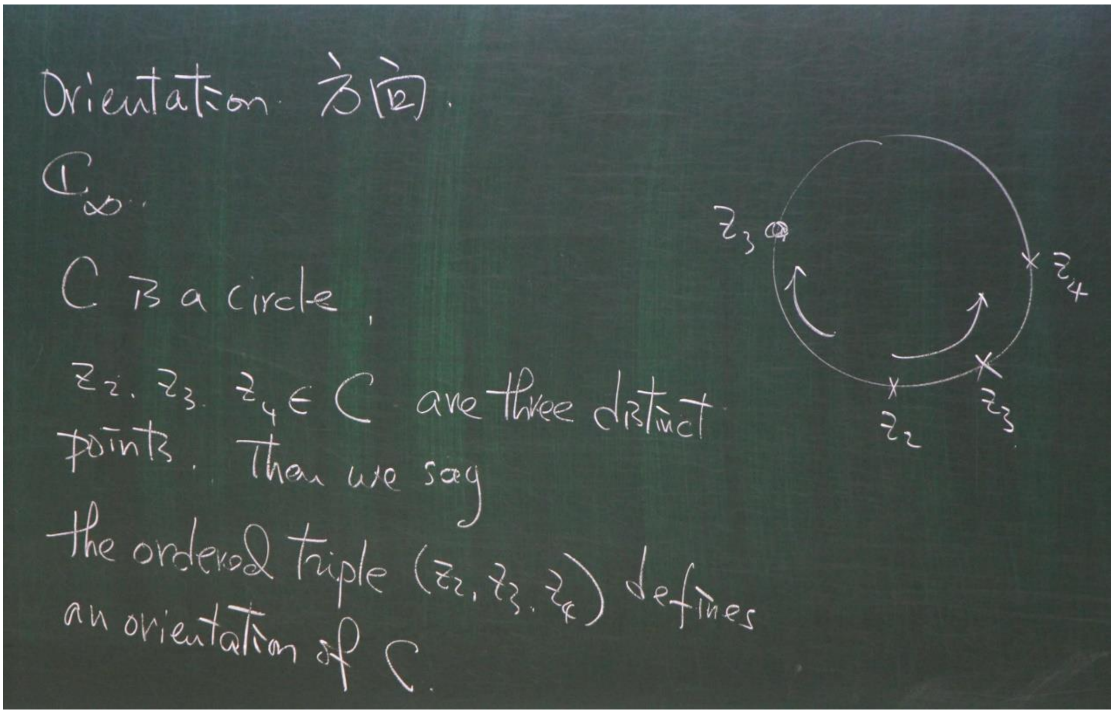
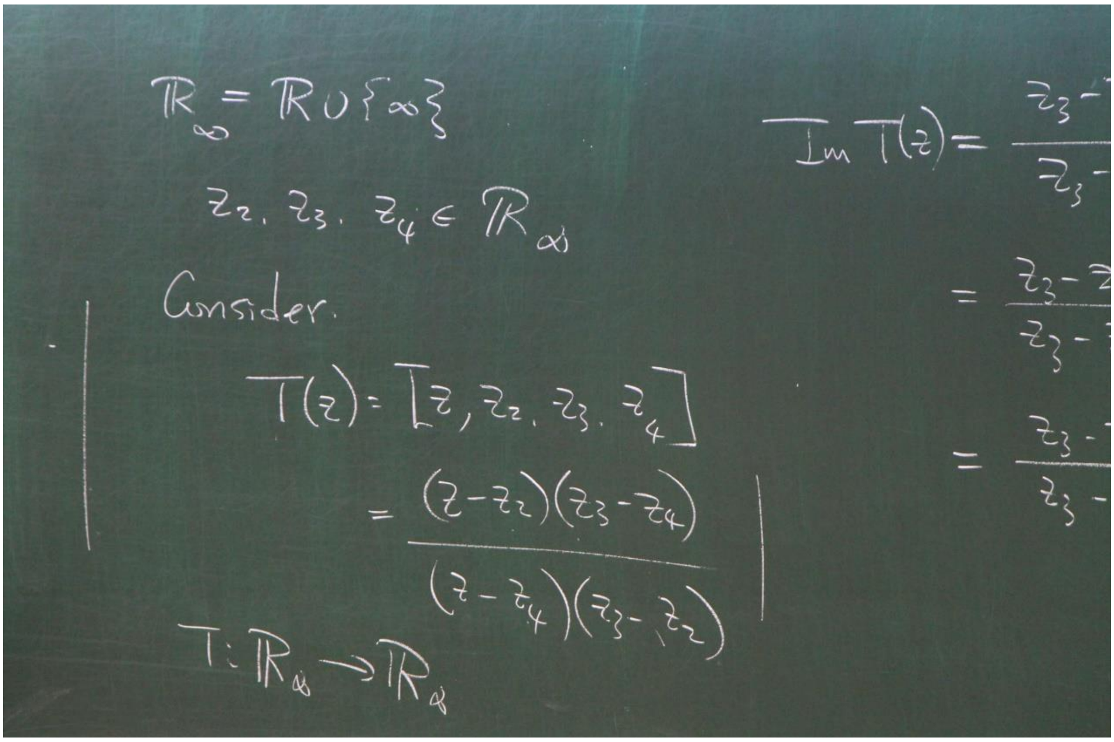
【10920 程守慶教授複變數函數論 / 第 28 堂版書】





$$\operatorname{Im} T(z) = \frac{z_3 - z_4}{z_3 - z_2} \operatorname{Im} \frac{z - z_2}{z - z_4}$$

$$= \frac{z_3 - z_4}{z_3 - z_2} \operatorname{Im} \frac{(z - z_2)(\bar{z} - z_4)}{|z - z_4|^2}$$

$$= \frac{z_3 - z_4}{z_3 - z_2} \operatorname{Im} \frac{|z|^2 + z_2 z_4 - z_2 \bar{z} - z_4 z}{|z - z_4|^2} = (z$$

$$\frac{z - z_2}{z - z_4}$$

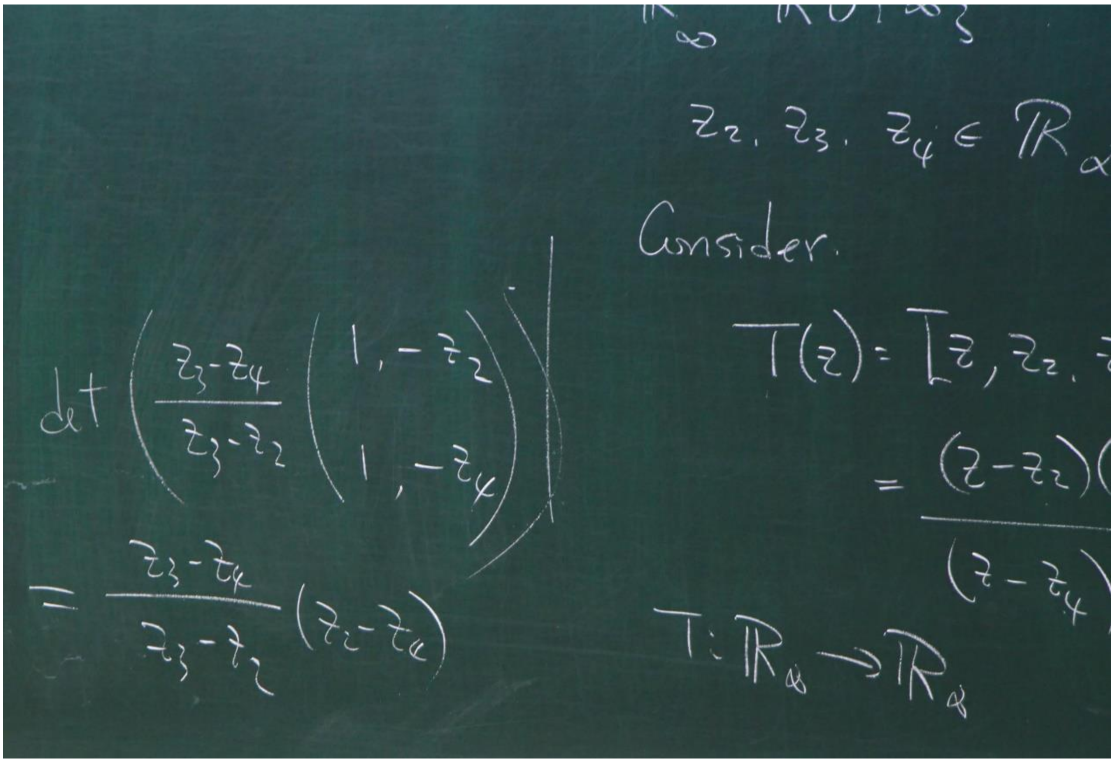
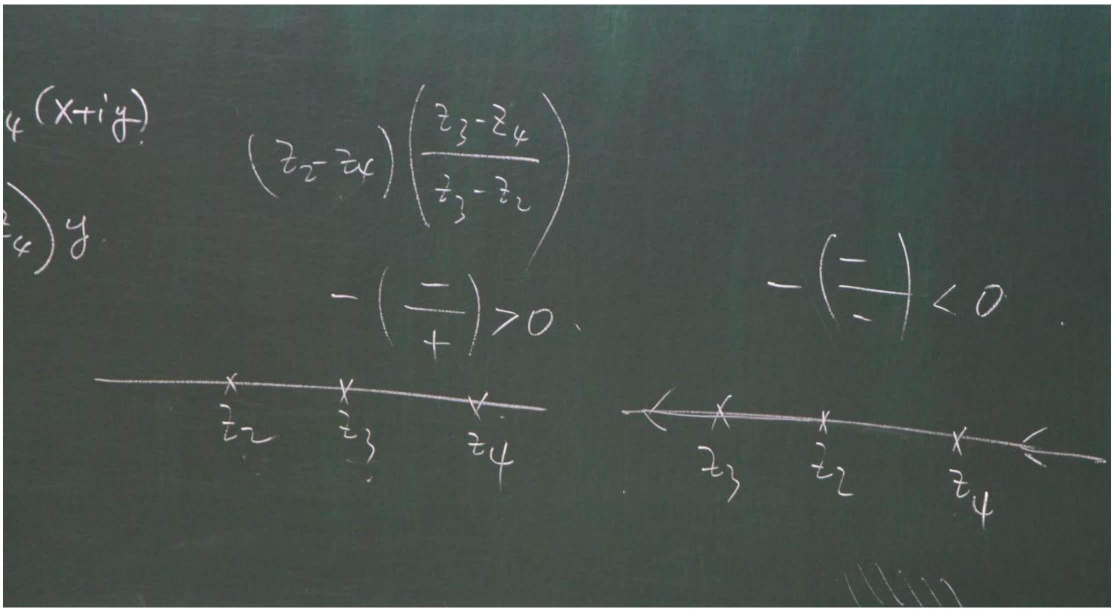
$$z = x + iy.$$

$$-z_2 \bar{z} - z_4 z = -z_2 (x - iy) - z_4 (x + iy)$$

$$= -z_2 x - z_4 x + i (z_2 - z_4) y$$

$$\frac{(z - z_2)(\bar{z} - z_4)}{|z - z_4|^2}$$

$$\frac{|z|^2 + z_2 z_4 - z_2 \bar{z} - z_4 z}{|z - z_4|^2}$$ $$= \boxed{(z_2 - z_4)\left(\frac{z_3 - z_4}{z_3 - z_2}\right)} \operatorname{Im} z$$


z2, z3, z4 ∈ R∞
Consider.
T(z) = [z, z2,
= (z − z2)(
(z − z4)
det ((z3 − z4)/(z3 − z2) (1, −z2 / 1, −z4))
= (z3 − z4)/(z3 − z2) (z2 − z4)
T: R∞ → R∞

4 (x+iy)
(z2 − z4) ((z3 − z4)/(z3 − z2))
4) y
− (−/+) > 0
− (−/−) < 0
z2 z3 z4
z3 z2 z4

C : circle

$z_2, z_3, z_4 \in C$ distinct

$\varphi : C \longrightarrow \mathbb{R}_\infty$

$T(z) = [z, z_2, z_3, z_4]$

$$\{z \mid \operatorname{Im} T(z) > 0\} = \{z \mid \operatorname{Im}[\varphi(z), \varphi(z_2), \varphi(z_3), \varphi(z_4)]\}$$

$$= \varphi^{-1}\left(\{w \mid \operatorname{Im}[w, \varphi(z_2), \varphi(z_3), \varphi(z_4)]\}\right)$$

Define w.r.t. (z_2, z_3, z_4).

the left side of C is $\{z \mid \operatorname{Im}[z, z_2, z_3, z_4] > 0\}$.

right 〃 $\{z \mid \operatorname{Im}[z, z_2, z_3, z_4] < 0\}$.

$[\varphi(z), \varphi(z_2), \varphi(z_3), \varphi(z_4)]\}$

(Orientation Principle)

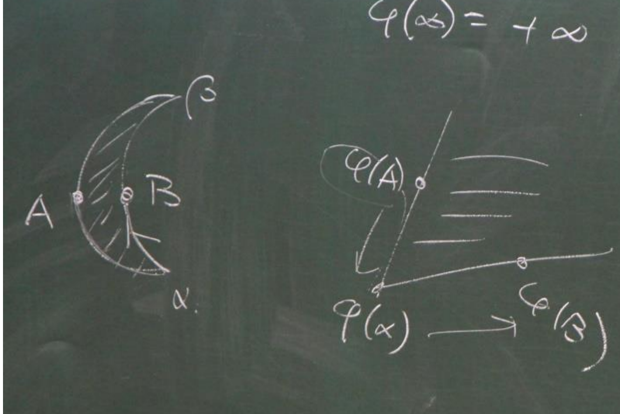
C: circle $z_2, z_3, z_4 \in C$ distinct.

φ: linear fractional transformation.

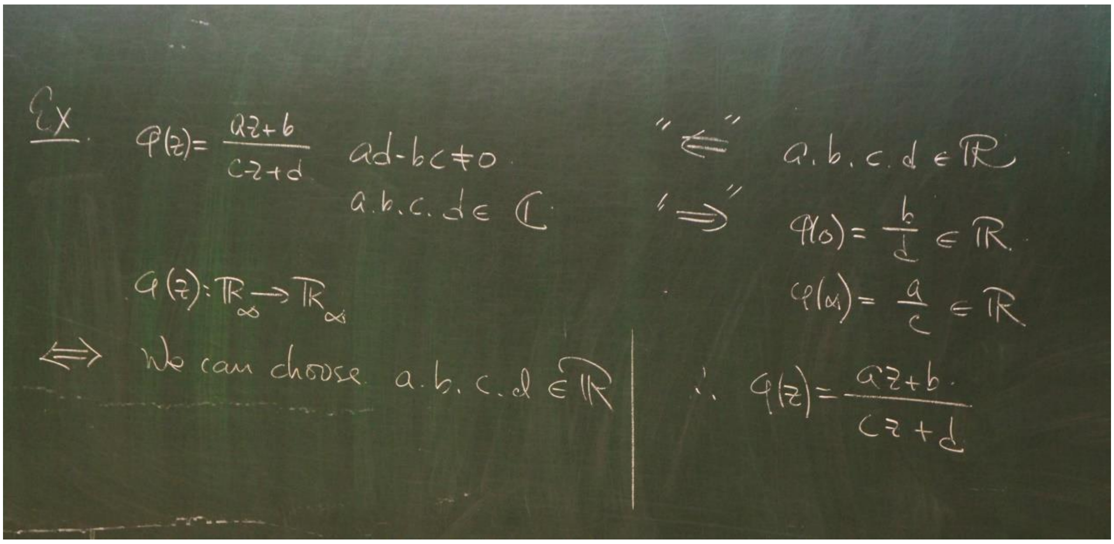
$\Rightarrow$ w.r.t. (z_2, z_3, z_4) of C

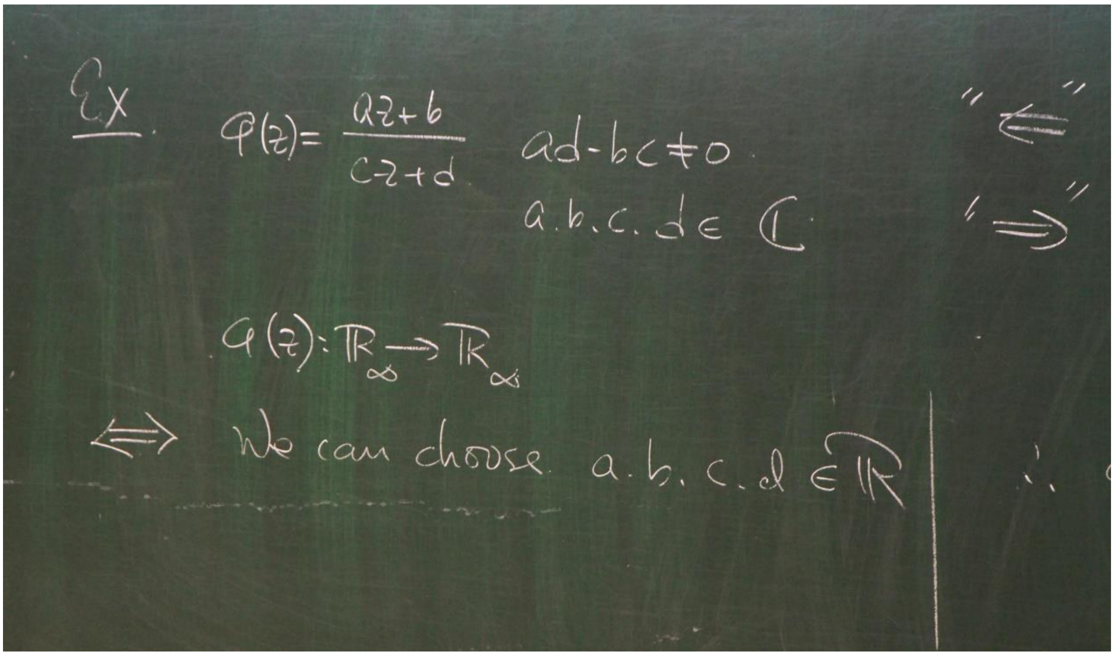
$=$ $(\varphi(z_2), \varphi(z_3), \varphi(z_4))$ of $\varphi(C)$

Then the left side of C corresponds to the left side of $\varphi(C)$



(Orientation Principl
C: circle z_2.
φ: linear fractio
$\Rightarrow$ w.r.t. $(z_2, z_3, z_4$
$=$ $(\varphi(z_2), \varphi(z_3)$
Then the left side o
left side of φ


Ex. $\varphi(z) = \frac{az+b}{cz+d}$ $ad-bc \neq 0$
$a.b.c.d \in \mathbb{C}$
$\varphi(z): \mathbb{R}_\infty \to \mathbb{R}_\infty$
$\Longleftrightarrow$ We can choose $a.b.c.d \in \mathbb{R}$
" $\Longleftarrow$ " $a.b.c.d \in \mathbb{R}$
" $\Longrightarrow$ " $\varphi(0) = \frac{b}{d} \in \mathbb{R}$.
$\varphi(\infty) = \frac{a}{c} \in \mathbb{R}$
$\therefore \varphi(z) = \frac{az+b}{cz+d}$


Ex. $\varphi(z) = \frac{az+b}{cz+d}$ $ad-bc \neq 0$.
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$\varphi(z): \mathbb{R}_\infty \to \mathbb{R}_\infty$
$\Longleftrightarrow$ We can choose $a.b.c.d \in \mathbb{R}$

"$\Leftarrow$" $a, b, c, d \in \mathbb{R}$

"$\Rightarrow$" $\varphi(0) = \frac{b}{d} \in \mathbb{R}$

$\varphi(\infty) = \frac{a}{c} \in \mathbb{R}$

$\therefore\ \varphi(z) = \frac{az+b}{cz+d}$

$\varphi^{-1}(0) = -\frac{b}{a} \in \mathbb{R}$

$\varphi^{-1}(\infty) = -\frac{d}{c} \in \mathbb{R}$

$\varphi(z) = \frac{az+b}{cz+d} = \frac{a\left(z+\frac{b}{a}\right)}{c\left(z+\frac{d}{c}\right)}$

$\therefore\ \frac{a}{c} \in \mathbb{R}$

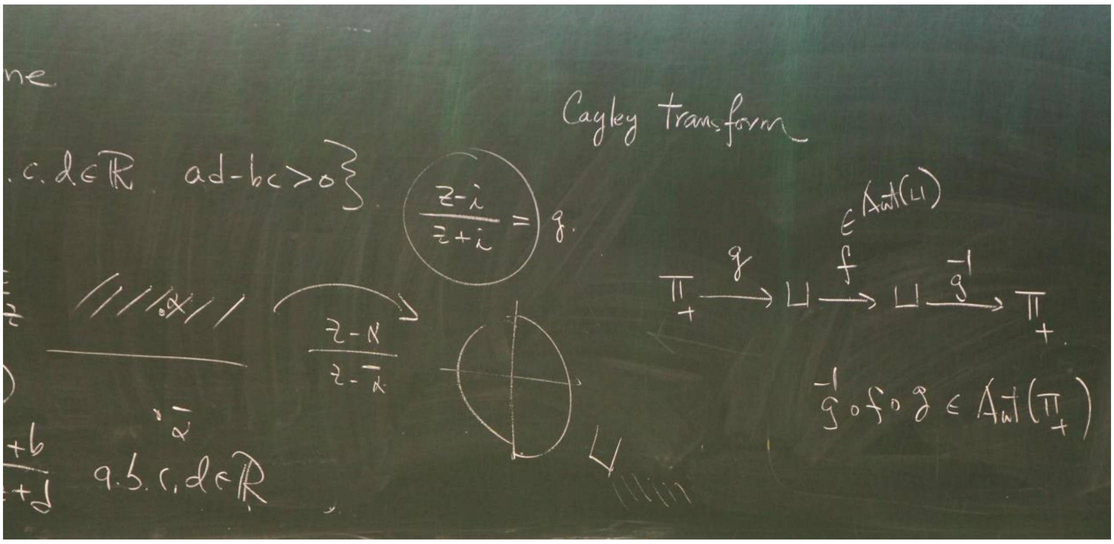
$\Pi_+ = \{ z = x+iy \mid y>0 \}$ upper half plane

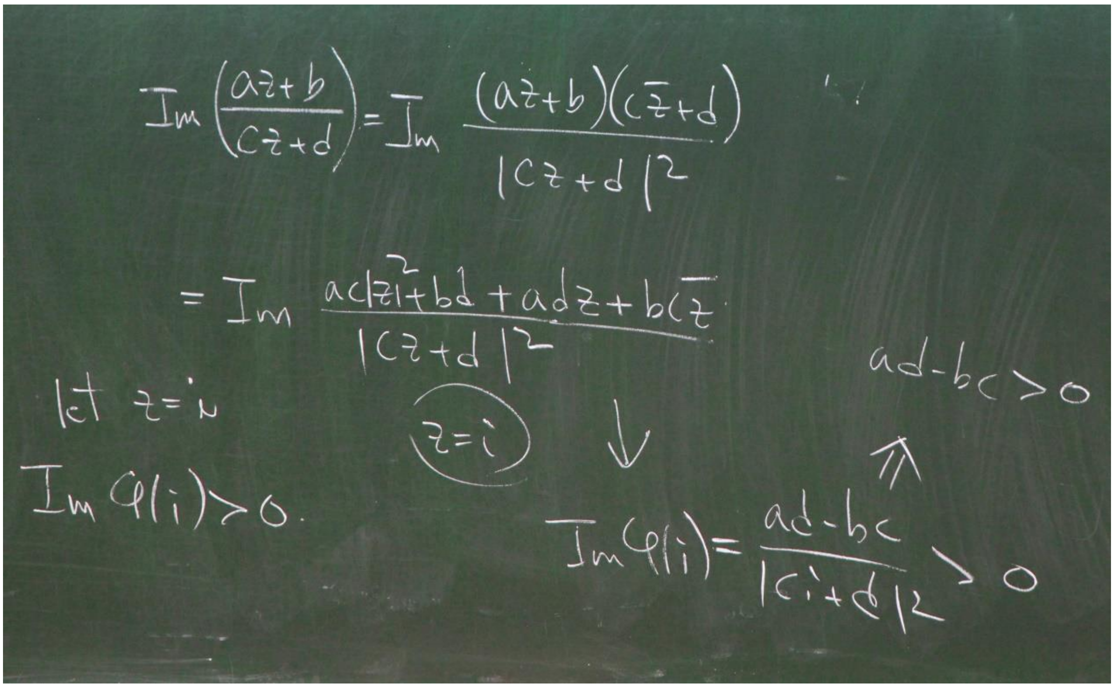
<u>thm</u> $\mathrm{Aut}(\Pi_+) = \left\{ \varphi(z) = \frac{az+b}{cz+d} \;\middle|\; a, b, c, d \in \mathbb{R},\ ad-bc>0 \right\}$

pf. "$\Leftarrow$" o.k. $\Longleftrightarrow$

"$\Rightarrow$" $\psi \in \mathrm{Aut}(\Pi_+)$ $\quad f(z) = e^{i\theta}\varphi_a(z) = e^{i\theta}\frac{a-z}{1-\bar{a}z}$

$\psi = g^{-1} \circ f \circ g \Rightarrow \psi(z) = \frac{az+b}{cz+d}$ $\quad a, b, c, d \in \mathbb{R}$


Cayley transform
a.b.c.d ∈ ℝ
ad-bc > 0
(z-i)/(z+i) = g
(z-α)/(z-ᾱ)
∈ Aut(⊔)
g⁻¹ ∘ f ∘ g ∈ Aut(Π+)


Im((az+b)/(cz+d)) = Im ((az+b)(cz̄+d))/|cz+d|²
= Im (ac|z|²+bd+adz+bcz̄)/|cz+d|²
let z=i
z=i
Im φ(i) > 0.
ad-bc > 0
Im φ(i) = (ad-bc)/|ci+d|² > 0