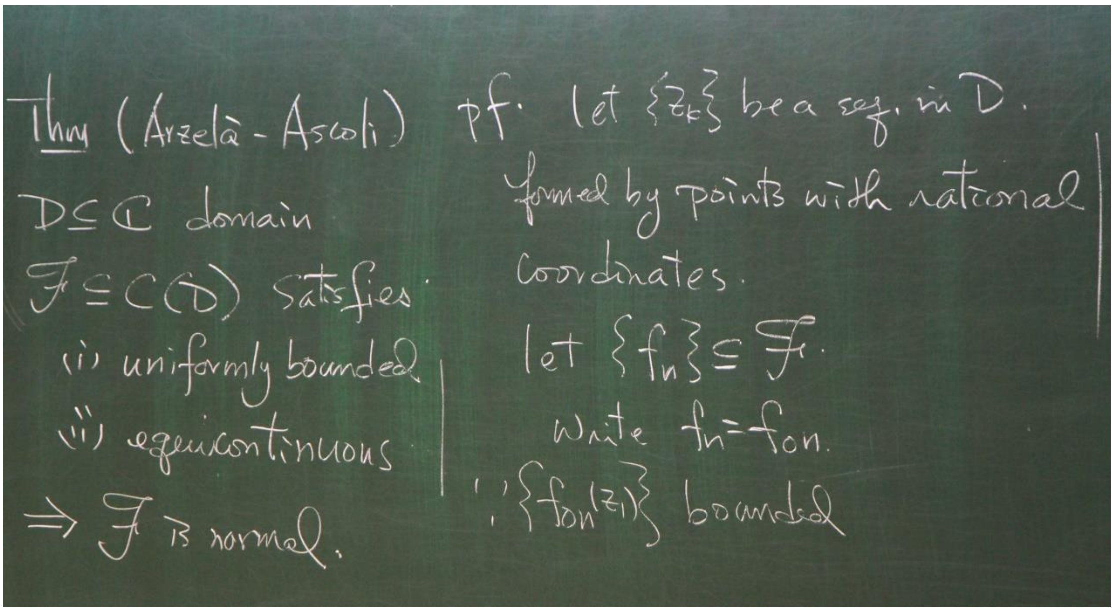
【10920 程守慶教授複變數函數論 / 第 22 堂版書】



Thm (Arzelà - Ascoli)

$D \subseteq \mathbb{C}$ domain

$\mathcal{F} \subseteq C(D)$ satisfies.

(i) uniformly bounded

(ii) equicontinuous

$\Rightarrow \mathcal{F}$ is normal.

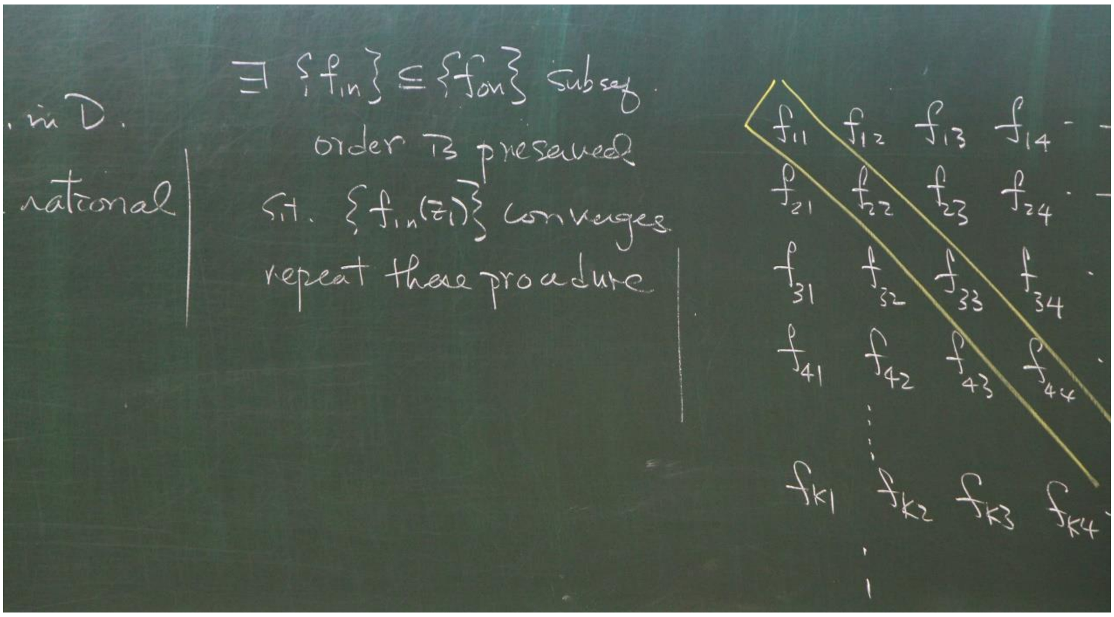
pf. let $\{z_k\}$ be a seq. in D.

formed by points with rational coordinates.

let $\{f_n\} \subseteq \mathcal{F}$.

Write $f_n = f_{0n}$.

$\because \{f_{0n}(z_1)\}$ bounded



. in D.

rational

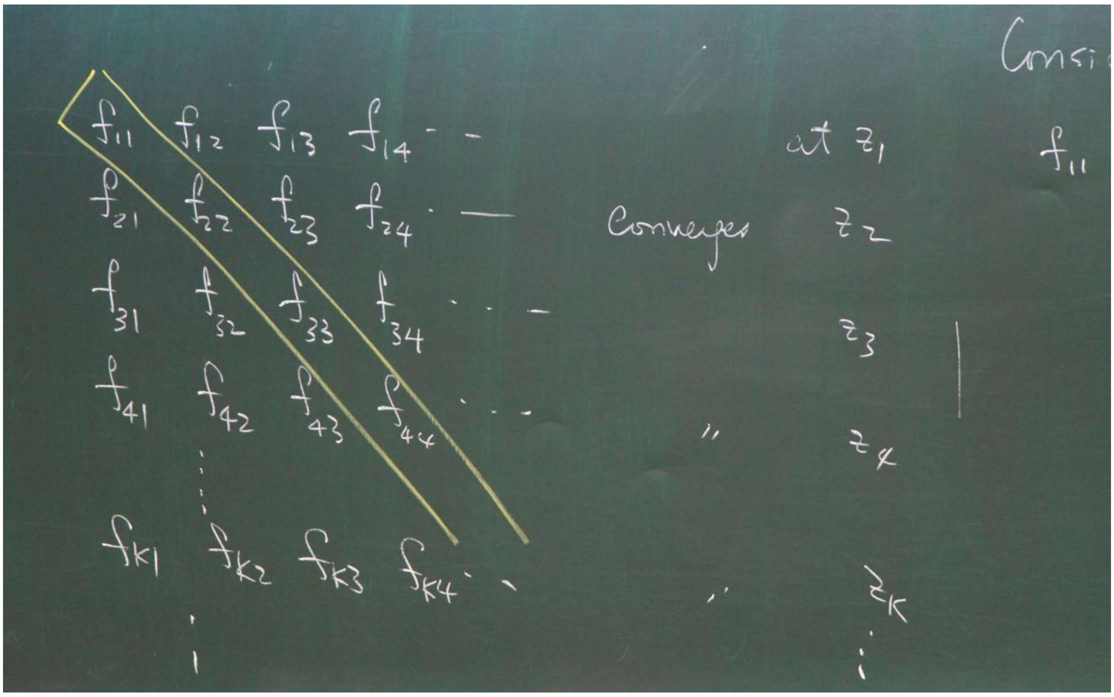
$\exists \{f_{1n}\} \subseteq \{f_{0n}\}$ subseq.

order is preserved

s.t. $\{f_{1n}(z_1)\}$ converges.

repeat these procedure

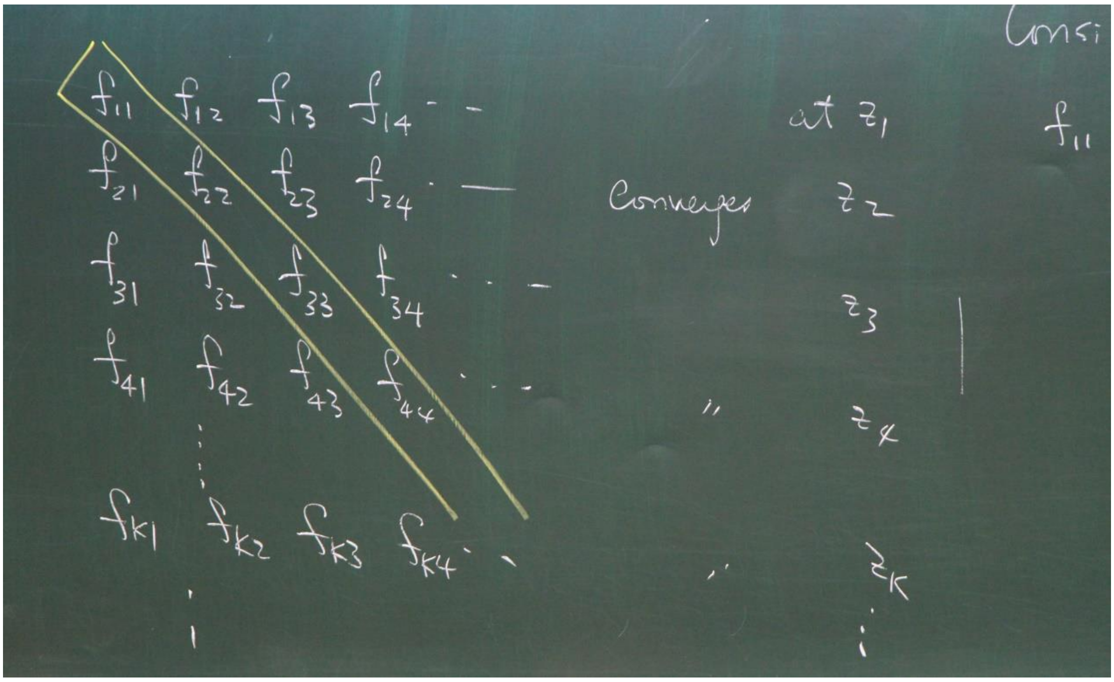
$$\begin{array}{cccc} f_{11} & f_{12} & f_{13} & f_{14} \cdots \\ f_{21} & f_{22} & f_{23} & f_{24} \cdots \\ f_{31} & f_{32} & f_{33} & f_{34} \cdots \\ f_{41} & f_{42} & f_{43} & f_{44} \cdots \\ & \vdots & & \\ f_{k1} & f_{k2} & f_{k3} & f_{k4} \cdots \\ & \vdots & & \end{array}$$



Consi

$$\begin{array}{llllll}
f_{11} & f_{12} & f_{13} & f_{14} & \cdots & \text{at } z_1 \\
f_{21} & f_{22} & f_{23} & f_{24} & \cdots & \text{Converges } z_2 \\
f_{31} & f_{32} & f_{33} & f_{34} & \cdots & z_3 \\
f_{41} & f_{42} & f_{43} & f_{44} & \cdots & '' \quad z_4 \\
\vdots & & & & & \\
f_{K1} & f_{K2} & f_{K3} & f_{K4} & \cdots & '' \quad z_K \\
\vdots & & & & & \vdots
\end{array}$$

f_{11}



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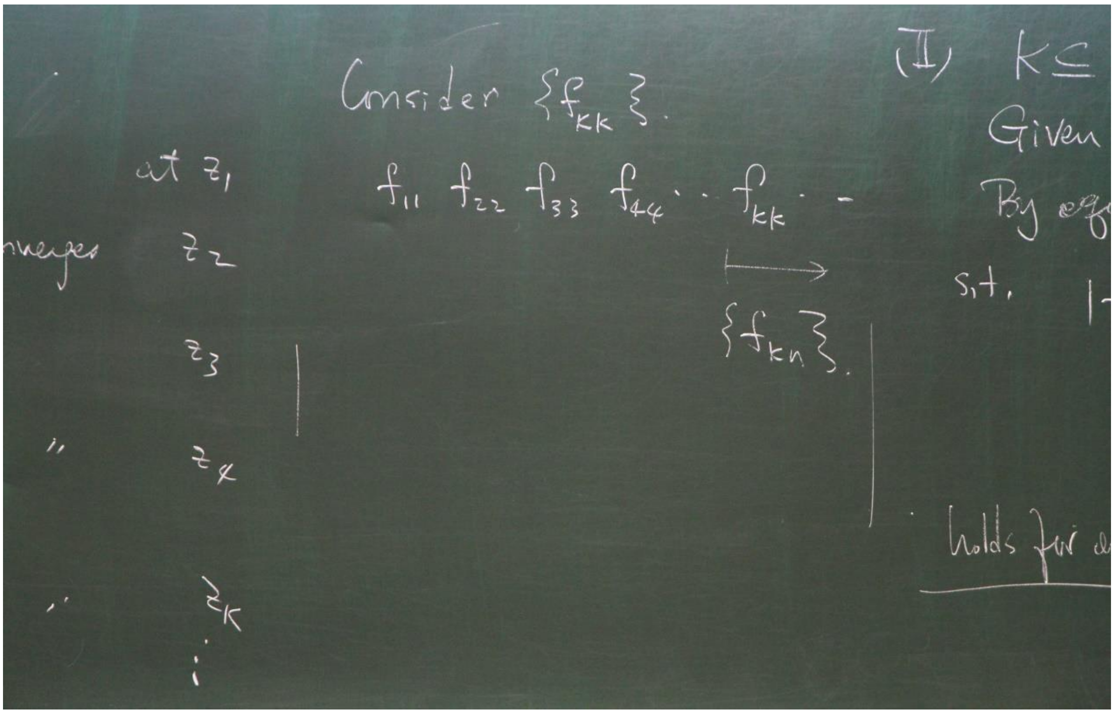
\end{array}$$

f_{11}



at z_1

nverges z_2

z_3

" z_4

" z_k

$\vdots$

Consider $\{f_{kk}\}$.

$f_{11}\ f_{22}\ f_{33}\ f_{44} \cdots f_{kk} \cdots$

$\longmapsto$

$\{f_{kn}\}$.

(II) $K \subseteq$

Given

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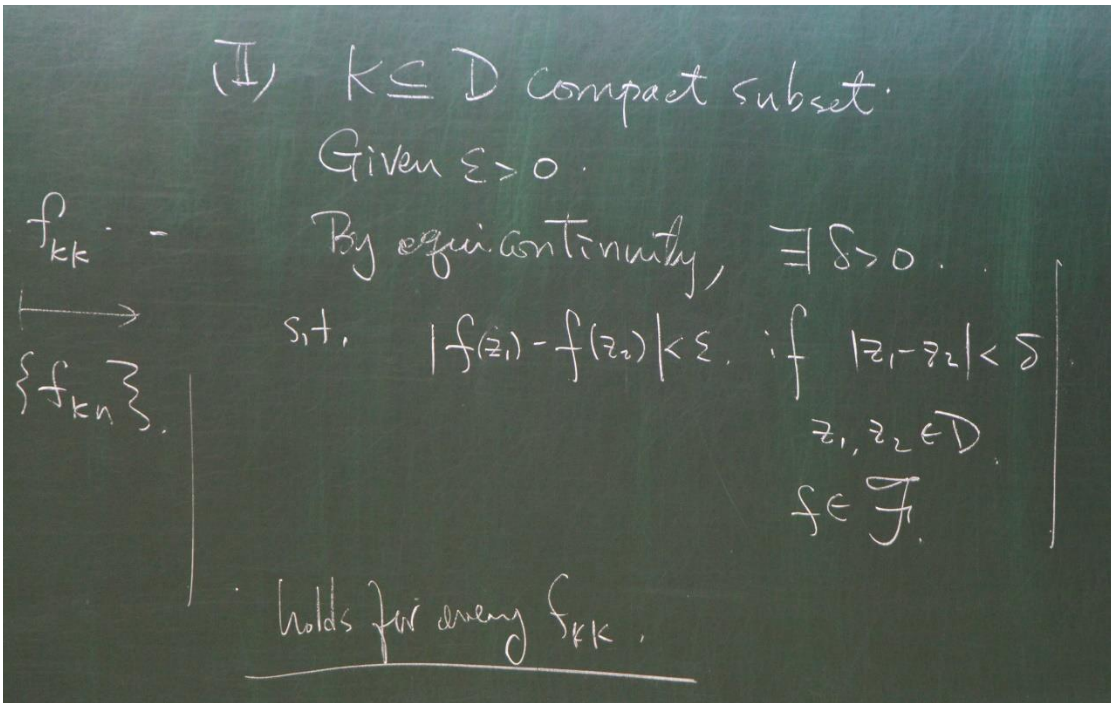
s.t. |

holds for e



$f_{kk} \cdots$

$\longmapsto$

$\{f_{kn}\}$.

(II) $K \subseteq D$ compact subset.

Given $\varepsilon > 0$.

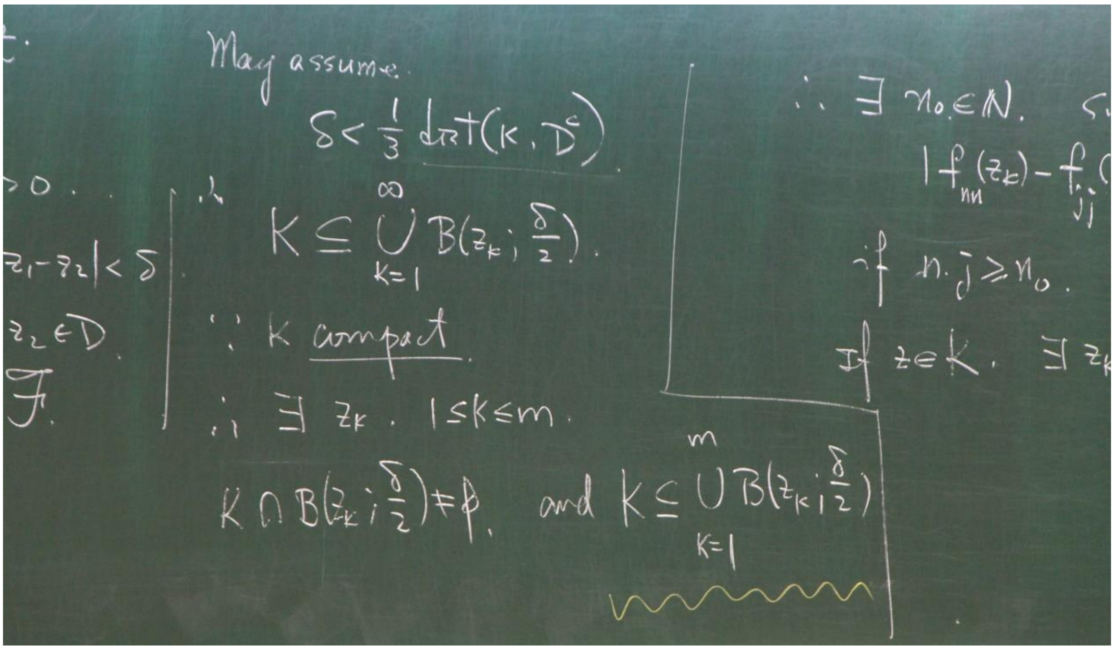
By equicontinuity, $\exists\, \delta > 0$.

s.t. $|f(z_1) - f(z_2)| < \varepsilon$. if $|z_1 - z_2| < \delta$

$z_1, z_2 \in D$.

$f \in \mathcal{F}$.

holds for every f_{kk}.



$t.$

$> 0.$

$z_1 - z_2| < \delta$

$z_2 \in D.$

$\mathcal{F}.$

May assume

$$\underline{\delta < \frac{1}{3}\,\mathrm{dist}(K, D^c)}.$$

$$\therefore\ K \subseteq \bigcup_{k=1}^{\infty} B\left(z_k ; \frac{\delta}{2}\right).$$

$\because$ K $\underline{\text{compact}}$.

$\therefore\ \exists\ z_k,\ 1 \le k \le m.$

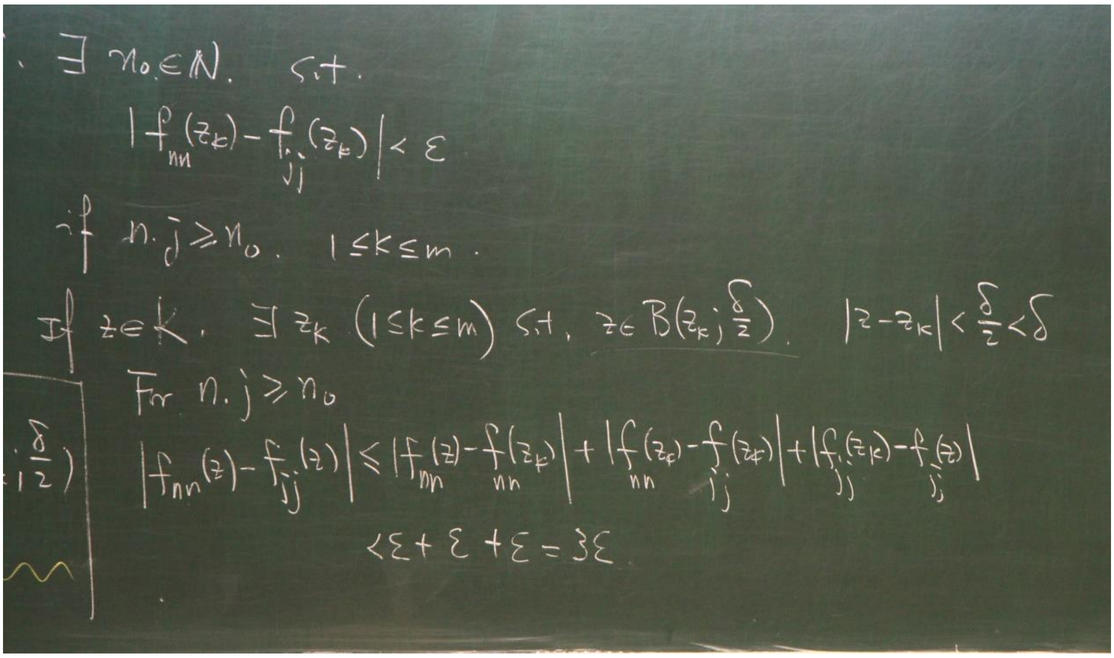
$$K \cap B\left(z_k ; \frac{\delta}{2}\right) \neq \phi, \text{ and } \underbrace{K \subseteq \bigcup_{k=1}^{m} B\left(z_k ; \frac{\delta}{2}\right)}$$

$\therefore\ \exists\ n_0 \in \mathbb{N}.\ \ s$

$|f_{nn}(z_k) - f_{jj}($

if $n, j \ge n_0.$

If $z \in K.\ \ \exists\ z_k$



$\therefore\ \exists\ n_0 \in \mathbb{N}.\ \ s.t.$

$$|f_{nn}(z_k) - f_{jj}(z_k)| < \varepsilon$$

if $n, j \ge n_0.\ \ 1 \le k \le m.$

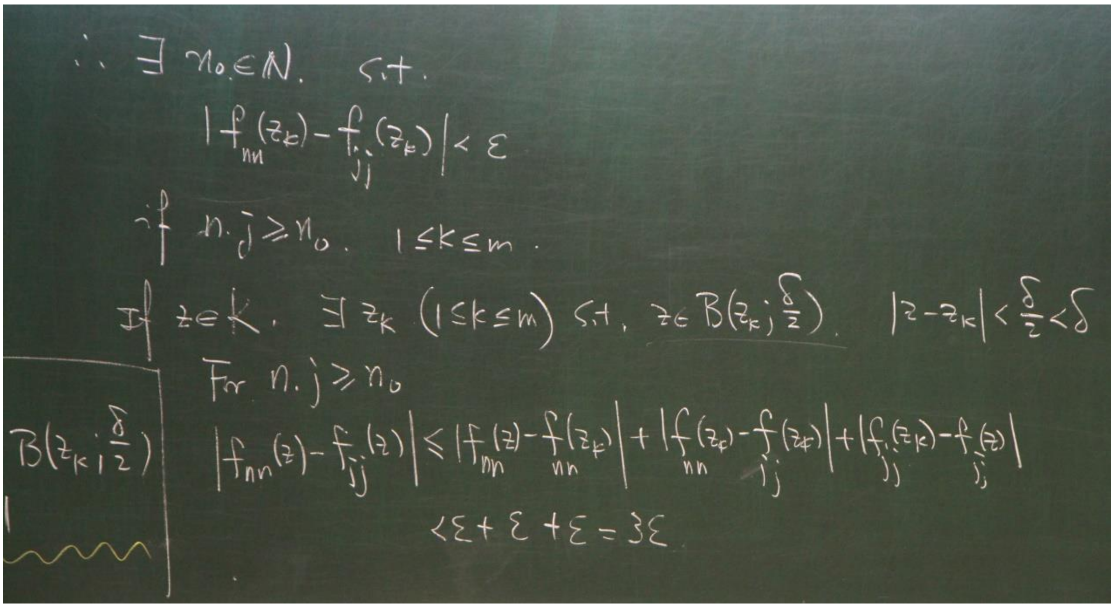
If $z \in K.\ \ \exists\ z_k\ (1 \le k \le m)$ s.t. $\underline{z \in B\left(z_k ; \frac{\delta}{2}\right)}.\ \ |z - z_k| < \frac{\delta}{2} < \delta$

$; \frac{\delta}{2})$

For $n, j \ge n_0$

$$|f_{nn}(z) - f_{jj}(z)| \le |f_{nn}(z) - f_{nn}(z_k)| + |f_{nn}(z_k) - f_{jj}(z_k)| + |f_{jj}(z_k) - f_{jj}(z)|$$

$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$



$\therefore \exists\, n_0 \in \mathbb{N}$. s.t.

$|f_{nn}(z_k) - f_{jj}(z_k)| < \varepsilon$

if $n, j \geq n_0$. $1 \leq k \leq m$.

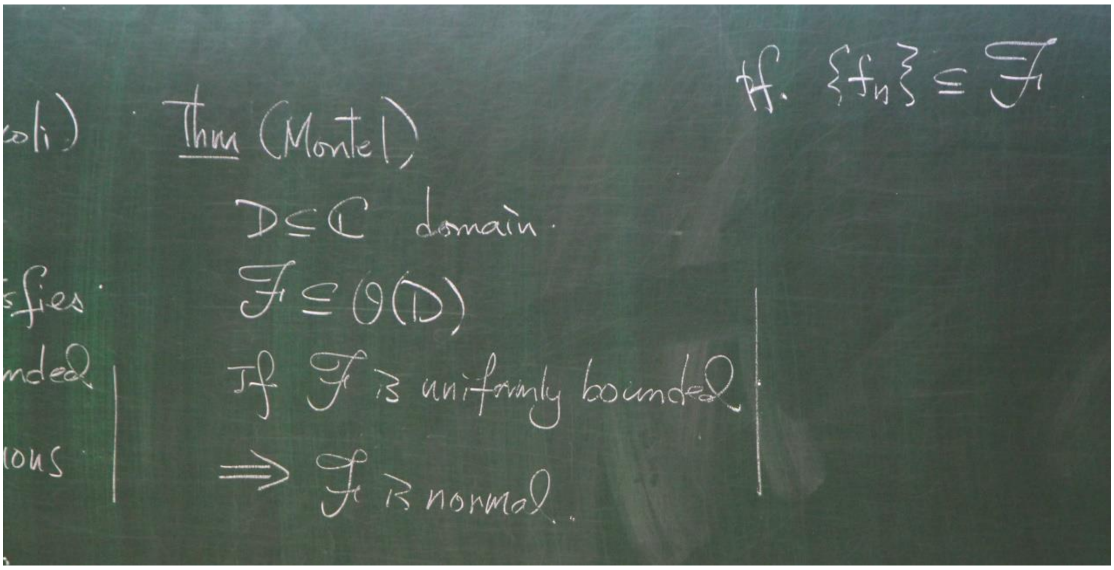
If $z \in K$. $\exists\, z_k$ $(1 \leq k \leq m)$ s.t. $z \in B(z_k; \frac{\delta}{2})$. $|z - z_k| < \frac{\delta}{2} < \delta$

For $n, j \geq n_0$

$|f_{nn}(z) - f_{jj}(z)| \leqslant |f_{nn}(z) - f_{nn}(z_k)| + |f_{nn}(z_k) - f_{jj}(z_k)| + |f_{jj}(z_k) - f_{jj}(z)|$

$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$

$B(z_k; \frac{\delta}{2})$



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Thm (Montel)

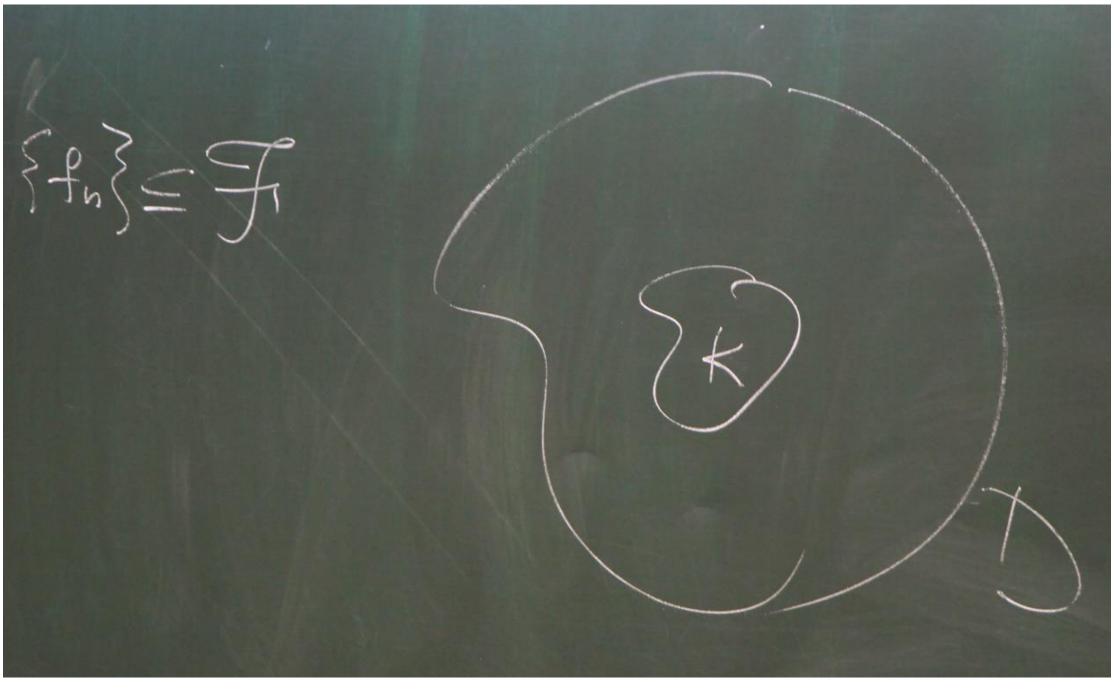
$D \subseteq \mathbb{C}$ domain.

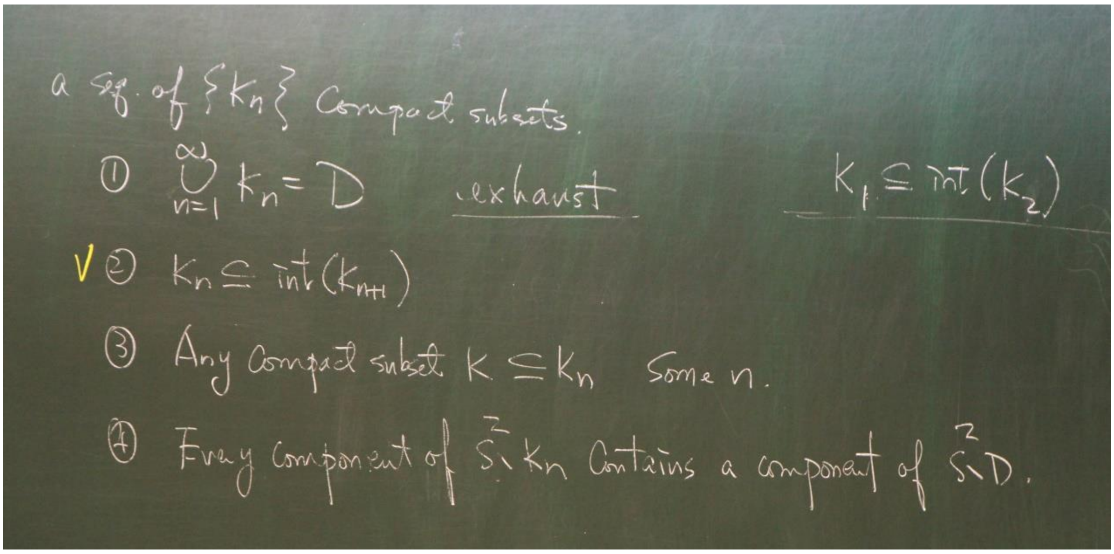
$\mathcal{F} \subseteq \mathcal{O}(D)$

If $\mathcal{F}$ is uniformly bounded

$\Longrightarrow$ $\mathcal{F}$ is normal.

Pf. $\{f_n\} \subseteq \mathcal{F}$





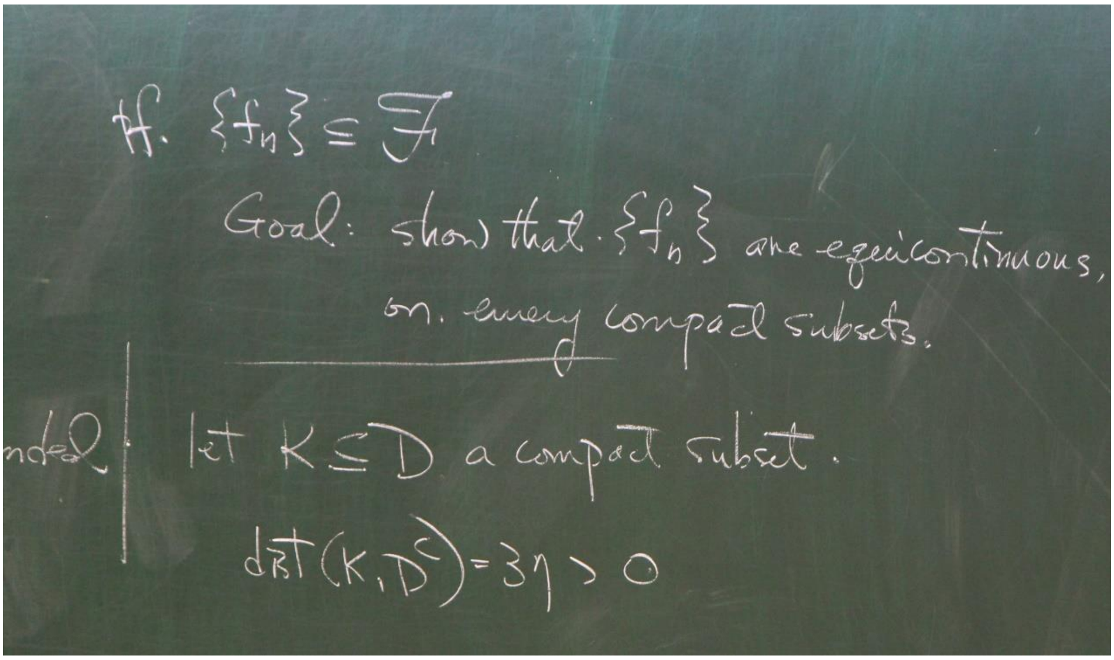
a seq. of $\{K_n\}$ Compact subsets.

① $\bigcup_{n=1}^{\infty} K_n = D$ exhaust $K_1 \subseteq \text{int}(K_2)$

✓ ② $K_n \subseteq \text{int}(K_{n+1})$

③ Any Compact subset $K \subseteq K_n$ Some n.

④ Every component of $S^2 \setminus K_n$ Contains a component of $S^2 \setminus D$.



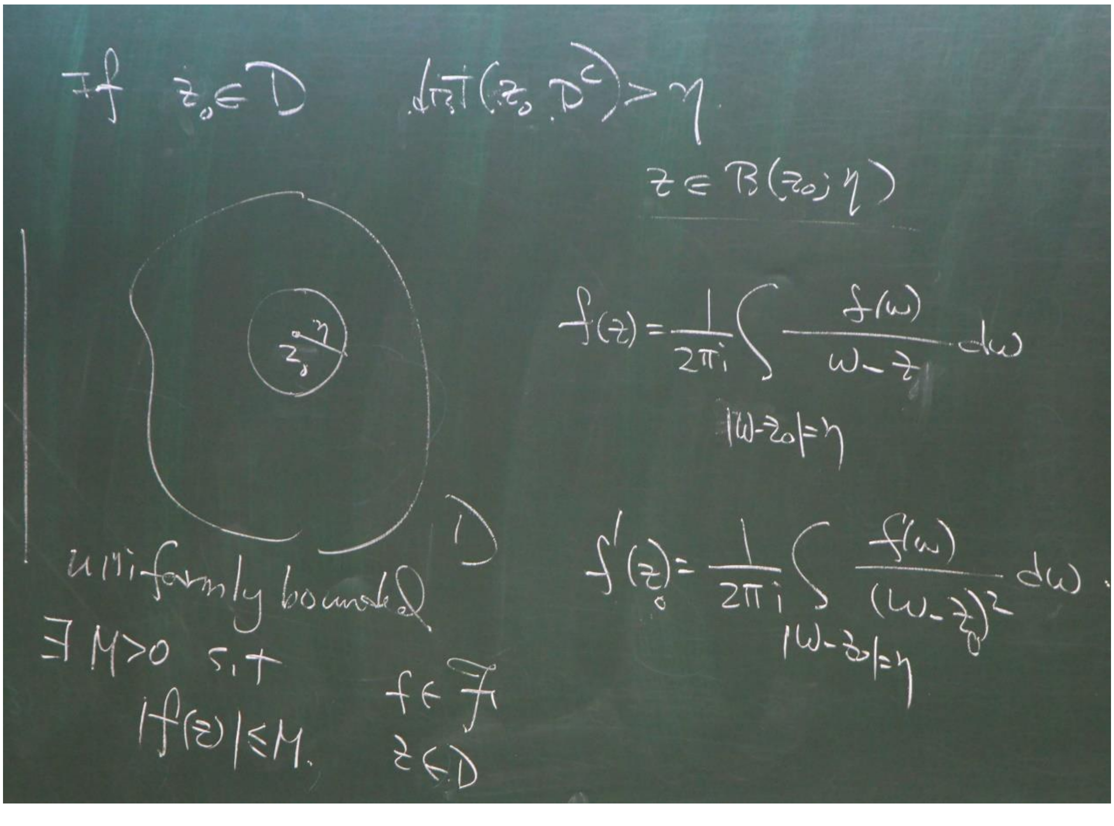
Pf. $\{f_n\} \subseteq \mathcal{F}$

Goal: show that $\{f_n\}$ are equicontinuous, on every compact subsets.

nded

let $K \subseteq D$ a compact subset.

$\mathrm{dist}(K, D^c) = 3\eta > 0$



Fix $z_0 \in D$ $\quad \mathrm{dist}(z_0, D^c) > \eta$

$z \in B(z_0; \eta)$

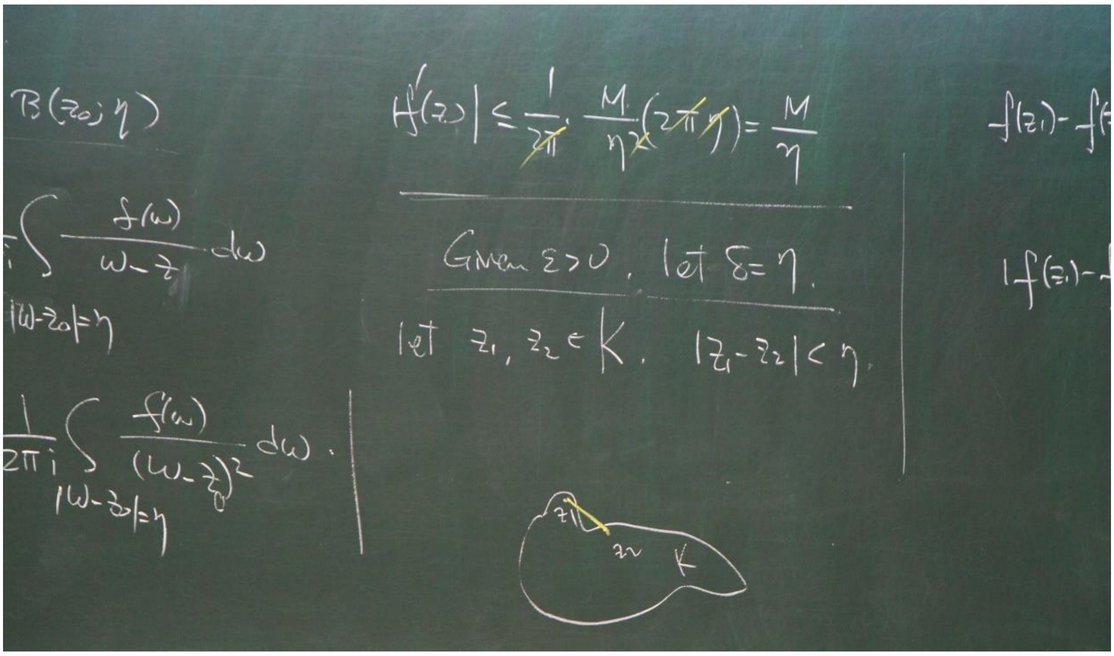
$$f(z) = \frac{1}{2\pi i}\int_{|w-z_0|=\eta} \frac{f(w)}{w-z}\,dw$$

$$f'(z_0) = \frac{1}{2\pi i}\int_{|w-z_0|=\eta} \frac{f(w)}{(w-z_0)^2}\,dw$$

uniformly bounded

$\exists\, M>0$ s.t.

$|f(z)| \leq M$. $\quad f \in \mathcal{F}$, $z \in D$



$B(z_0; \eta)$

$$i \int_{|w-z_0|=\eta} \frac{f(w)}{w-z}\,dw$$

$$\frac{1}{2\pi i} \int_{|w-z_0|=\eta} \frac{f(w)}{(w-z_0)^2}\,dw .$$

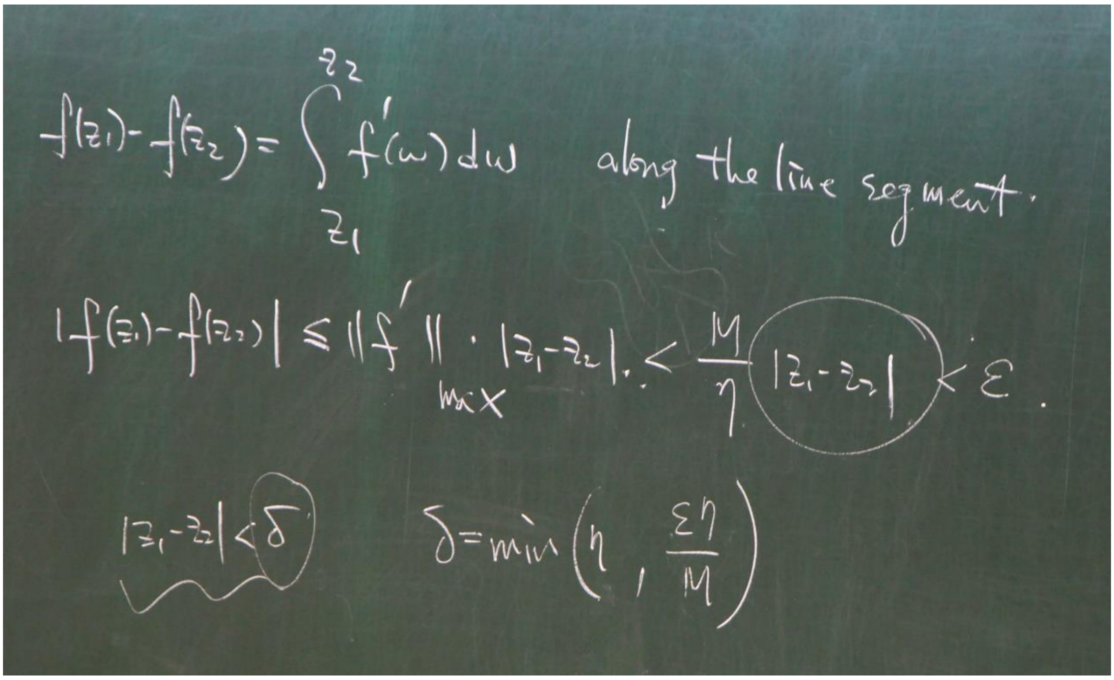
$$|f'(z)| \leq \frac{1}{\cancel{2\pi}} \cdot \frac{M}{\eta^{\cancel{2}}}(\cancel{2\pi}\cancel{\eta}) = \frac{M}{\eta}$$

Given $\varepsilon > 0$, let $\delta = \eta$.

let $z_1, z_2 \in K$. $|z_1 - z_2| < \eta$.

$f(z_1) - f($

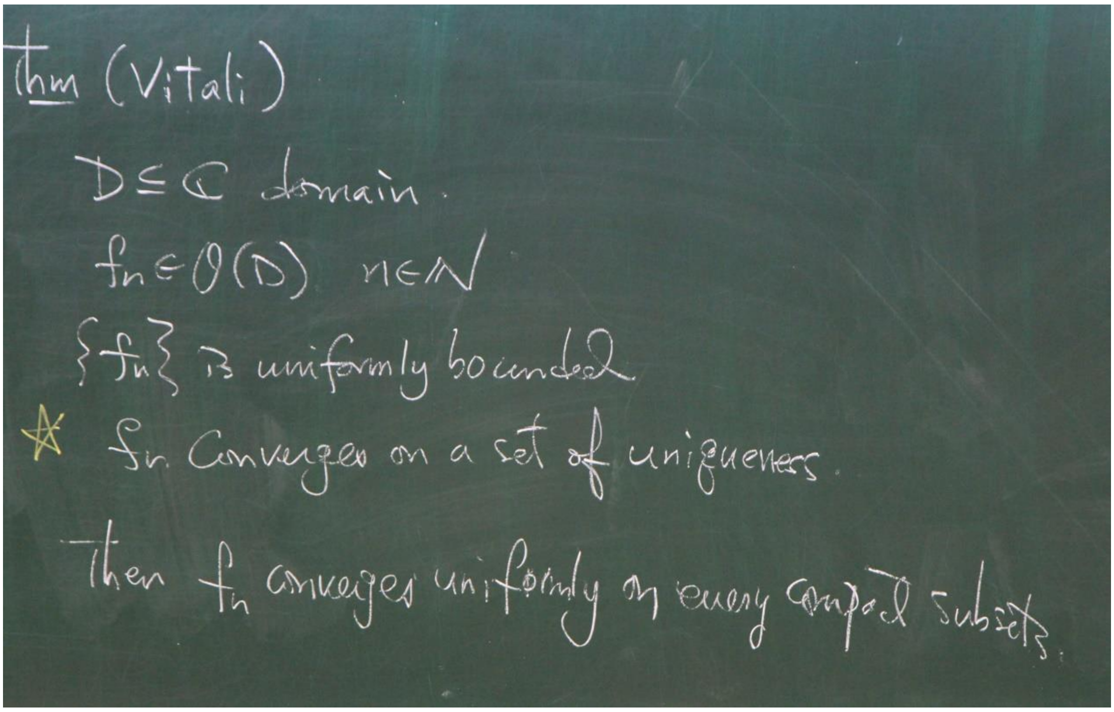
$|f(z_1) -$



$$f(z_1) - f(z_2) = \int_{z_1}^{z_2} f'(w)\,dw$$ along the line segment.

$$|f(z_1) - f(z_2)| \leq \|f'\|_{\max} \cdot |z_1 - z_2| < \frac{M}{\eta}\,|z_1 - z_2| < \varepsilon .$$

$|z_1 - z_2| < \delta$ $\qquad$ $\delta = \min\left(\eta, \frac{\varepsilon\eta}{M}\right)$



thm (Vitali)

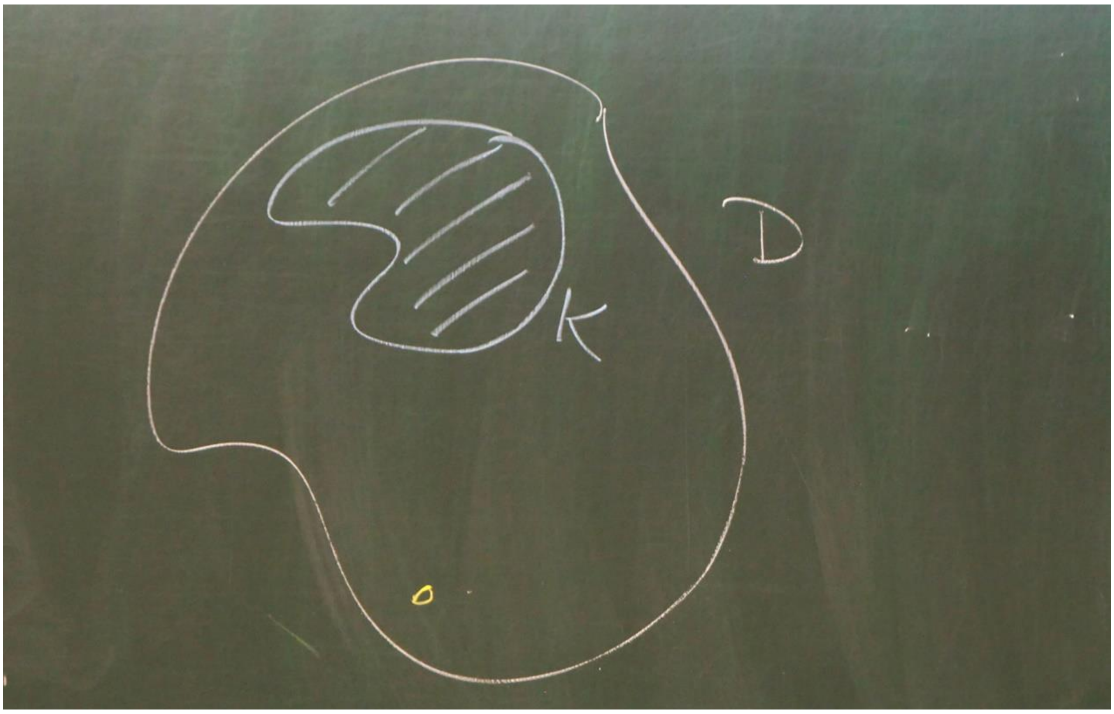
$D \subseteq \mathbb{C}$ domain.

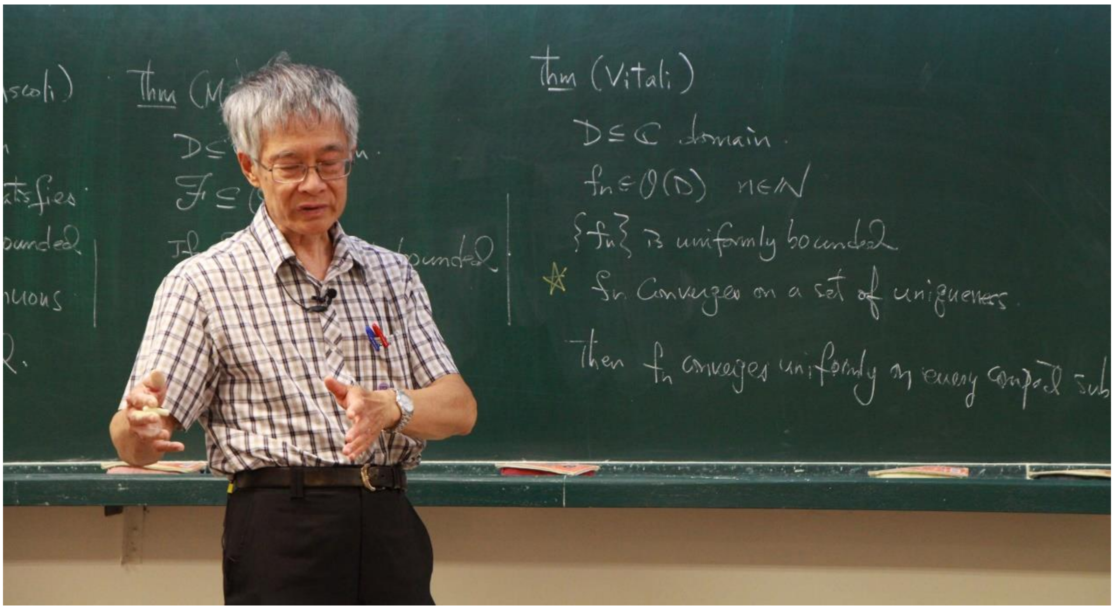
$f_n \in \mathcal{O}(D)$ $n \in \mathbb{N}$

$\{f_n\}$ is uniformly bounded

☆ f_n Converges on a set of uniqueness.

Then f_n converges uniformly on every compact subsets.

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Thm (Vitali)
D ⊆ ℂ domain.
fn ∈ O(D) n ∈ ℕ
{fn} is uniformly bounded
☆ fn Converges on a set of uniqueness.
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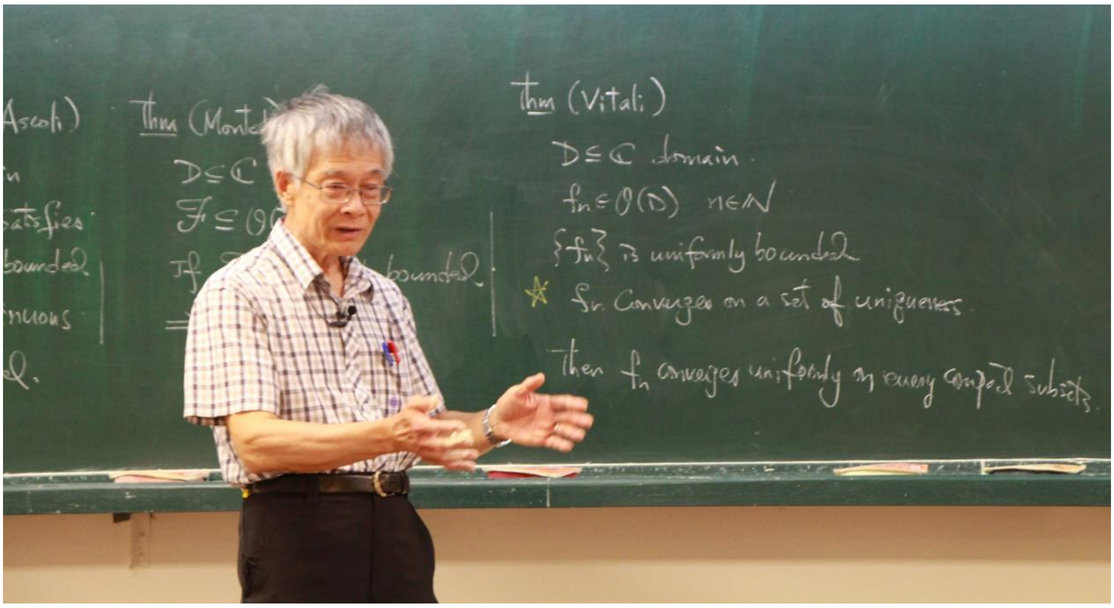
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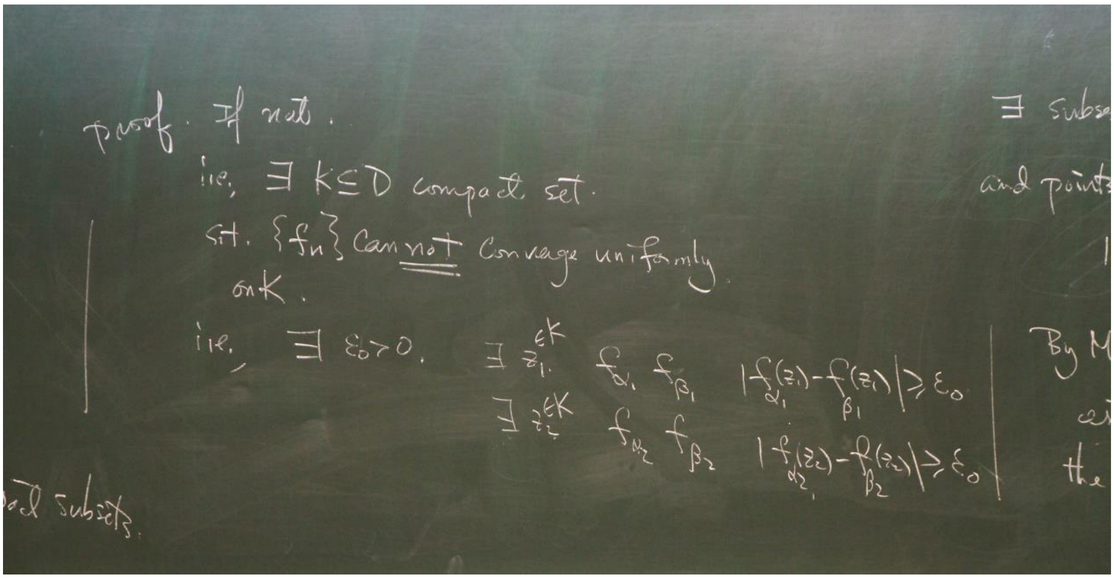

proof. If not.

i.e., $\exists\ K \subseteq D$ compact set.

s.t. $\{f_n\}$ can not converge uniformly on K.

i.e., $\exists\ \varepsilon_0 > 0$. $\exists\ z_1 \in K$ $f_{\alpha_1}, f_{\beta_1}$ $|f_{\alpha_1}(z_1) - f_{\beta_1}(z_1)| \geq \varepsilon_0$

$\exists\ z_2 \in K$ $f_{\alpha_2}\ f_{\beta_2}$ $|f_{\alpha_2}(z_2) - f_{\beta_2}(z_2)| \geq \varepsilon_0$

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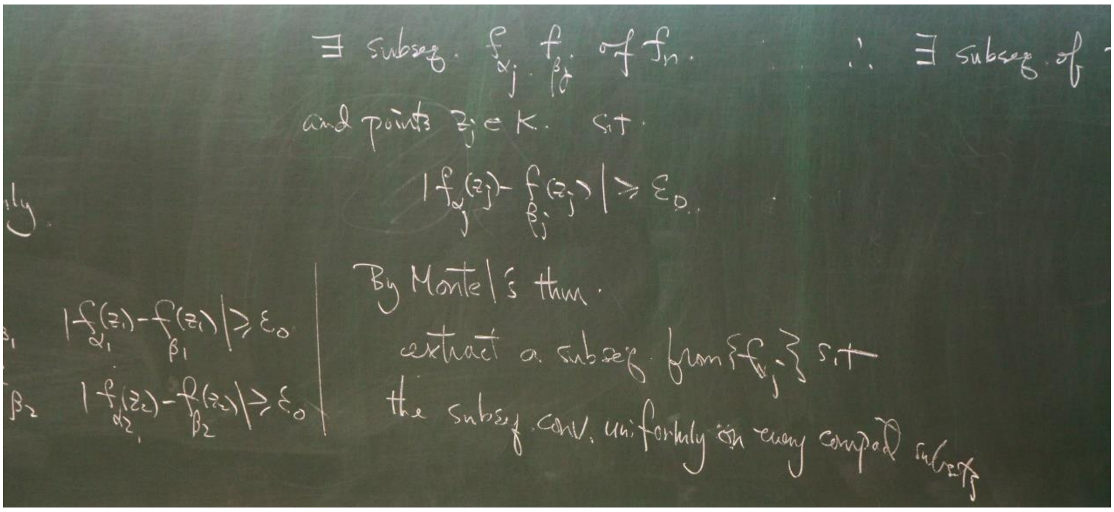


$\exists$ subseq. f_{α_j} f_{β_j} of f_n. $\therefore\ \exists$ subseq of

and points $z_j \in K$. s.t.

$|f_{\alpha_j}(z_j) - f_{\beta_j}(z_j)| \geq \varepsilon_0$.

ly.

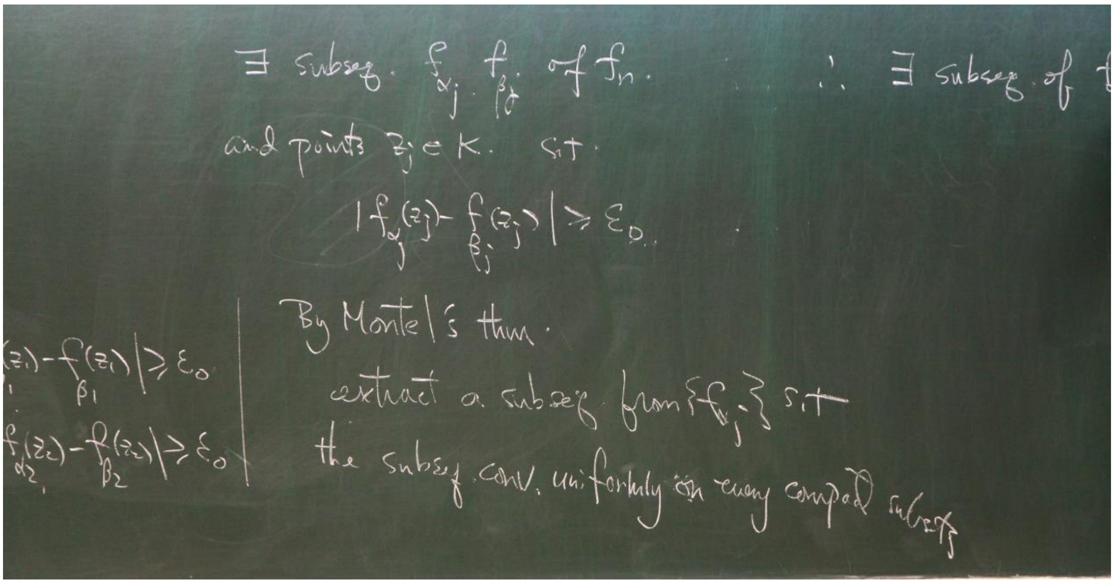
$_1$ $|f_{\alpha_1}(z_1) - f_{\beta_1}(z_1)| \geq \varepsilon_0$

β_2 $|f_{\alpha_2}(z_2) - f_{\beta_2}(z_2)| \geq \varepsilon_0$

By Montel's thm.

extract a subseq from $\{f_{\alpha_j}\}$ s.t

the subseq conv. uniformly on every compact subsets



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$(z_1) - f_{\beta_1}(z_1)| \geq \varepsilon_0$

$f_{\alpha_2}(z_2) - f_{\beta_2}(z_2)| \geq \varepsilon_0$

By Montel's thm.

extract a subseq from $\{f_{\alpha_j}\}$ s.t

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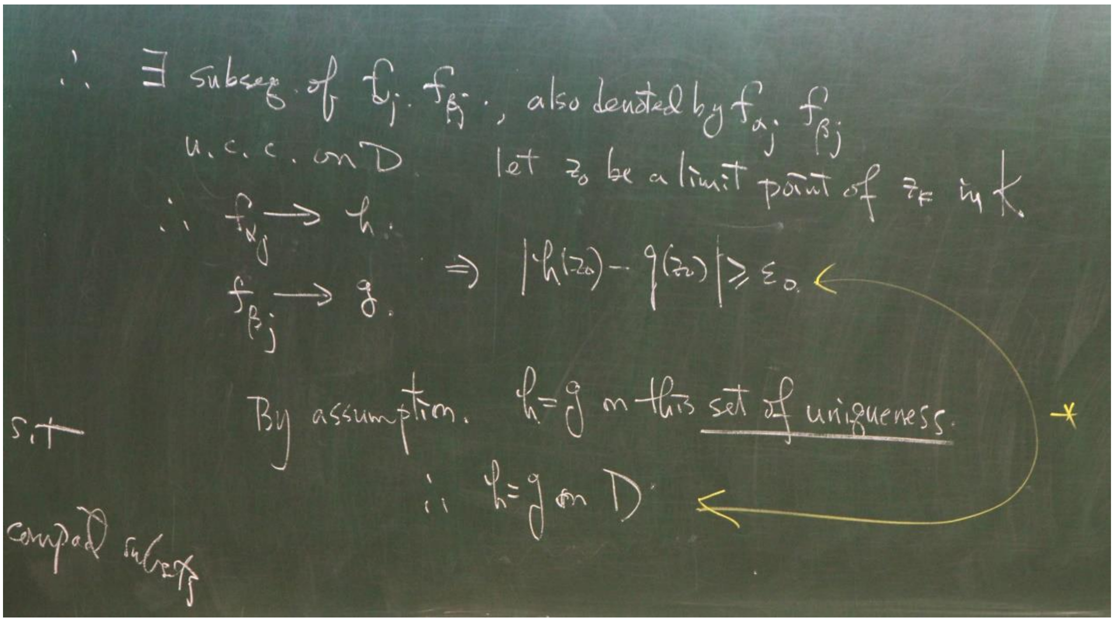


∴ ∃ subseq. of f_j. f_{β_j}, also denoted by f_{α_j}, f_{β_j}

u.c.c. on D. let z_0 be a limit point of z_F in K.

∴ $f_{\alpha_j} \longrightarrow h$.

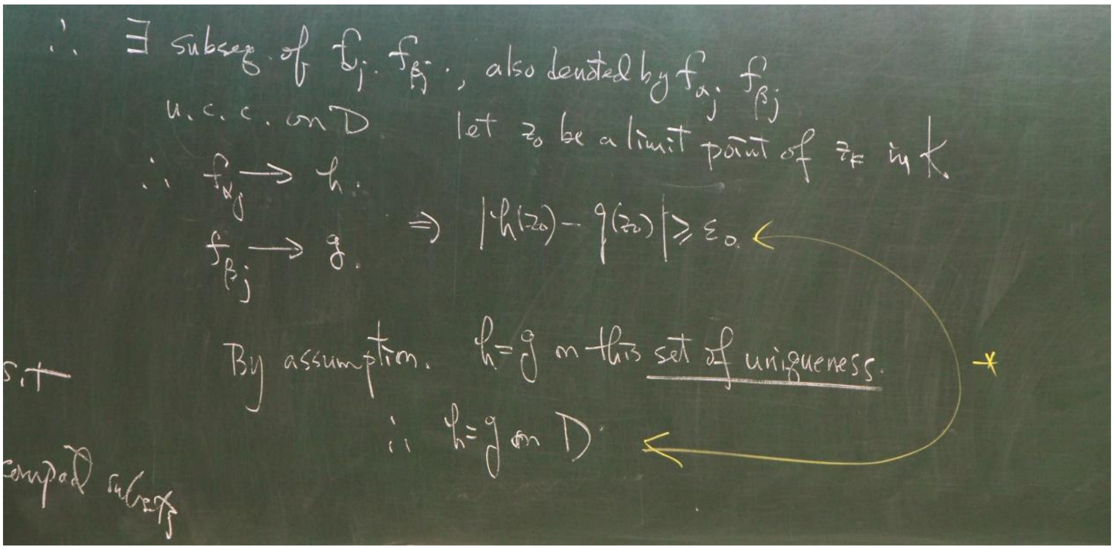
$f_{\beta_j} \longrightarrow g$. $\Rightarrow |h(z_0) - g(z_0)| \geq \varepsilon_0$.

s.t

By assumption. $h = g$ on this set of uniqueness. *

∴ $h = g$ on D.

compact subsets



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compact subsets