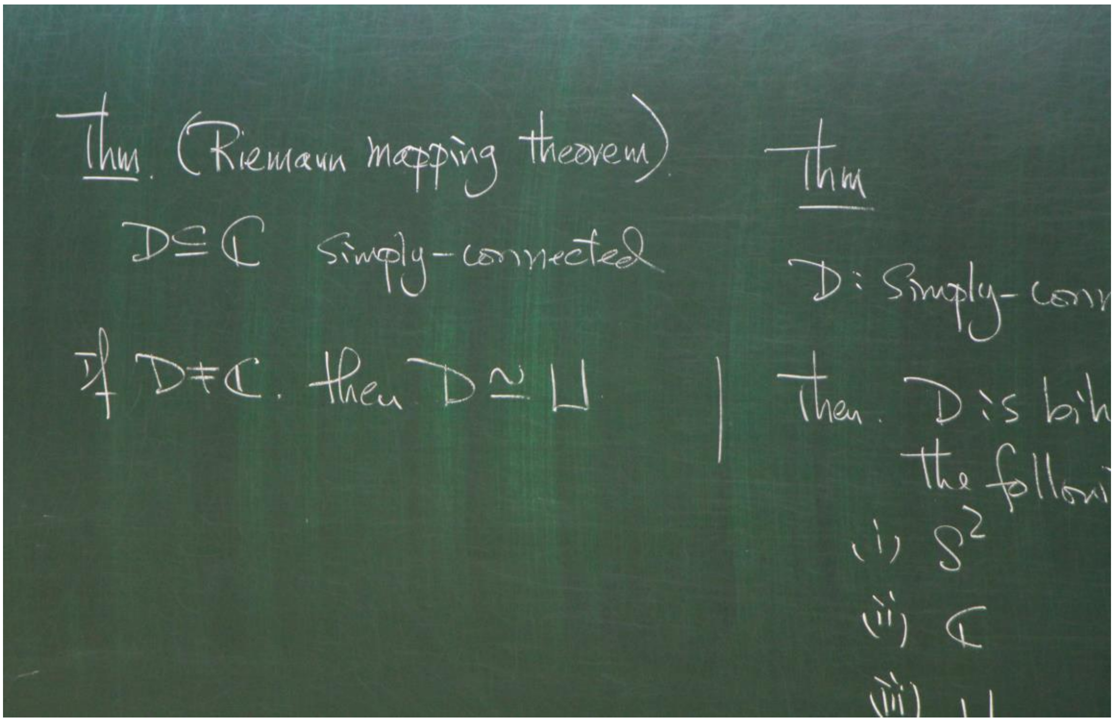
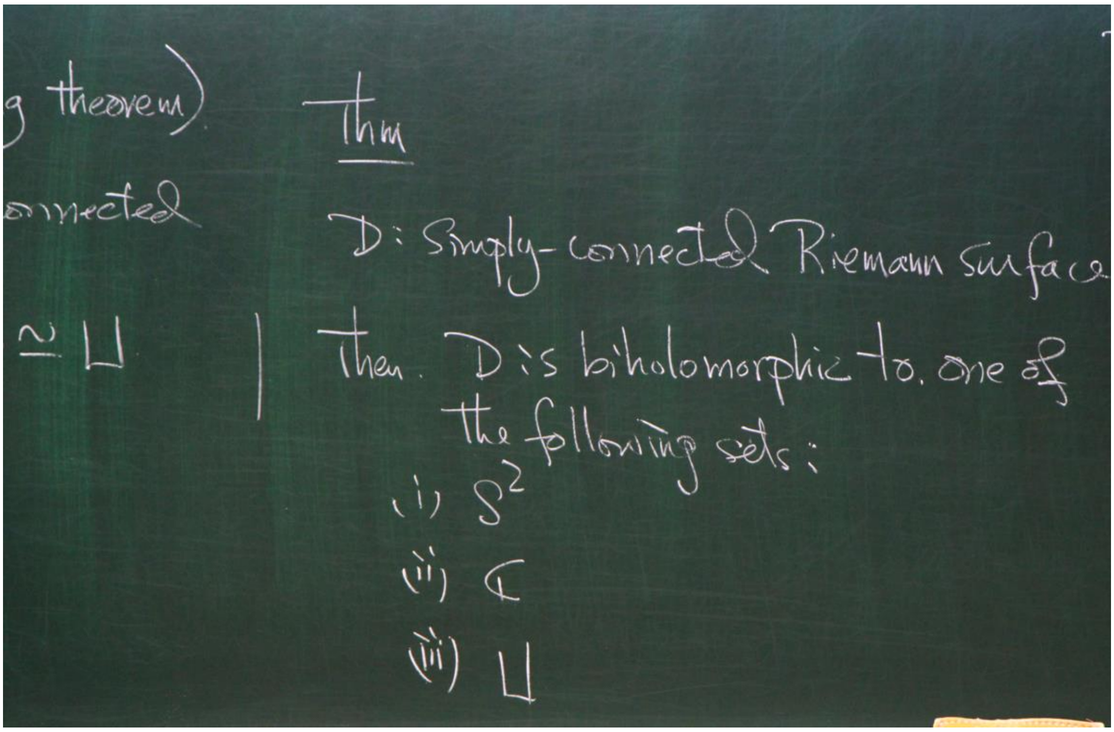
【10920 程守慶教授複變數函數論 / 第 23 堂版書】





Pf: Consider

$$\mathcal{G} = \{ f \in \mathcal{O}(D) \mid f \text{ is bounded and } 1\text{-}1 \}.$$

surface claim: $\mathcal{G} \neq \emptyset$.

one of

(i) If $\exists\, B(a;r) \subseteq \mathbb{C} \setminus D$.

Set $f(z) = \dfrac{1}{z-a} \in \mathcal{G}$.

(ii) $\nexists\, B(a;r) \subseteq \mathbb{C} \setminus D$.

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Consider the boundary points of D

i) $D \neq \mathbb{C}$.

ii) $\partial D = \{b\}$. $\times$

may assume. $\exists\ p \neq q$.

$p, q \in \partial D$

Pf. Consider

$$\mathcal{G} = \{ f \in \mathcal{O}(D) \mid f \text{ is bounded and } 1-1 \}.$$

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$\because D \neq \mathbb{C}$.

$\because \partial D = \{b\}$. $\divideontimes$

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$p, q \in \partial D$

$z - p \neq 0$ on D.

$z - q \neq 0$ 〃

Choose branches and define

$\log(z-p)$ and $\log(z-q)$

ry points of D

$\therefore$ We can define.

$$h(z) = \sqrt{\frac{z-p}{z-q}} := \exp\left(\frac{1}{2}\left(\log(z-p) - \log(z-q)\right)\right)$$

q.

$\therefore h \in \mathcal{O}(D)$

$\because p \neq q$

h is 1-1

$\because \sqrt{\dfrac{z_1 -}{z_1 -}}$

$\dfrac{z_1 - p}{z_1 - q}$

$\cancel{z_1 z_2} - p z$

$= \cancel{z_1 z_2} - q$

$(p - q) z_1$

l define

$\log(z-q)$

$\exp\left(\frac{1}{2}\left(\log(z-p)-\log(z-q)\right)\right)$

h is 1-1

$\because \sqrt{\dfrac{z_1-p}{z_1-q}} = \sqrt{\dfrac{z_2-p}{z_2-q}}$

$\dfrac{z_1-p}{z_1-q} = \dfrac{z_2-p}{z_2-q}$

$\cancel{z_1 z_2} - pz_2 - qz_1 + \cancel{pq}$

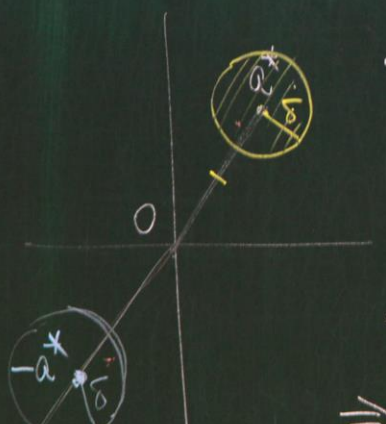
$= \cancel{z_1 z_2} - qz_2 - pz_1 + \cancel{pq}$

$(p-q)z_1 = (p-q)z_2$

$\because p \neq q$

$\Rightarrow z_1 = z_2$

$0, \infty \notin Im(h)$.



Choose a point $a^* \in Im(h)$.

$a^* \neq 0$

$\because h$: open mapping

$\Rightarrow Im(h)$: open

$\therefore \exists\, \delta < \frac{1}{2}|a^*|$

s.t. $B(a^*, \delta) \subseteq Im(h)$

$\Rightarrow B(-a^*; \delta) \subseteq \mathbb{C} \setminus Im(h)$

Pf: Consider

(I) $\mathcal{G} = \{ f \in \mathcal{O}(D) \mid f$ is bounded and 1–1 $\}$.

<u>claim</u>: $\mathcal{G} \neq \emptyset$.

(II) Pick $c \in D$. Consider

$\mathcal{M} = \{ f \in \mathcal{O}(D) \mid f$ is bounded and 1–1 and $f(c)=0$, $f'(c)=1 \}$

<u>claim</u>: $\mathcal{M} \neq \emptyset$.

theorem)

nected

$\sqcup$.

Choose $g \in \mathcal{G}$.

Set:

$$f(z) = \frac{g(z) - g(c)}{g'(c)}$$

l and 1-1 }

and 1-1 }
$f'(c)=1$ }

For each $f \in \mathcal{M}$

$$f : D \to D_f$$

set

$$m_f = \sup_{z \in D} |f| < \infty .$$

and.

$$s = \inf_{f \in \mathcal{M}} m_f . \quad 0 \le s < \infty$$

$|f| < \infty .$

$m_f . \quad 0 \le s < \infty$

$n \in \mathbb{N}$.

$\therefore \exists f_n \in \mathcal{M}$ s.t.

$$m_{f_n} \le s + \frac{1}{n} \le s + 1 .$$

Get $\{f_n\} \subseteq \mathcal{M}$.

$\therefore \{f_n\}$: uniformly bounded.

By Montel's thm: $\exists$ a subseq.

$\{f_{n_j}\}$ s.t. $f_{n_j} \xrightarrow{u.c.c.} f$ in D.

① $f(c)=0$

② f is not

and H_f

as $n_j \to$

$\Rightarrow$

s.t.

$\leq s+1$.

$\mathcal{M}$.

...rmly bounded.

... : $\exists$ a subseq.

$\xrightarrow{u.c.c.} f$ on D.

① $f(c)=0$. $f'(c)=1$.

② f is not a constant function

and $|f_{n_j}| \leq s + \frac{1}{n_j}$

as $n_j \to \infty$

$\Rightarrow |f| \leq s$.

$\because s>0$.

③ f is 1-1. By Hurwitz

$\therefore f \in \mathcal{M}$.

$f: D \to U(s) = \{ z \mid |z|<s \}$

$c \mapsto f(c)=0$

<u>Claim</u>: f is onto $U(s)$.

$\boxed{f(z)/s \text{ onto } U}$

If not. $\exists\ \beta \in U(s)$ and $\beta \notin \mathrm{Im}(f)$.

Set $w=f(z)$

$$w(c) = f(c) = 0$$

Set

$$w_1 = s\sqrt{\frac{\frac{w}{s}-\frac{\beta}{s}}{1-\frac{\bar{\beta}}{s}\frac{w}{s}}} = s\sqrt{\frac{s(w-\beta)}{s^2-\bar{\beta}w}} : \mathbb{U}(s) \to \mathbb{U}(s).$$

$$w_1(c) = (-s\beta)^{1/2} \neq 0.$$

$$w_2 = s\cdot\frac{\frac{w_1}{s}-\frac{w_1(c)}{s}}{1-\frac{\overline{w_1(c)}}{s}\frac{w_1}{s}} = \frac{s^2(w_1-w_1(c))}{s^2-\overline{w_1(c)}\,w_1}$$

Compute.

$$\frac{dw_2}{dz}(c) = \frac{dw_2}{dw_1}\bigg|_{w_1=w_1(c)} \cdot \frac{dw_1}{dw}\bigg|_{w=w(c)} \cdot \frac{dw}{dz}\bigg|_{z=c}.$$

Compute.

$$\frac{dw_2}{dz}(c) = \frac{dw_2}{dw_1}\Big|_{w_1=w_1(c)} \cdot \frac{dw_1}{dw}\Big|_{w=w(c)} \cdot \frac{dw}{dz}\Big|_{z=c}.$$

$$= \frac{s^2}{s^2-|w_1(c)|^2}\cdot \frac{1}{s}(-s\beta)^{-\frac{1}{2}}(s^2-|\beta|^2)\cdot 1$$

$\leq s < \infty$

$$= \frac{s^2}{s^2-|s\beta|}\cdot\frac{1}{s}(-s\beta)^{-\frac{1}{2}}\cdot(s^2-|\beta|^2)$$

$$= (-s\beta)^{-\frac{1}{2}}\cdot(s+|\beta|)$$

$$w_2 = \frac{s^2(w_1 - w_1(c))}{s^2-\overline{w_1(c)}\,w_1}$$

$$\frac{dw_2}{dw_1} = \frac{s^2(s^2-\overline{w_1(c)}\,w_1) + \overline{w_1(c)}\cdot s^2\cancelto{0}{(w_1 - w_1(c))}}{(s^2-\overline{w_1(c)}\,w_1)^2}\Bigg|_{w_1=w_1(c)}$$

$$= \frac{s^2(s^2-|w_1(c)|^2)}{(s^2-|w_1(c)|^2)^2} = \frac{s^2}{s^2-|w_1(c)|^2}$$

$$w_1 = s\left(\frac{s(w-\beta)}{s^2-\bar{\beta}w}\right)^{1/2}$$

$$\frac{dw_1}{dw} = s\left(\frac{s(w-\beta)}{s^2-\bar{\beta}w}\right)^{-1/2}\cdot\frac{s(s^2-\bar{\beta}w)+\bar{\beta}\cdot s(w-\beta)}{(s^2-\bar{\beta}w)^2}\Bigg|_{w=w(c)=0}$$

$$= s^2(-s\beta)^{-\frac{1}{2}}\cdot\frac{s^3-s|\beta|^2}{s^4} = \frac{1}{s}(-s\beta)^{-\frac{1}{2}}\cdot(s^2-|\beta|^2)$$

Set

$$w_3 = \frac{w_2}{\frac{dw_2}{dz}(c)}$$

$\therefore\ w_3 \in \mathcal{H}$

$$\sup_{z\in D}|w_3| = \frac{\sup_{z\in D}|w_2|}{\left|\frac{dw_2}{dz}(c)\right|} < \sup_{z\in D}|w_2| \leq s.$$

$$\left|\frac{dw_2}{dz}(c)\right| - 1 = \frac{s+|\beta|}{\sqrt{s|\beta|}} - 1 = \frac{s+|\beta| - \sqrt{s|\beta|}}{\sqrt{s|\beta|}} = \frac{\left(\sqrt{s}-\sqrt{|\beta|}\right)^2}{\sqrt{s|\beta|}} > 0.$$

$$\therefore \left|\frac{dw_2}{dz}(c)\right| > 1.$$

$\times$

Set $m_f = \sup_{z\in D}|f| < \infty$.

and $s = \inf_{f\in\mathcal{H}} m_f$. $\quad 0 \leq s < \infty$

Compute.

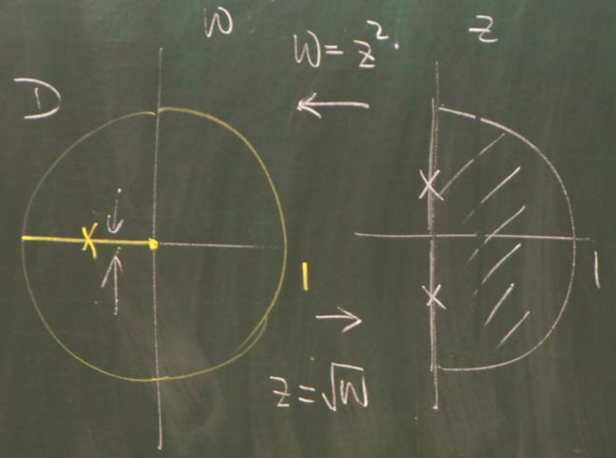
$$\frac{dw_2}{dz}(c) = \frac{dw\ldots}{dw\ldots}$$

$$= \frac{s^2\ldots}{s^2 - |w_1\ldots}$$

$$= \frac{s\ldots}{s^2 - |s\ldots}$$

$$= (-s\beta)\ldots$$

$$W_2 = \frac{s^2(w_1 - w_1(c))}{s^2 - \overline{w_1(c)}\, w_1} \qquad \frac{dw}{du}$$



$$w_1 = s\left(\frac{s(w-\beta)}{s^2 - \bar{\beta} w}\right)^{1/2}$$

Thm (Caratheodory)

$D \subsetneq \mathbb{C}$ simply connected $\Rightarrow$ $D \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} \mathbb{U}$. [Riemann mapping theorem]

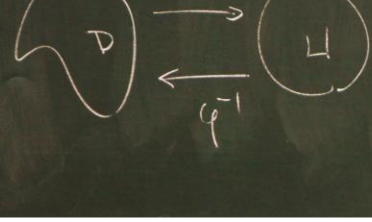
If ∂D is a Jordan curve

then φ can be extended continuously up to the boundary.

s.t. $\varphi : \overline{D} \longrightarrow \overline{\mathbb{U}}$ is a homeomorphism.

Solve Dirichlet problem on such D (∂D : Jordan)

Assume g is real defined on ∂D.
continuous



$\therefore$ $g \circ \varphi^{-1}$ defined on $\partial\mathbb{U}$.

By $P[g\circ\varphi^{-1}]\circ\varphi$.

Boundary value $(g\circ\varphi^{-1})\circ\varphi = g$.