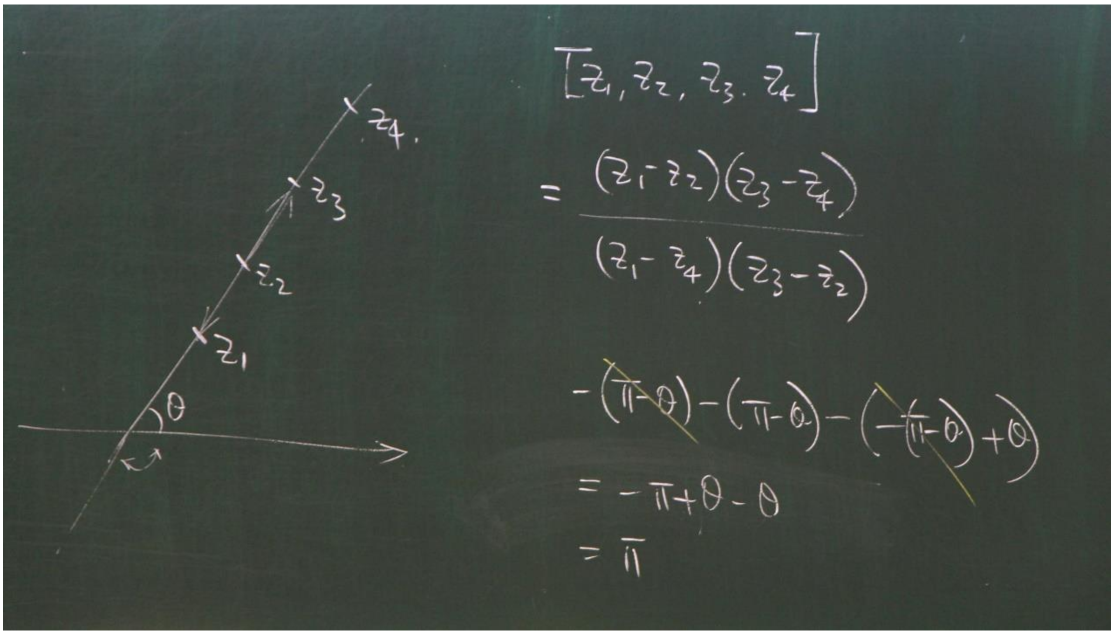
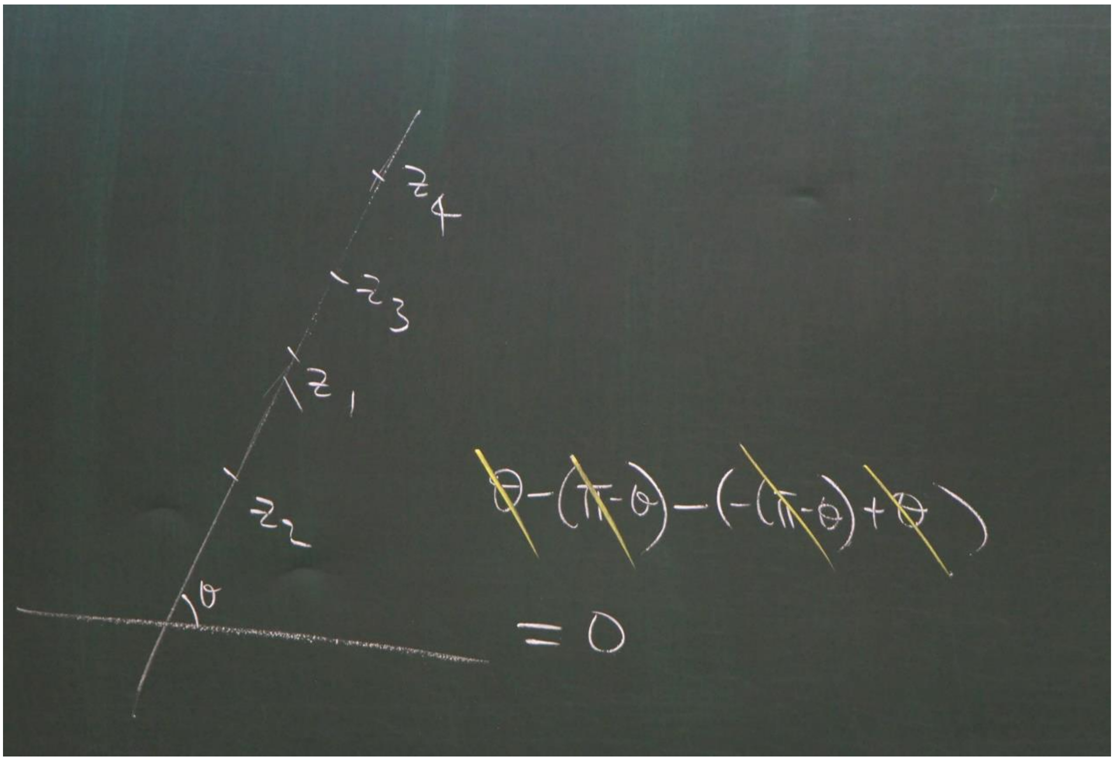
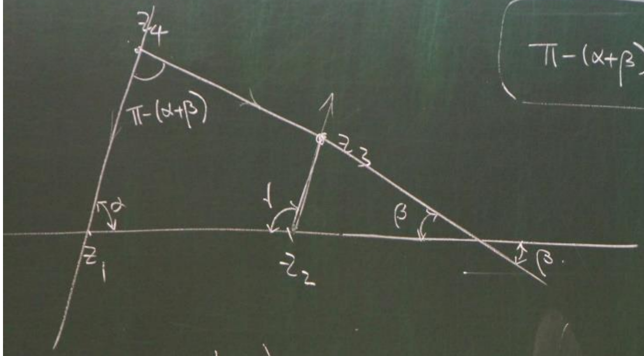
【10920 程守慶教授複變數函數論 / 第 26 堂版書】







$$\pi-(\alpha+\beta)+\gamma=\pi$$

$$\operatorname{Arg}(\text{cross ration}) = \pi-\beta-\big(-(\pi-\alpha)+\pi-\gamma\big) = \cancel{\pi}-\beta+\pi-\alpha-\cancel{\pi}+\gamma = \pi+\gamma-(\beta+\alpha)$$

Thm. Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$.
Then z_1, z_2, z_3, z_4 lie in the same circle if and only if

$$[z_1, z_2, z_3, z_4] \in \mathbb{R}.$$

proof
$\mathbb{R}_\infty = \mathbb{R}$
φ: linear f
(l.f.t.)
show $\varphi(\mathbb{R}$

proof

$\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$

φ: linear fractional transformation (l.f.t.)

show $\varphi(\mathbb{R}_\infty)$ is a generalized circle. (a finite circle or a line)

Circle C:

let z_2, z_3, z_4 distinct in $\mathbb{C}$

Consider

$$T(z) = [z, z_2, z_3, z_4]$$

$z_2 \mapsto 0$

$z_3 \mapsto 1$

$z_4 \mapsto \infty$

$$T(z_1) = [z_1, z_2, z_3, z_4]$$

$w \in \Sigma_1$. Consider $\varphi^{-1}(w)$

$$\varphi^{-1}(w) = \frac{aw+b}{cw+d}$$

Assume $\varphi^{-1}(w) \in \mathbb{R}$

$$\therefore \quad \frac{aw+b}{cw+d} = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}$$

$$T(z_1) = [z_1, z_2, z_3, z_4]$$

$a\bar{c}|w|^2 + b\bar{c}\bar{w} + \ldots$

$(a\bar{c}-\bar{a}c)|w|^2 + ($

① $a\bar{c} \in \mathbb{R}$.

$\therefore \operatorname{Im}(($

$$a\bar{c}|w|^2 + b\bar{c}\bar{w} + a\bar{d}w + b\bar{d} = \bar{a}c|w|^2 + \bar{a}d\bar{w} + \bar{b}cw + \bar{b}d.$$

$$(a\bar{c}-\bar{a}c)|w|^2 + (a\bar{d}-\bar{b}c)w - (\bar{a}d-b\bar{c})\bar{w} + b\bar{d}-\bar{b}d = 0$$

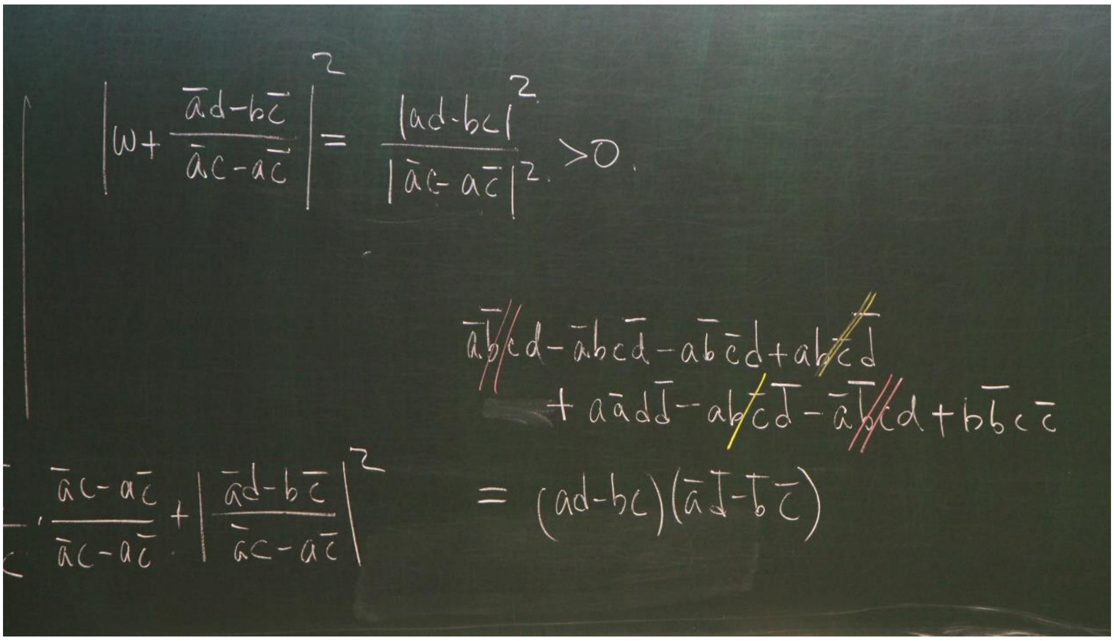
① $a\bar{c} \in \mathbb{R}$. $a\bar{c} = \bar{a}c$

$\therefore \operatorname{Im}\left((a\bar{d}-\bar{b}c)w + b\bar{d}\right) = 0$. equation of a line.

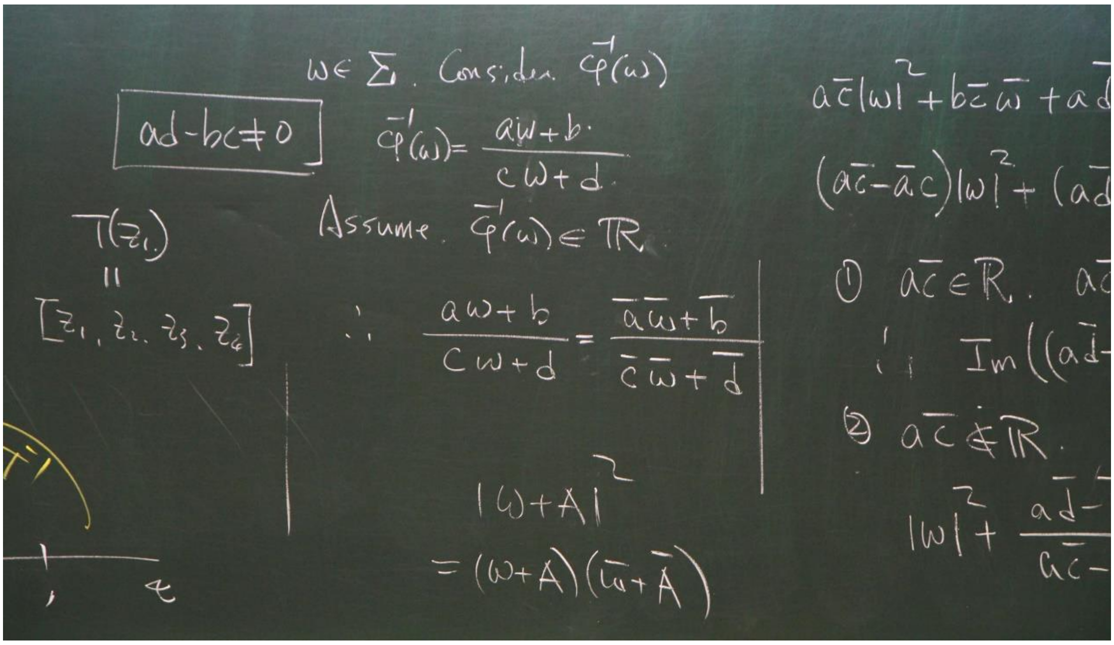
② $a\bar{c} \notin \mathbb{R}$.

$$|w|^2 + \frac{a\bar{d}-\bar{b}c}{a\bar{c}-\bar{a}c}w - \frac{\bar{a}d-b\bar{c}}{a\bar{c}-\bar{a}c}\bar{w} + \left|\frac{\bar{a}d-b\bar{c}}{\bar{a}c-a\bar{c}}\right|^2 = \frac{\bar{b}d-b\bar{d}}{a\bar{c}-\bar{a}c}\cdot\frac{\bar{a}c-a\bar{c}}{\bar{a}c-a\bar{c}} + \left|\frac{\bar{a}d-b\bar{c}}{\bar{a}c-a\bar{c}}\right|^2$$

$$\left|w + \frac{\bar{a}d-b\bar{c}}{\bar{a}c-a\bar{c}}\right|^2 = \frac{\quad}{|\bar{a}c-\ldots}$$



$\left|w + \frac{\bar a d - b\bar c}{\bar a c - a\bar c}\right|^2 = \frac{|ad-bc|^2}{|\bar a c - a\bar c|^2} > 0.$
$\cancel{\bar a b c d} - \bar a b c \bar d - a\bar b \bar c d + \cancel{a b \bar c d}$
$+ a\bar a d\bar d - a \cancel{b} c\bar d - \bar a \cancel{b} c d + b\bar b c\bar c$
$\frac{\bar a c - a\bar c}{\bar a c - a\bar c} + \left|\frac{\bar a d - b\bar c}{\bar a c - a\bar c}\right|^2$
$= (ad-bc)(\bar a\bar d - \bar b\bar c)$

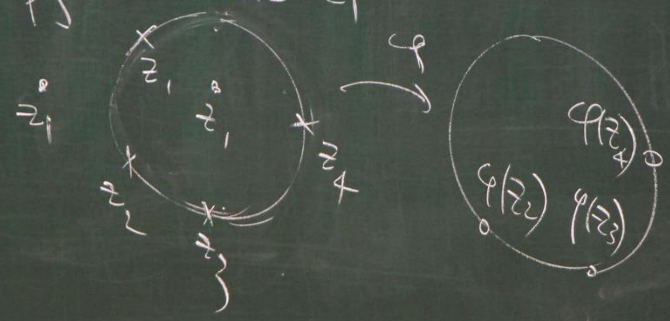


$w \in \Sigma$. Consider $\varphi^{-1}(w)$
$\boxed{ad-bc\neq 0}$
$\varphi^{-1}(w) = \frac{aw+b}{cw+d}$
Assume $\varphi^{-1}(w)\in\mathbb{R}$.
$\therefore \frac{aw+b}{cw+d} = \frac{\bar a\bar w+\bar b}{\bar c\bar w+\bar d}$
$|w+A|^2$
$= (w+A)(\bar w+\bar A)$
$T(z_1) = [z_1, z_2, z_3, z_4]$
$a\bar c|w|^2 + b\bar c\bar w + a\bar d$
$(a\bar c - \bar a c)|w|^2 + (a\bar d$
① $a\bar c\in\mathbb{R}$. $a\bar c$
$\therefore$ $\mathrm{Im}((a\bar d$
② $a\bar c\notin\mathbb{R}$.
$|w|^2 + \frac{a\bar d -}{a\bar c -}$

Thm. φ: l.f.t.

Then φ maps circles to circles

pf. circle C_1

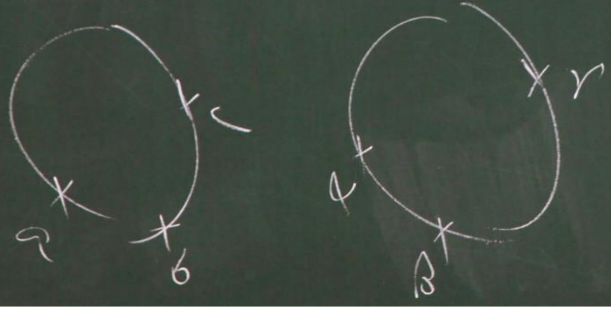


$[z_1, z_2, z_3, z$

$\therefore$

$$[z_1, z_2, z_3, z_4] = [\varphi(z_1), \varphi(z_2), \varphi(z_3), \varphi(z_4)]$$

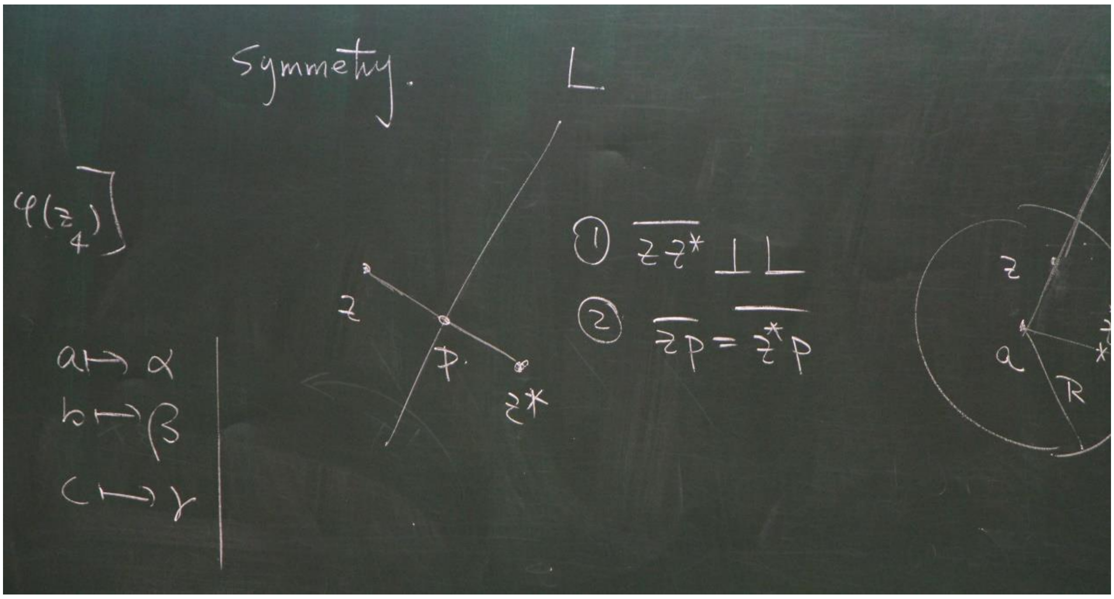
C_1 C_2

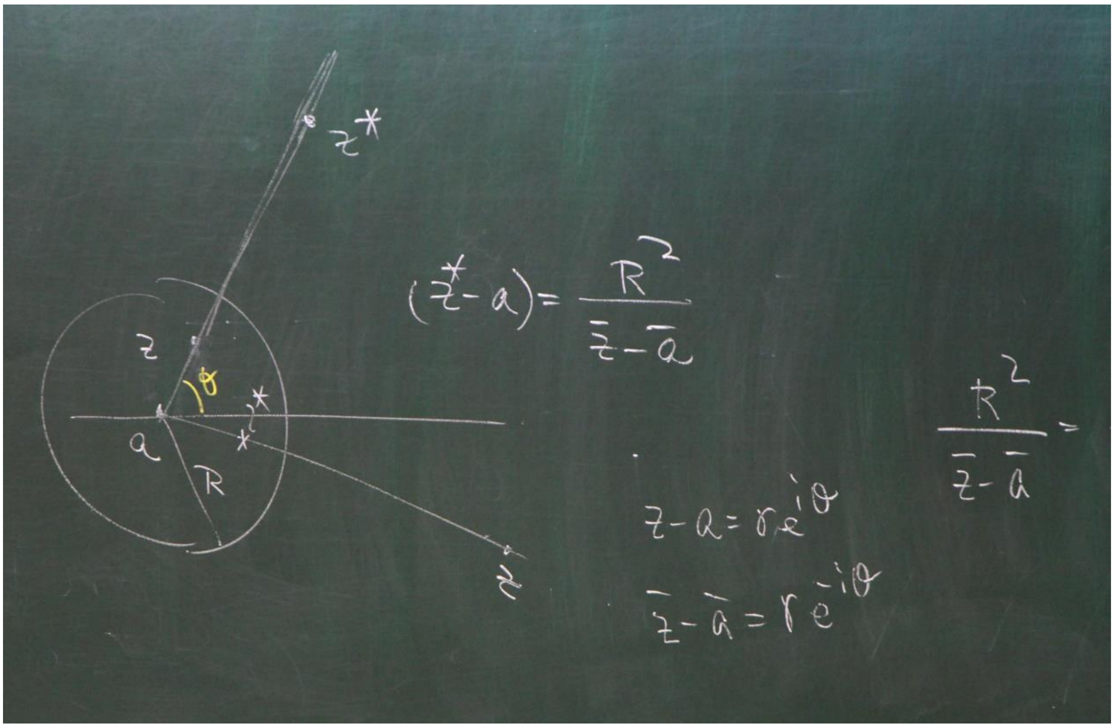


$a \mapsto \alpha$

$b \mapsto \beta$

$c \mapsto \gamma$

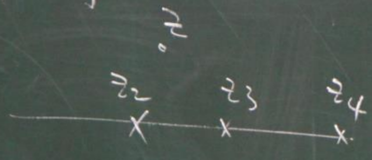

Symmetry.
L
$\varphi(z_4)$
① $\overline{zz^*} \perp L$
② $\overline{zp} = \overline{z^*p}$
z
p
z^*
$a \mapsto \alpha$
$b \mapsto \beta$
$c \mapsto \gamma$
z
a
R


z^*
$(z^* - a) = \frac{R^2}{\bar{z} - \bar{a}}$
z
θ
z^*
a
R
z
$z - a = re^{i\theta}$
$\bar{z} - \bar{a} = re^{-i\theta}$
$\frac{R^2}{\bar{z} - \bar{a}} =$

Thm. Let C be a circle or a line. $z_2, z_3, z_4 \in C$ distinct.
Let z, z^* be two distinct points,
which are symmetric w.r.t. C
if and only if

$$[z^*, z_2, z_3, z_4] = \overline{[z, z_2, z_3, z_4]}$$

proof. (i) C: x-axis.

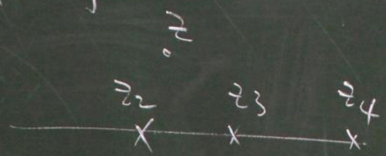


$\therefore z^* = \bar{z}$

$$\therefore [z^*, z_2, z_3, z_4] = [\bar{z}, z_2, z_3, z_4] = \overline{[z, z_2, z_3, z_4]}$$

distinct

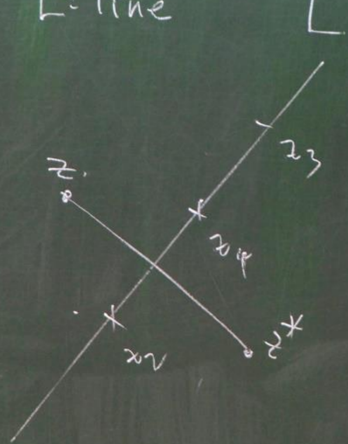
proof. (i) C: x-axis.



$z^* = \bar{z}$

$$\therefore [z^*, z_2, z_3, z_4] = [\bar{z}, z_2, z_3, z_4] = \overline{[z, z_2, z_3, z_4]}$$

(ii) L: line



By a rigid ...

$[z^*, z_2, z_3, z_4] = [$

$=$

$=$

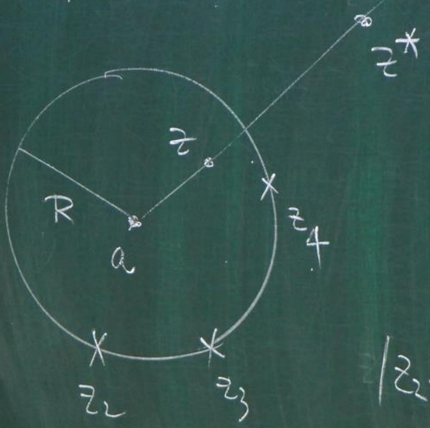
By a rigid motion φ (l.f.t)

$$[z^*, z_2, z_3, z_4] = [\varphi(z^*), \varphi(z_2), \varphi(z_3), \varphi(z_4)] = [\overline{\varphi(z)}, \varphi(z_2), \varphi(z_3), \varphi(z_4)]$$

$$= \overline{[\varphi(z), \varphi(z_2), \varphi(z_3), \varphi(z_4)]}$$

$$= \overline{[z, z_2, z_3, z_4]}$$

(iii) C: finite circle



$$[z^*, z_2, z_3, z_4] = [z^* - a,$$

$$= \Big[\frac{R^2}{\bar z - \bar a}$$

$$= [\bar z - \bar a$$

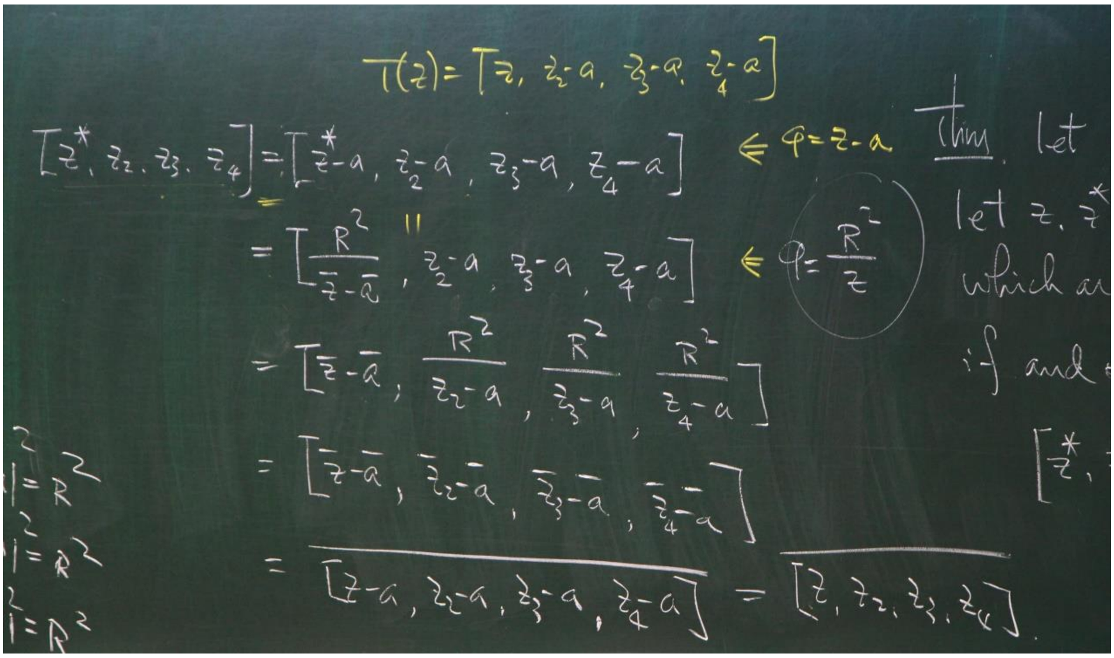
$$= [\bar z - \bar a$$

$$= \overline{[z - a}$$

$|z_2 - a|^2 = R^2$

$|z_3 - a|^2 = R^2$

$|z_4 - a|^2 = R^2$


$T(z) = [z, z_2 - a, z_3 - a, z_4 - a]$
$[z^*, z_2, z_3, z_4] = [z^* - a, z_2 - a, z_3 - a, z_4 - a]$ $\Leftarrow \varphi = z - a$
$= \left[\frac{R^2}{\bar{z} - \bar{a}}, z_2 - a, z_3 - a, z_4 - a\right]$ $\Leftarrow \varphi = \frac{R^2}{z}$
$= \left[\bar{z} - \bar{a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a}, \frac{R^2}{z_4 - a}\right]$
$= [\bar{z} - \bar{a}, \bar{z}_2 - \bar{a}, \bar{z}_3 - \bar{a}, \bar{z}_4 - \bar{a}]$
$= \overline{[z - a, z_2 - a, z_3 - a, z_4 - a]} = \overline{[z, z_2, z_3, z_4]}.$
thm. let
let z,
if and