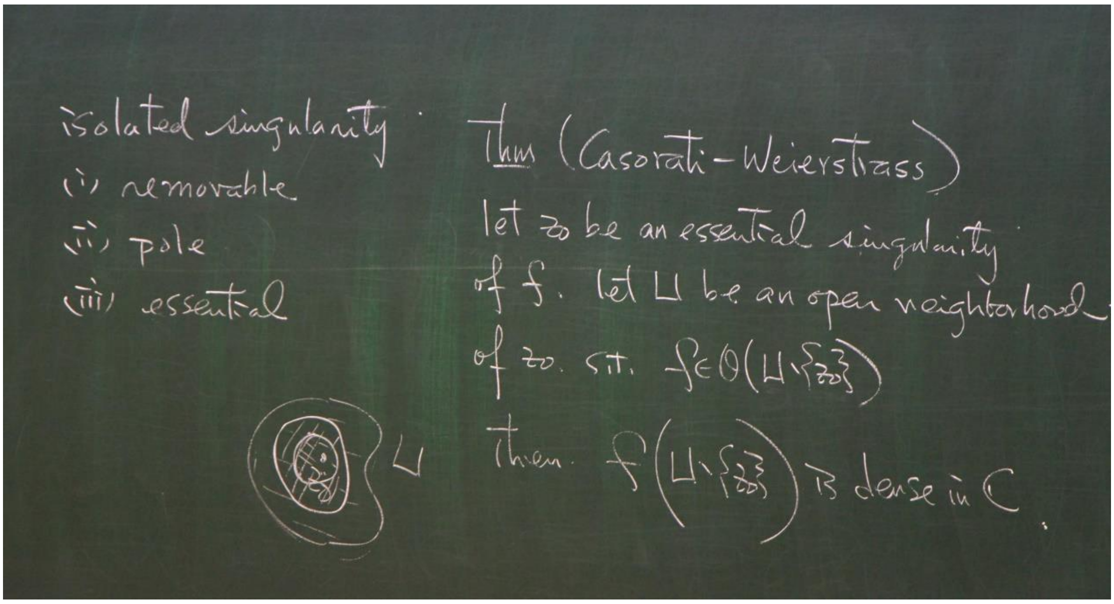

Pf. f ∈ C(
compact subs
u.s.c
f_n
D


let D be a domain in ℂ.
f_n ∈ O(D), n ∈ N.
f_n uniformly converges on compact subsets.
D to f. Then
f ∈ O(D) and f_n' → f' u.s.c on D
Pf. f ∈ C(D)
choose
B̄(z0; ρ) ⊆ D
z ∈ B(z0; ρ)
f_n
D
T_ρ
D


choose
B̄(z0; ρ) ⊆ D
z ∈ B(z0; ρ)
f_n(z) = 1/(2πi) ∫_{T_ρ} f_n(w)/(w − z) dw
lim n→∞
f(z) = 1/(2πi) ∫_{T_ρ} f(w)/(w − z) dw
f_n'(z) = 1/(2πi) ∫_{T_ρ} f_n(w)/(w − z)^2 dw
n→∞
uniformly on a smaller disc
f'(z) = 1/(2πi) ∫_{T_ρ} f(w)/(w − z)^2 dw



isolated singularity.
Thm (Casorati-Weierstrass)
let z0 be an essential singularity
of f. let U be an open neighborhood
of z0. s.t. f∈O(U∖{z0})
U



isolated singularity.
(i) removable
(ii) pole
(iii) essential
Thm (Casorati-Weierstrass)
let z0 be an essential singularity
of f. let U be an open neighborhood
of z0. s.t. f∈O(U∖{z0})
then. f(U∖{z0}) is dense in ℂ.
U

Picard's great theorem:

Same hypotheses as before

Then

$f(\Delta \setminus \{z_0\})$ misses at most one point.

Ex. $f(z) = e^{\frac{1}{z}}$ 0: essential singularity

misses only 0.

Picard's little theorem.

Let $f \in \mathcal{O}(\mathbb{C})$. $f(z) = e^{z}$ misses 0.

If $f(\mathbb{C})$ misses two points $f(z) = z$

then f is a constant function.

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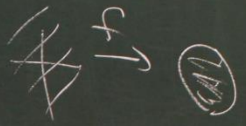
singularity

Liouville: Any bounded entire ft.



is a constant function.

...n.

...entire ft.

...t function.

$f(z)=e^z$ misses 0.

$f(z)=z$

Pf. (Casorati-Weierstrass)

If not. i.e., $\overline{f(U\setminus\{z_0\})}\neq\mathbb{C}$.

$\mathbb{C}\setminus\overline{f(U\setminus\{z_0\})}$ contains $B(p;\delta)$ $\delta>0$.

Consider.

$$g(z)=\frac{1}{f(z)-p},\quad z\in U\setminus\{z_0\}.$$

$$|g(z)|=\frac{1}{|f(z)-p|}\leq\frac{1}{\delta}.$$

...eorem.

...s two points

...nstant function.

...: Any bounded entire ft

is a constant function.

$f(z)=e^z$ misses 0.

$f(z)=z$

Pf. (Casorati-Weierstrass)

If not. i.e., $\overline{f(U\setminus\{z_0\})}\neq\mathbb{C}$.

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Consider.

$$g(z)=\frac{1}{f(z)-p},\quad z\in U\setminus\{z_0\}.$$

$$|g(z)|=\frac{1}{|f(z)-p|}\leq\frac{1}{\delta}.$$

∴ z_0 is a removable singularity of g.

∴ $\lim_{z\to z_0} g(z)=L$ exists.

① $L=0$. ∴ $\lim_{z\to z_0}|f(z)|=\infty$ $\Rightarrow$ z_0 is a pole

※

② $L\neq 0$. $\dfrac{1}{\lim_{z\to z_0} f(z)-p}=L\neq 0$ $\Rightarrow$ $\lim_{z\to z_0} f(z)=p+\dfrac{1}{L}$

※

∴ $\Rightarrow$ z_0 is a removable sing. of f

Residue (留数)

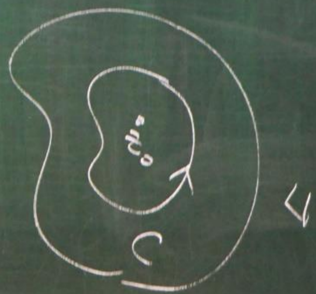
let z_0 be a singularity of f

let C be a simple closed curve (positively oriented)

s.t. z_0 is surrounded by C

and f is holomorphic on $U \setminus \{z_0\}$

and $C \subseteq U \setminus \{z_0\}$



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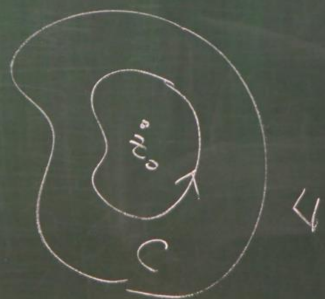


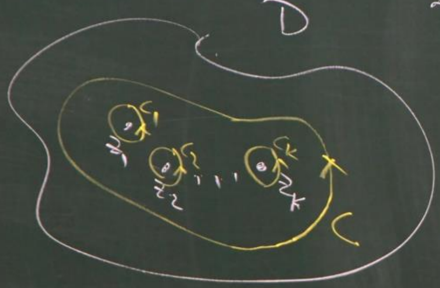
...ue of f

Thm. (residue)

C surrounds $z_1, \dots z_k$.

Then

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \sum_{j=1}^{k} \frac{1}{2\pi i}\int_{C_j} f(z)\,dz$$

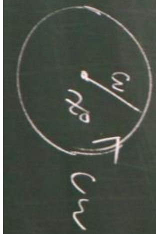


z_0: pole.

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \dots + \frac{c_{-1}}{z-z_0} + h(z).$$

$$\text{residue} = \frac{1}{2\pi i}\int_{C_\varepsilon} f(z)\,dz = \frac{1}{2\pi i}\int_{C_\varepsilon}\left(\frac{c_{-m}}{(z-z_0)^m} + \dots + \frac{c_{-1}}{z-z_0}\right)dz = \frac{1}{2\pi i}\int_0^{2\pi}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{c_{-m}}{(\varepsilon e^{i\dots}}$$



$z \in C_\varepsilon$

$z = z_0 + \varepsilon e^{i\theta}$ $\quad dz = i\varepsilon e^{i\theta}\,d\theta$

$z - z_0 = \varepsilon e^{i\theta}$

$$\ldots^{m} + \cdots + \frac{c_{-1}}{z-z_0} + h(z).$$

$$\frac{1}{2\pi i}\int_{C_\varepsilon} f(z)\,dz = \frac{1}{2\pi i}\int_{C_\varepsilon}\left(\frac{c_{-m}}{(z-z_0)^m} + \cdots + \frac{c_{-1}}{z-z_0}\right)dz = \frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{c_{-m}}{(\varepsilon e^{i\theta})^m} + \cdots + \frac{c_{-1}}{\varepsilon e^{i\theta}}\right) i\varepsilon e^{i\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{c_{-m}}{(\varepsilon e^{i\theta})^{m-1}} + \cdots + \frac{c_{-2}}{\varepsilon e^{i\theta}} + c_{-1}\right)d\theta$$

$z \in C_\varepsilon$

$z = z_0 + \varepsilon e^{i\theta}$ $dz = i\varepsilon e^{i\theta}\,d\theta$

$z - z_0 = \varepsilon e^{i\theta}$

Simple pole $f(z) = \frac{c_{-1}}{z-z_0} + h(z)$

residue at $z_0 = \lim_{z\to z_0}(z-z_0)f(z)$ | $(z-z_0)f(z) = c_{-1} + (z-z_0)h(z)$

order of the pole $m > 1$.

residue at z_0 $= \lim_{z\to z_0}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$

$f(z) = \frac{c_{-1}}{z-z_0} + h(z)$

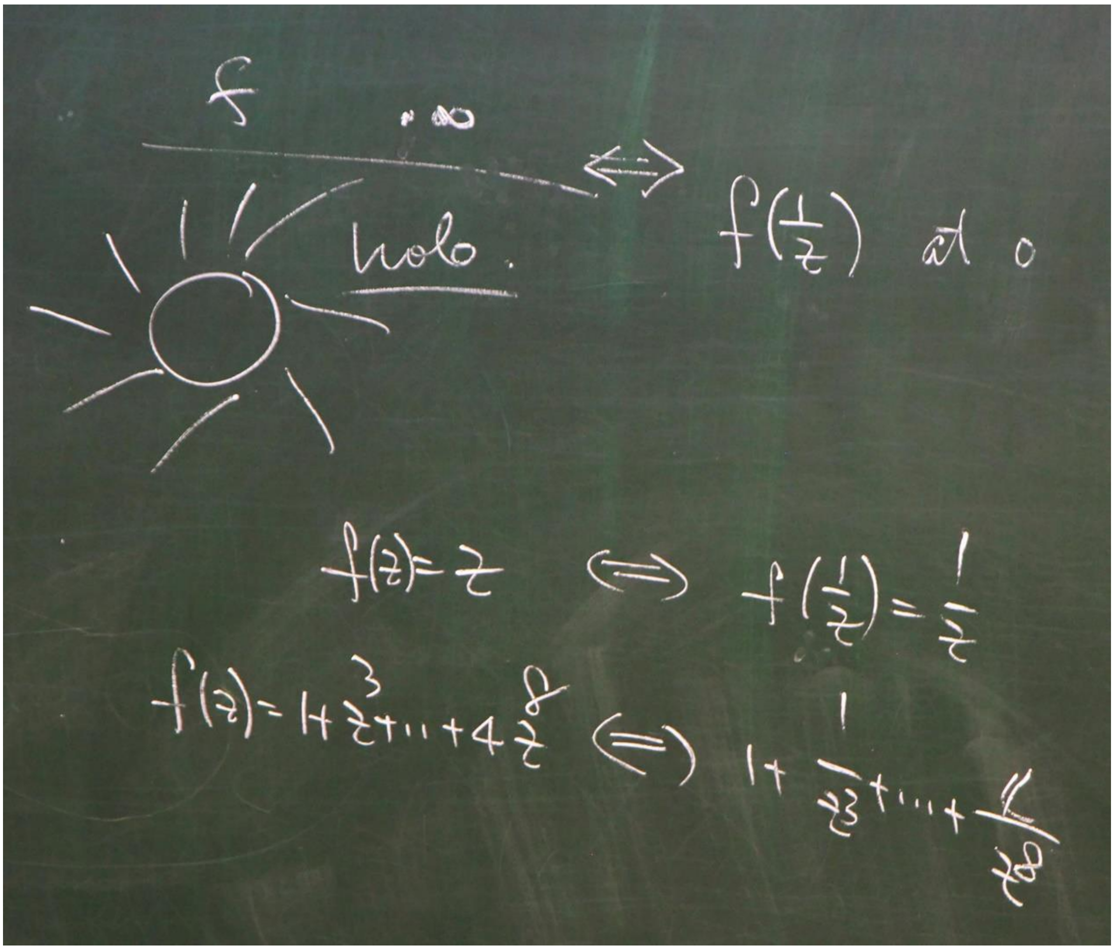
$\lim_{z\to z_0}(z-z_0)f(z)$ | $(z-z_0)f(z) = c_{-1} + (z-z_0)h(z)$

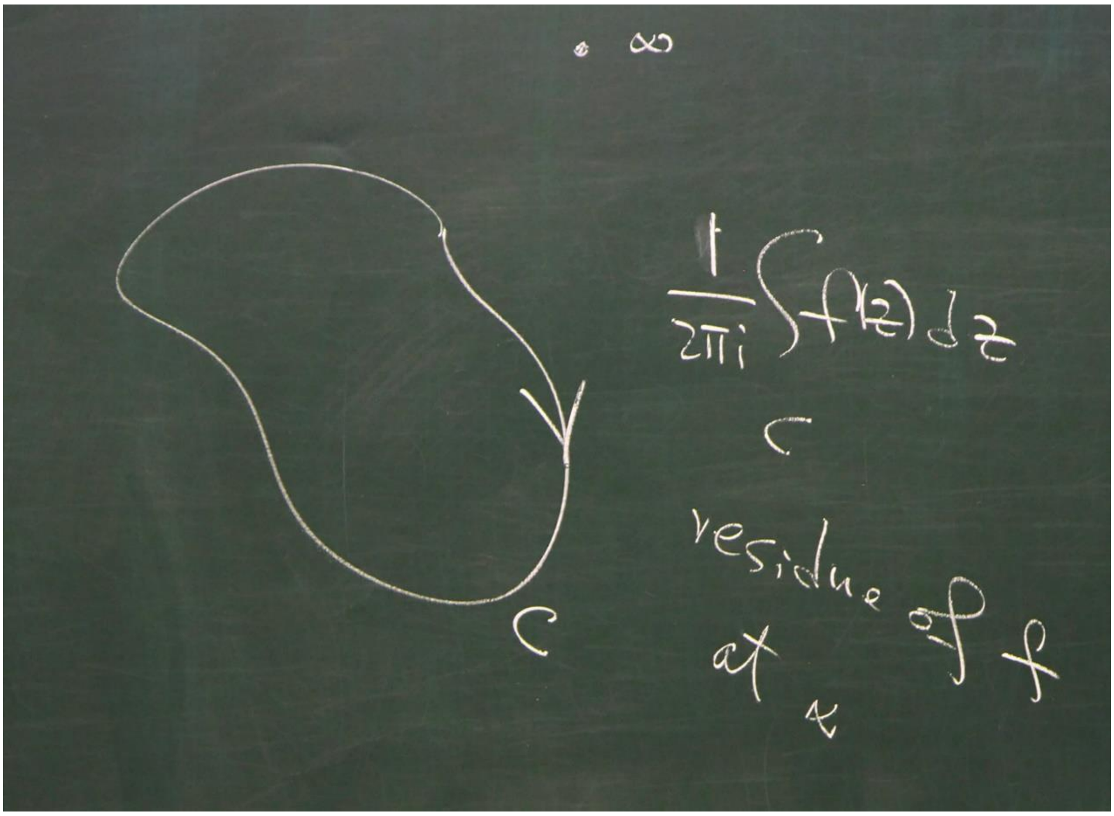
le $m > 1$.

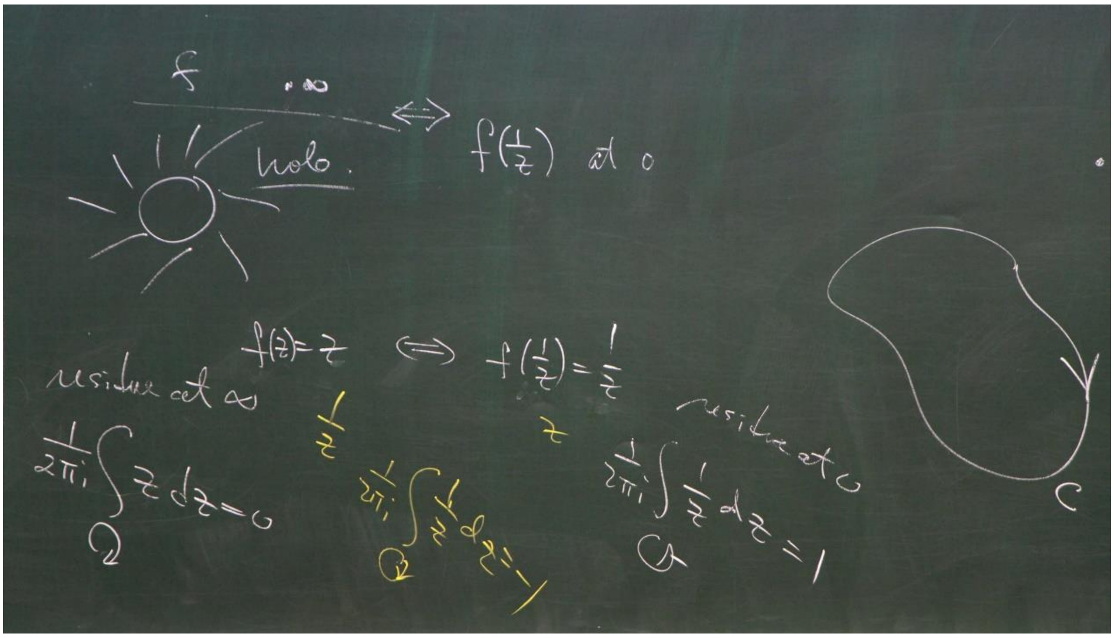
$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \cdots + \frac{c_{-1}}{z-z_0} + h(z).$$

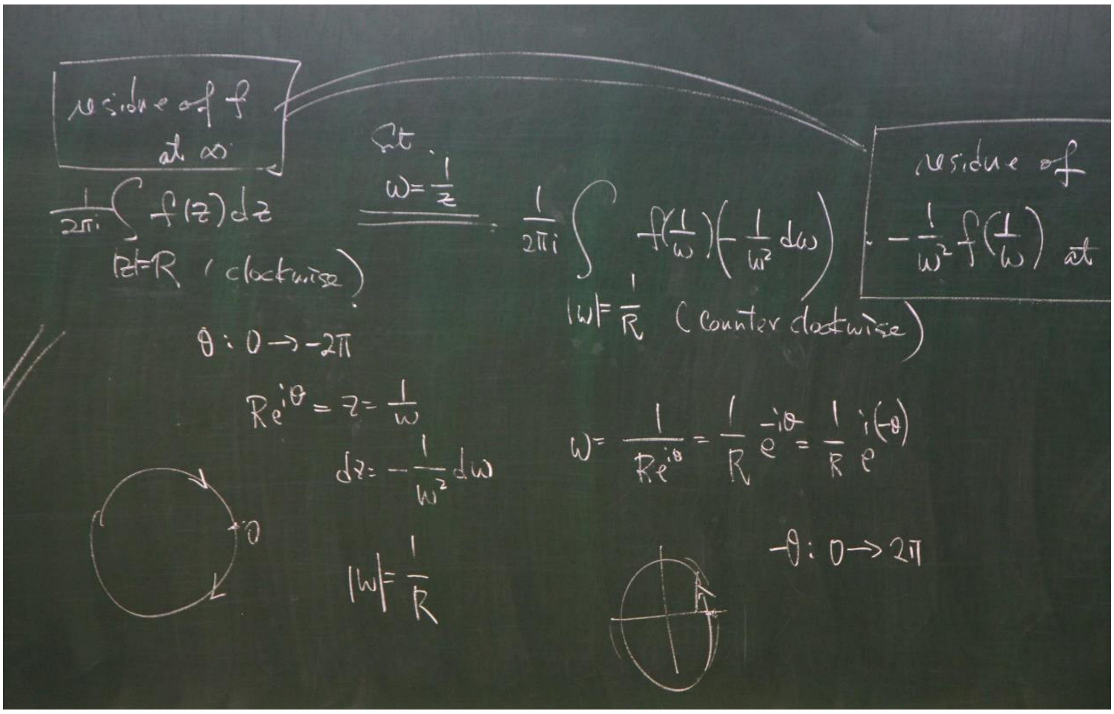
$\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$

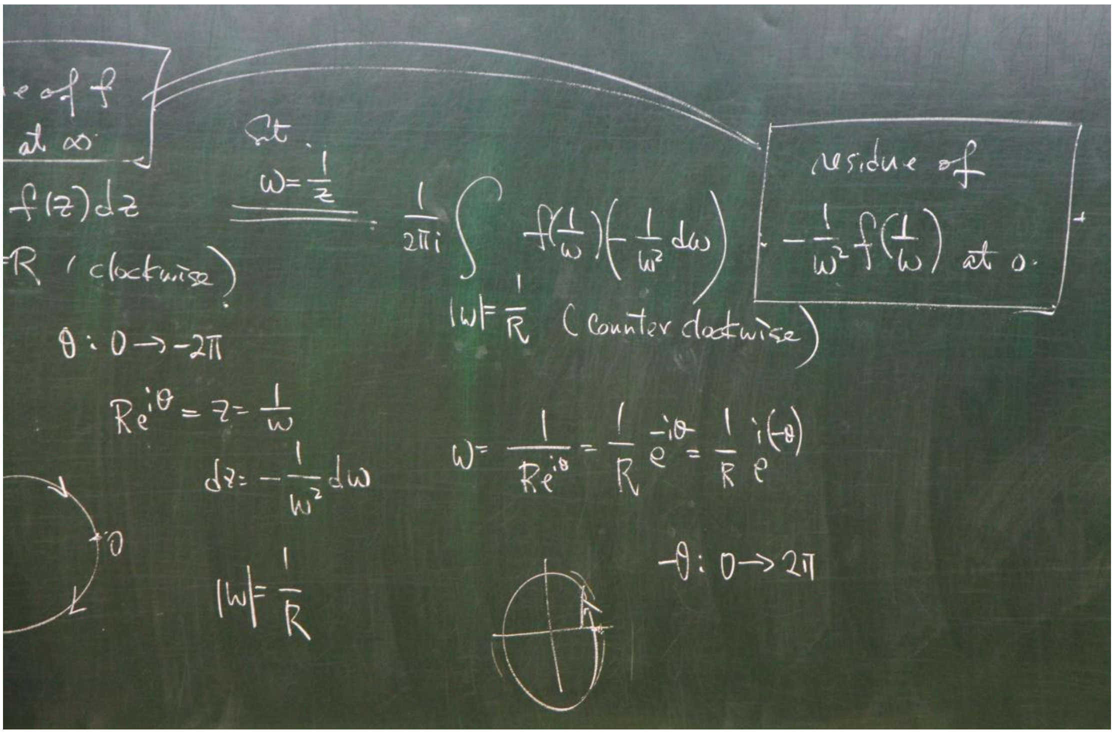
$$(z-z_0)^m f(z) = c_{-m} + c_{-m+1}(z-z_0) + \cdots + c_{-1}(z-z_0)^{m-1} + (z-z_0)^m h(z)$$

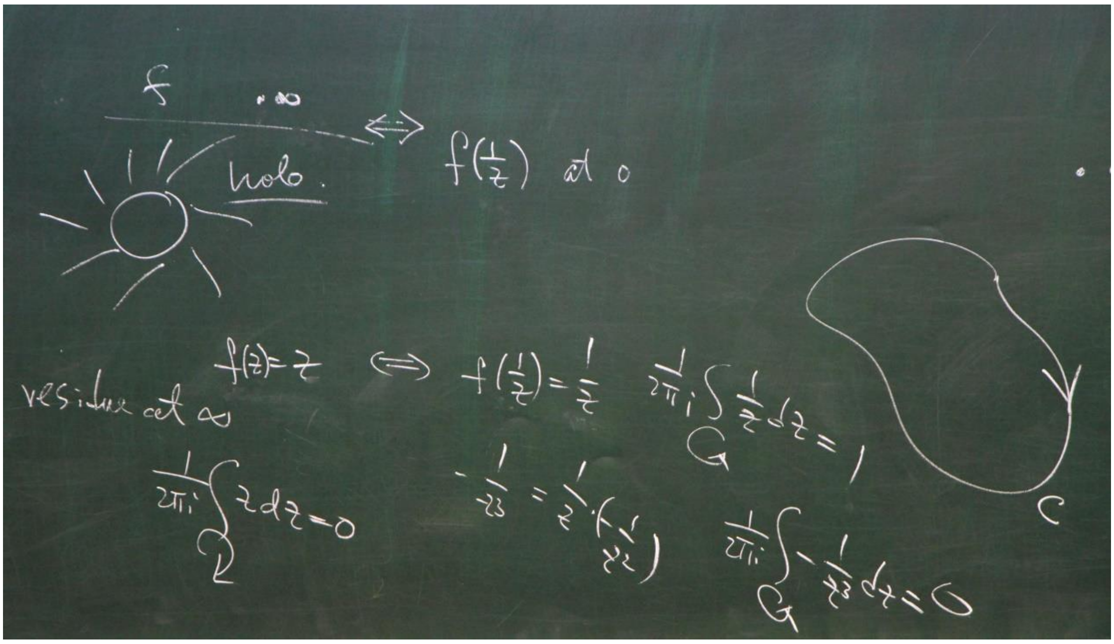

f
∞
holo.
⟺ f(1/z) at 0
f(z) = z ⟺ f(1/z) = 1/z
f(z) = 1 + z^3 + ... + 4z^8 ⟺ 1 + 1/z^3 + ... + 4/z^8

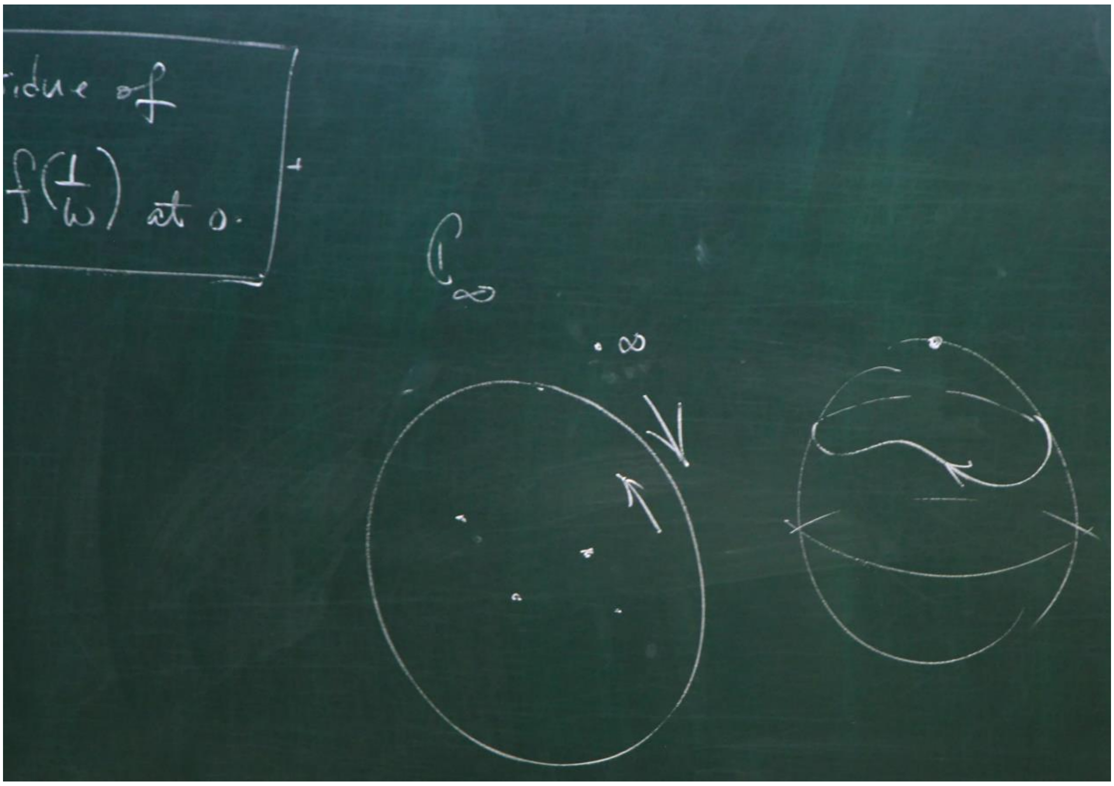

∞
1/(2πi) ∫_C f(z) dz
C
residue of f
at ∞


f
∞
holo.
⟺ f(1/z) at 0
f(z)=z ⟺ f(1/z) = 1/z
residue at ∞
1/2πi ∫ z dz = 0
1/z
z
1/2πi ∫ 1/z dz = -1
residue at 0
1/2πi ∫ 1/z dz = 1
C


residue of f at ∞
1/2πi ∫ f(z) dz
|z|=R (clockwise)
set w = 1/z
1/2πi ∫ f(1/w)(- 1/w² dw)
|w| = 1/R (counter clockwise)
residue of - 1/w² f(1/w) at
θ : 0 → -2π
Re^{iθ} = z = 1/w
dz = - 1/w² dw
|w| = 1/R
θ
w = 1/Re^{iθ} = 1/R e^{-iθ} = 1/R e^{i(-θ)}
-θ : 0 → 2π


e of f
at ∞
Set $w = \frac{1}{z}$
$f(z)\,dz$
$=R$ (clockwise)
$\theta : 0 \to -2\pi$
$Re^{i\theta} = z = \frac{1}{w}$
$dz = -\frac{1}{w^2}dw$
$|w| = \frac{1}{R}$
$\frac{1}{2\pi i}\int_{|w|=\frac{1}{R}} f\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}dw\right)$
(counter clockwise)
residue of
$-\frac{1}{w^2}f\left(\frac{1}{w}\right)$ at 0.
$w = \frac{1}{Re^{i\theta}} = \frac{1}{R}e^{-i\theta} = \frac{1}{R}e^{i(-\theta)}$
$-\theta : 0 \to 2\pi$


f ∞
holo.
⟺ $f\left(\frac{1}{z}\right)$ at 0
$f(z) = z$ ⟺ $f\left(\frac{1}{z}\right) = \frac{1}{z}$
residue at ∞
$\frac{1}{2\pi i}\int_C \frac{1}{z}dz = 1$
$\frac{1}{2\pi i}\int_C z\,dz = 0$
$-\frac{1}{z^3} = \frac{1}{z}\cdot\left(-\frac{1}{z^2}\right)$
$\frac{1}{2\pi i}\int_C -\frac{1}{z^3}dz = 0$
C


idue of
f(1/w) at 0.
C∞
∞