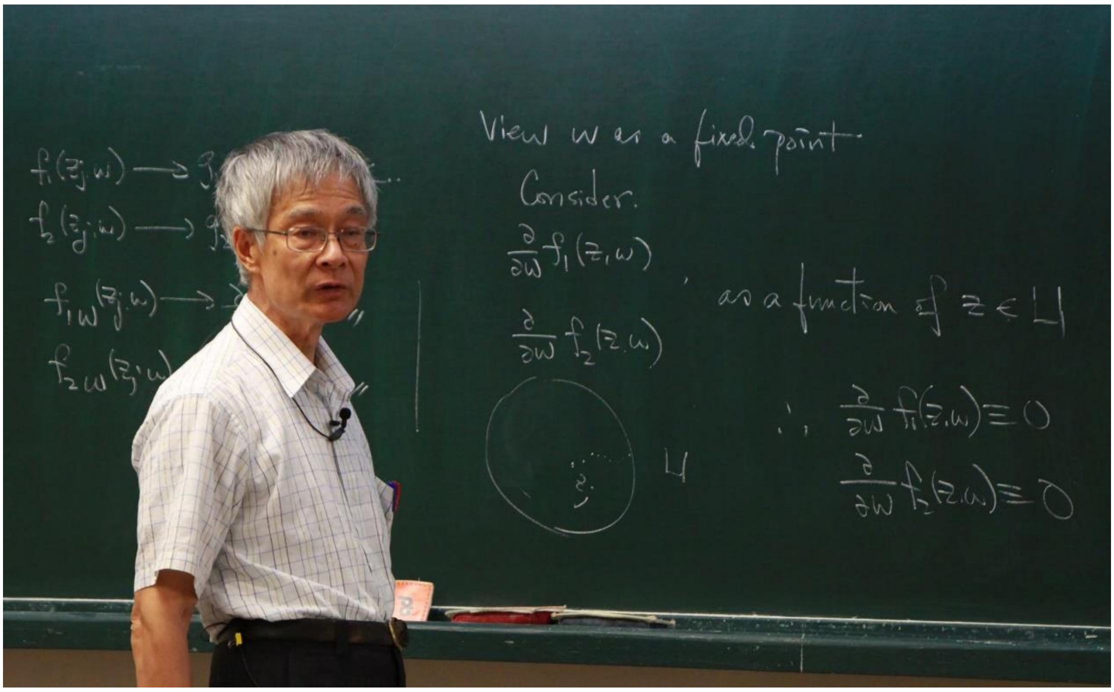
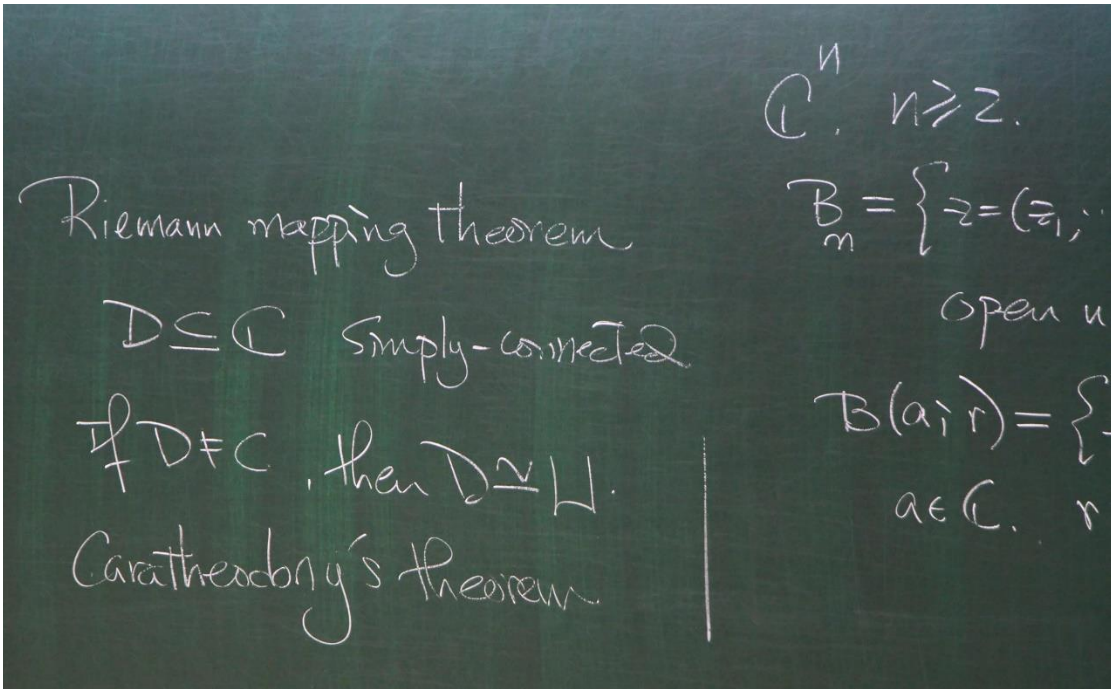
【10920 程守慶教授複變數函數論 / 第 24 堂版書】





$\mathbb{C}^n$. $n \geq 2$.

$$B_n = \left\{ z=(z_1,\cdots,z_n) \,\middle|\, |z_1|^2+\cdots+|z_n|^2<1 \right\}.$$

open unit ball

$$B(a;r) = \left\{ z\in\mathbb{C} \,\middle|\, |z-a|<r \right\}$$

$a\in\mathbb{C}$. $r>0$ disc. centered at a with radius r

Polydisc in $\mathbb{C}^n$.

多圓盤.

$$P(a;r) = \prod_{j=1}^{n} B(a_j;r_j)$$

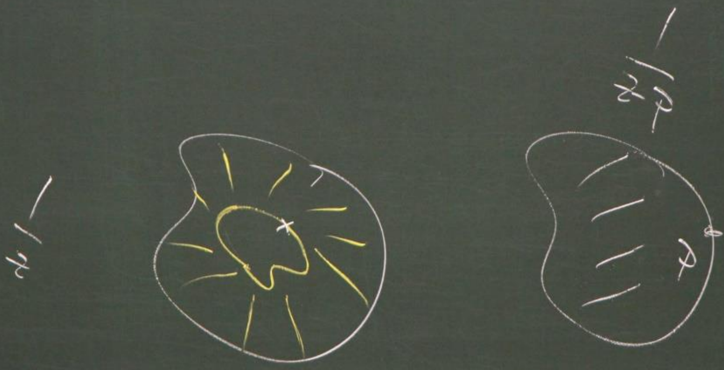
$a=(a_1,a_2,\cdots,a_n)\in\mathbb{C}^n$

$r=(r_1,r_2,\cdots,r_n)$.

$r_j>0$

Thm (Poincaré, 1907)

There is $\boxed{\text{no}}$ biholomorphism between B_n and $P(a;r)$.



$$\varphi(z) = e^{i\theta}\cdot\frac{a-z}{1-\bar{a}z} = -e^{i\theta}z = e^{i(\pi+\theta)}\cdot z$$

$$\varphi(0)=0=a\cdot e^{i\theta}$$

$$a=0$$

Thm (Poincaré, 1907)

There is $\boxed{\text{no}}$ biholomorphism between B_n and $P(a;r)$.

$n=2$,

$B_2 \not\cong U^2 = U \times U$ | R. Remmert
Karl Stein

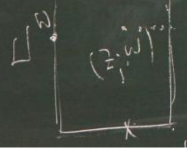
$U^2 \ni (z,w)$

Proof. If not.

i.e., $\exists\ f=(f_1,f_2): U^2 \longrightarrow B_2$
biholomorphism.

emmert | let $\{z_j\}$ be a seq. s.t. $|z_j| \to 1$

Stein | $\therefore\ f(z_j,w) = (f_1(z_j,w), f_2(z_j,w)) : U \to B_2$.



$(z_j,w) \to \partial(U^2)$

View as a f

$|f_1(z_j,w)| \le 1$

$|f_2(z_j,w)| \le 1$

By Montel's

$\mathbb{U}^2 \ni (z, w)$

. Remmert

arl Stein

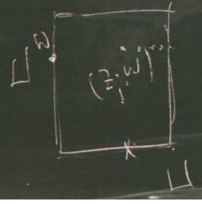
Proof. If not.

i.e., $\exists\ f = (f_1, f_2) : \mathbb{U}^2 \longrightarrow B_2$

biholomorphism.

let $\{z_j\}$ be a seq. s.t. $|z_j| \to 1$

$\therefore\ f(z_j, w) = \big(f_1(z_j, w), f_2(z_j, w)\big) : \mathbb{U} \to B_2.$



$(z_j, w) \to \partial(\mathbb{U}^2)$

View as a

$|f_1(z_j, w)| \le$

$|f_2(z_j, w)| \le$

By Monte

View as a fct. of w.

$|f_1(z_j, w)| \le 1$

$|f_2(z_j, w)| \le 1$

By Montel's thm., use the same notation,

$f_1(z_j, w) \longrightarrow g_1(w)$

$f_2(z_j, w) \longrightarrow g_2(w)$ u.c.c.

$\therefore\ |g_1(w)|^2 + |g_2(w)|^2 = 1$

use the same notation,

(w)

u.c.c.

$_2(w)$

$(w)|^2 \equiv 1$

$$\frac{\partial^2}{\partial w \partial \overline{w}}\left(|g_1(w)|^2 + |g_2(w)|^2\right) = 0$$

$$|g_1'(w)|^2 + |g_2'(w)|^2 \equiv 0$$

$$\Rightarrow \quad g_1'(w) \equiv 0 \qquad g_2'(w) \equiv 0$$

$$\underset{\displaystyle \frac{\partial g_1}{\partial w}}{\overset{\displaystyle g_1'(w)}{\|}}$$

$f_1(z_j, w) \longrightarrow g_1(w)$ u.c.c.

$f_2(z_j, w) \longrightarrow g_2(w)$ "

$f_{1w}(z_j, w) \longrightarrow \frac{\partial g_1}{\partial w} = 0$ "

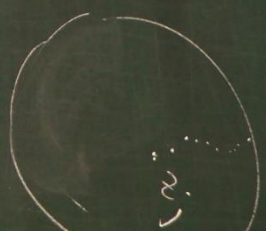
$f_{2w}(z_j, w) \longrightarrow \frac{\partial g_2}{\partial w} = 0$ "

View w as a

Consider.

$$\frac{\partial}{\partial w} f_1(z, w)$$

$$\frac{\partial}{\partial w} f_2(z, w)$$



View w as a fixed point

Consider.

$$\frac{\partial}{\partial w} f_1(z,w), \quad \frac{\partial}{\partial w} f_2(z,w) \quad \text{as a function of } z \in U$$

$$\therefore \quad \frac{\partial}{\partial w} f_1(z,w) \equiv 0$$

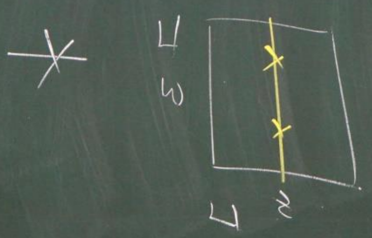
$$\frac{\partial}{\partial w} f_2(z,w) \equiv 0$$

$z \in U$

$w) \equiv 0$

$w) \equiv 0$

$$\frac{\partial f}{\partial w} = \left(\frac{\partial f_1}{\partial w}, \frac{\partial f_2}{\partial w}\right) \equiv 0.$$



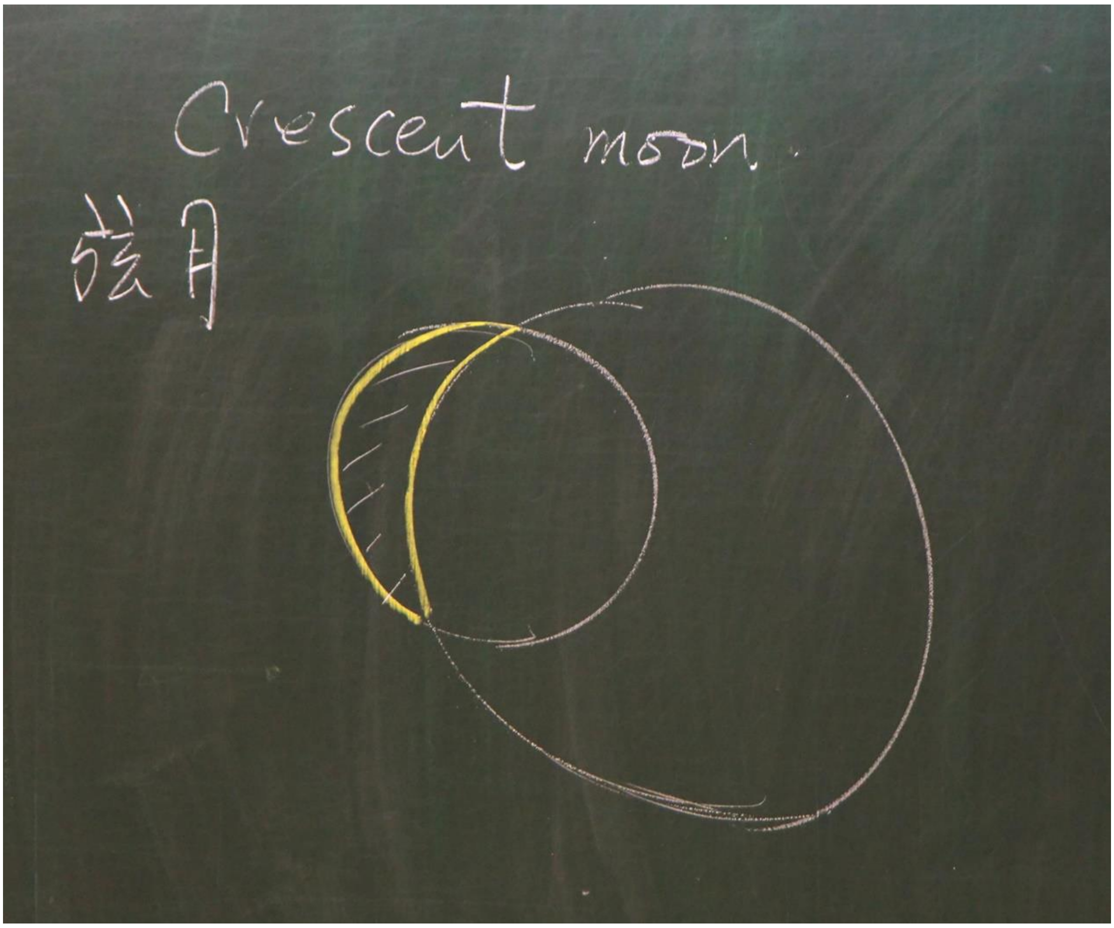
Thm (Poi

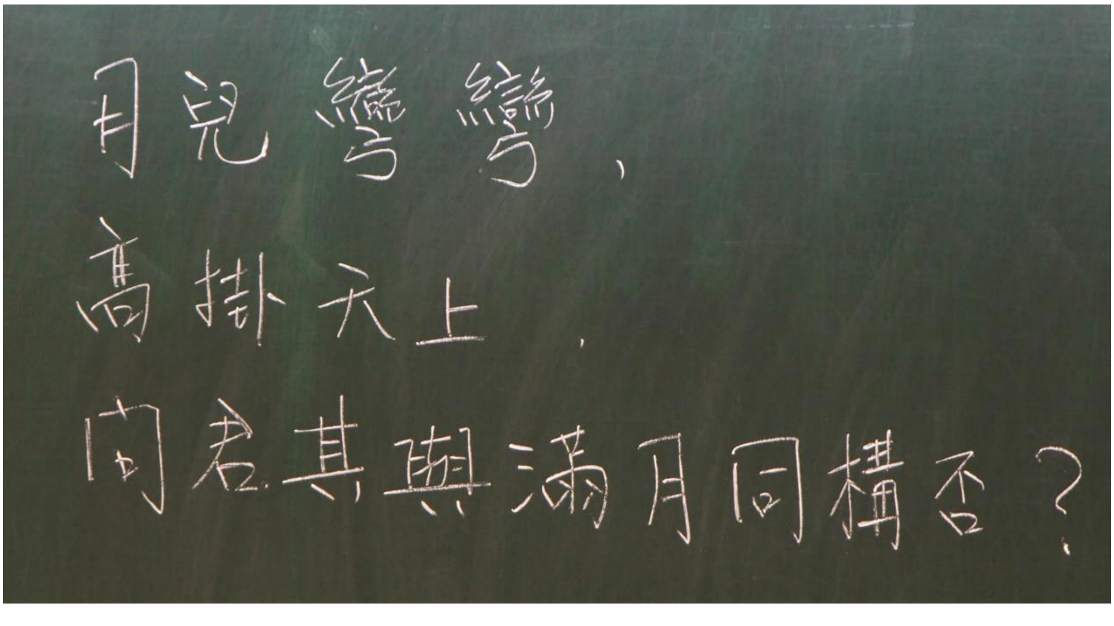
There is

between

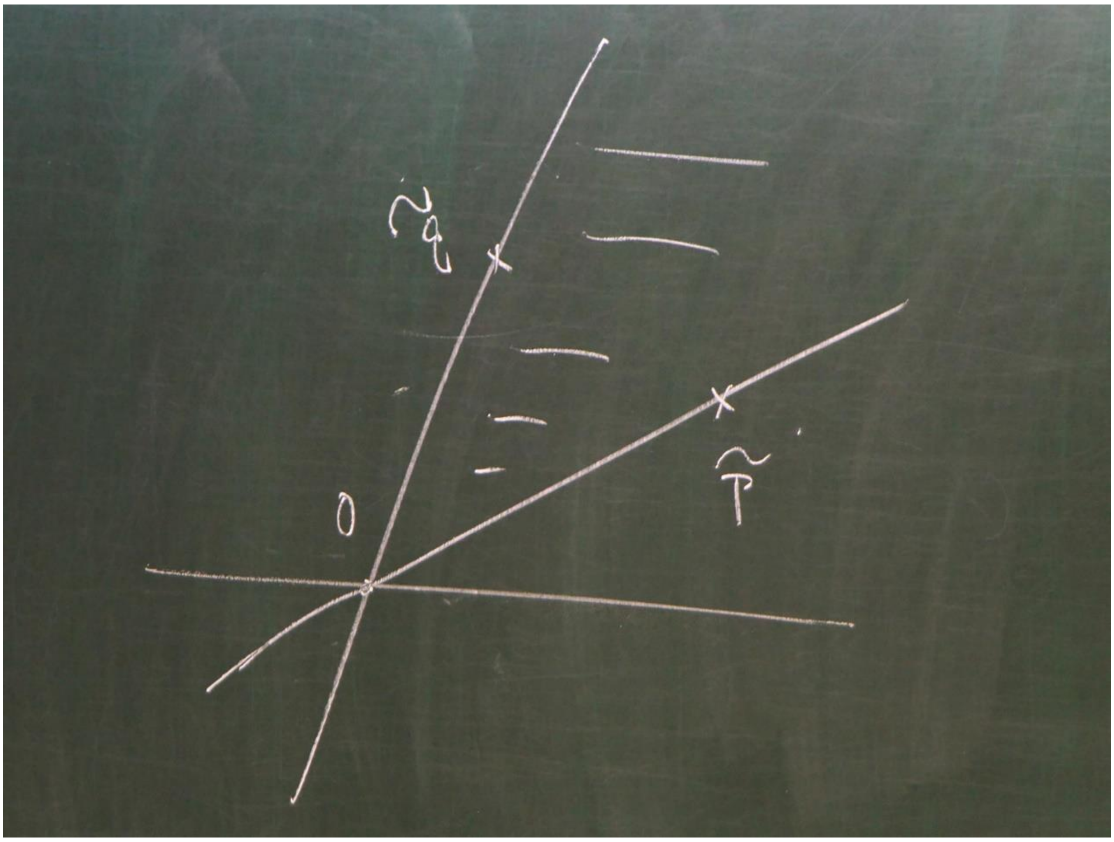
$n=2$,

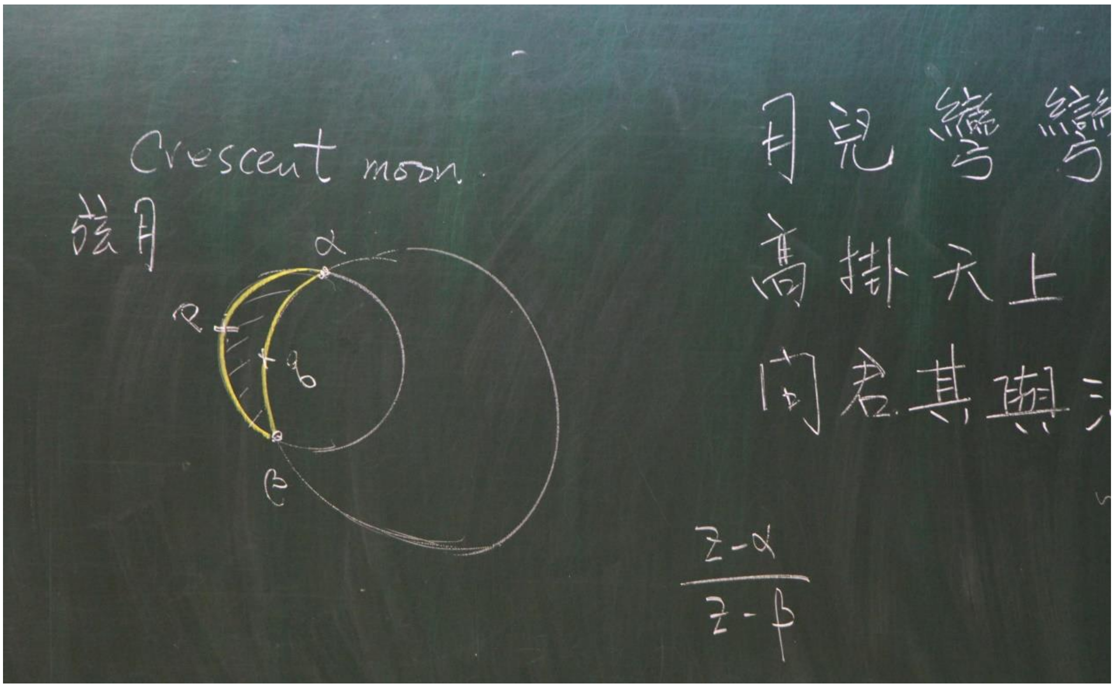
$B_2 \not\cong$

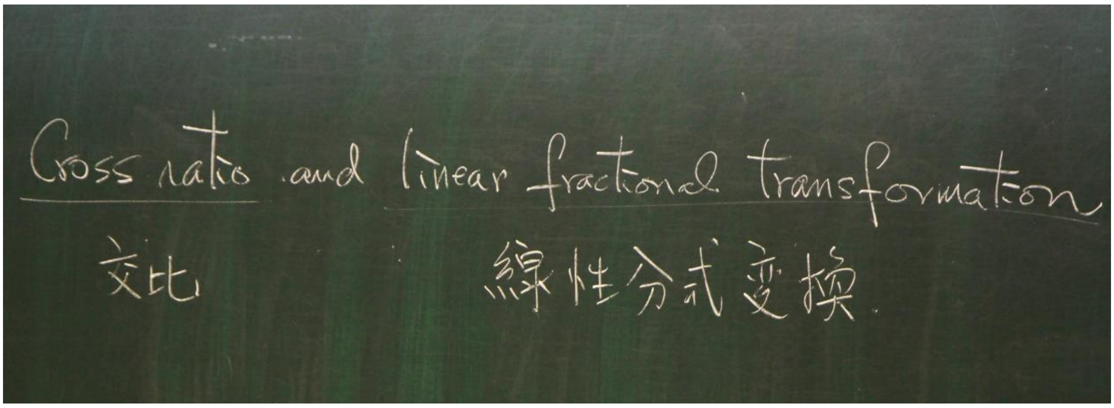

Crescent moon
弦月


月兒彎彎，
高掛天上，
問君其與滿月同構否？


Schwarz-Christoffel.


$\tilde{q}$
$\tilde{p}$
0


Crescent moon
弦月
α
β
月兒彎彎
高掛天上
問君其與
$\frac{z-\alpha}{z-\beta}$


Cross ratio and linear fractional transformation
交比
線性分式變換

linear fractional transformation.

By definition

$$\varphi(z) = \frac{az+b}{cz+d}.$$

$a, b, c, d \in \mathbb{C}$. $\quad ad - bc \neq 0$

$\varphi : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$.

1-1, onto.

$$(ad-bc)z_1 = (ad-bc)z_2$$

$$\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d} \qquad z_1 = z_2$$

$$\cancel{acz_1z_2} + bcz_2 + adz_1 + \cancel{bd} = \cancel{acz_1z_2} + bcz_1 + adz_2 + \cancel{bd}$$

$$\frac{az+b}{cz+d} = w$$

$$az+b = wcz+dw$$

$$(a-wc)z = dw-b$$

$$z = \frac{dw-b}{-cw+a}$$

$d-\cancel{bc})z_2$

$z_1 + ad\,z_2 + \cancel{bd}$

If $ad-bc=1$

$$g(z) = \frac{az+b}{cz+d} \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ ad-bc=1 \right\}$$

$$g^{-1}(w) = \frac{dw-b}{-cw+a} \longleftrightarrow \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}$$

$ad-bc=1$

$$\frac{A\left(\frac{az+b}{cz+d}\right)+B}{C\left(\frac{az+b}{cz+d}\right)+D} = \frac{Aaz+Ab+Bcz+Bd}{Caz+Cb+Dcz+dD} = \frac{(Aa+Bc)z+(Ab+Bd)}{(Ca+Dc)z+\boxed{Cb+dD}}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} Aa+Bc, & Ab+Bd \\ Ca+Dc & Cb+dD \end{pmatrix}$$

Special cases.

(i) $\varphi(z) = z+b$. (Translation).

(ii) $\varphi(z) = az$, $a \neq 0$. (dilation)

(iii) $\varphi(z) = e^{i\theta} z$. (rotation)

(iv) $\varphi(z) = \frac{1}{z}$ (inversion)

$$\varphi(z) = \frac{az+b}{cz+d}$$

(I) $c=0$.

(II) $c \neq 0$.

$$\varphi(z) = \frac{az+b}{cz+d}.$$

I) $c=0$. $\quad \varphi(z) = \frac{a}{d}z + \frac{b}{d}$ $\qquad \varphi_1(z) = \frac{a}{d}z \qquad \varphi_2(z) = z + \frac{b}{d}$.

$$= \varphi_2 \circ \varphi_1(z)$$

II) $c \neq 0$. $\quad \varphi(z) = \frac{az+b}{cz+d} = \frac{az+b}{c\left(z+\frac{d}{c}\right)} = \frac{\frac{bc-da}{c^2}}{z+\frac{d}{c}} + \frac{a}{c}$

$$= \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1(z)$$

$$az+b = Ac + Bz + \frac{d}{c}B$$

$B = a \qquad A = \frac{bc-da}{c^2}.$

$$Ac + \frac{da}{c} = b$$

$$Ac^2 = bc - da$$

$\varphi_1(z) = z + \frac{d}{c}.$

$\varphi_2(z) = \frac{1}{z}$

$\varphi_3(z) = \frac{bc-da}{c^2}z$

$\varphi_4(z) = z + \frac{a}{c}.$