COMPLEX ANALYSIS

ASSIGNMENT III; DUE APRIL 26, 2021.

Here U denotes the open unit disc in $\mathbb{C}$.

21. Evaluate $\int_0^\infty \frac{x^\lambda}{a^2+x^2}dx, -1<\lambda<1, a>0$.

22. Evaluate $\int_0^\infty \frac{\ln x}{x^\lambda(1+x)}dx, 0<\lambda<1$.

23. Let f be a holomorphic function defined on the open unit disc such that $|f(\frac{1}{n})| \le \frac{1}{3^n}$ for $n \ge 2$. Prove that f is identically zero.

24. Let $p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ be a polynomial with all a_j real and $0 \le a_0 \le a_1 \le \cdots \le a_n$. Show that all of the zeros of $p(z)$ lie inside the closed unit disc.

25. Let $p(z) = 1 + 2z - 18z^4$. Show that all the zeros of p lie within the open disc $D = (0; \frac{2}{3})$.

26. Show that the only univalent entire functions are the affine functions $f(z) = az + b$, $a, b \in \mathbb{C}$, $a \neq 0$.

27. Suppose g is holomorphic in the punctured plane $z \neq 0$ and satisfies $|g(z)| \le \sqrt{|z|} + \frac{1}{\sqrt{|z|}}$ for all $z \neq 0$. Prove g is a constant.

28. Let $\{m_1, m_2, \cdots, m_k\}$ be a set of positive integers and

$$R(z) = \frac{1}{(z^{m_1}-1)(z^{m_2}-1)\cdots(z^{m_k}-1)}.$$

Find the coefficient c_{-k} in the Laurent expansion for $R(z)$ about the point $z=1$.

29. Let $g \in \mathcal{O}(\Omega)$, where $\Omega = U \setminus \{0\}$. Suppose that $\iint_\Omega |g(z)|^2 dxdy < \infty$. Show that 0 is a removable singularity of g.

30. Show that the converse of Darboux-Picard's theorem is false: Find a simple closed curve C and a function f which is holomorphic on and inside C such that f is univalent inside C but not on C.

Thm. $D \subseteq \mathbb{C}$ domain

$f \in \mathcal{O}(D)$ and f is 1-1.

$\Rightarrow$ $f'(z) \neq 0$ $\forall z \in D$

and $f : D \to f(D)$

biholomorphism.

univalent

1-1

Schlicht

Ex.

$f(z) = e^z$ on

$f'(z) = e^z \neq 0$

$e^{2k\pi i} = 1$.

univalent

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$f(z) = e^z$ on $\mathbb{C}$.

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$e^{2k\pi i} = 1$. $k \in \mathbb{Z}$.

Thm. $D \subseteq \mathbb{C}$ domain

$z_0 \in D$. $f \in \mathcal{O}(D)$

Assume $f'(z_0) \neq 0$

Then, locally near z_0,

f is one-to-one.

univalent

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ℂ.

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Thm. $D \subseteq \mathbb{C}$ domain, $z_0 \in D$. $f \in \mathcal{O}(D)$.

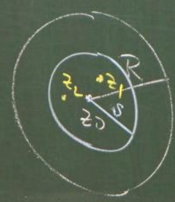
Assume $f'(z_0) \neq 0$.

Then, locally near z_0, f is one-to-one.

pf. near z_0.

$$f(z) = f(z_0) + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

holds on $|z - z_0| < R$



Consider $0 < s < R$

$z_1, z_2 \in B(z_0; s)$

$a_1 = f'(z_0) \neq 0$

$$\therefore \quad f(z_1) - f(z_2) = a_1(z_1 - z_2) + \sum_{k=2}^{\infty} \left(a_k (z_1 - z_0)^k - a_k (z_2 - z_0)^k \right)$$

$$= a_1(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1 - z_2) \left(\sum_{j=0}^{k-1} (z_1 - z_0)^j (z_2 - z_0)^{k-1-j} \right)$$

$$= (z_1 - z_2) \left[a_1 + \sum_{k=2}^{\infty} a_k \left(\sum_{j=0}^{k-1} (z_1 - z_0)^j (z_2 - z_0)^{k-1-j} \right) \right]$$

$\subseteq \mathbb{C}$ domain

D. $f \in \mathcal{O}(D)$

$f'(z_0) \neq 0$

locally near z_0,

one-to-one.

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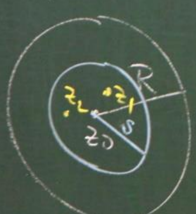
∴

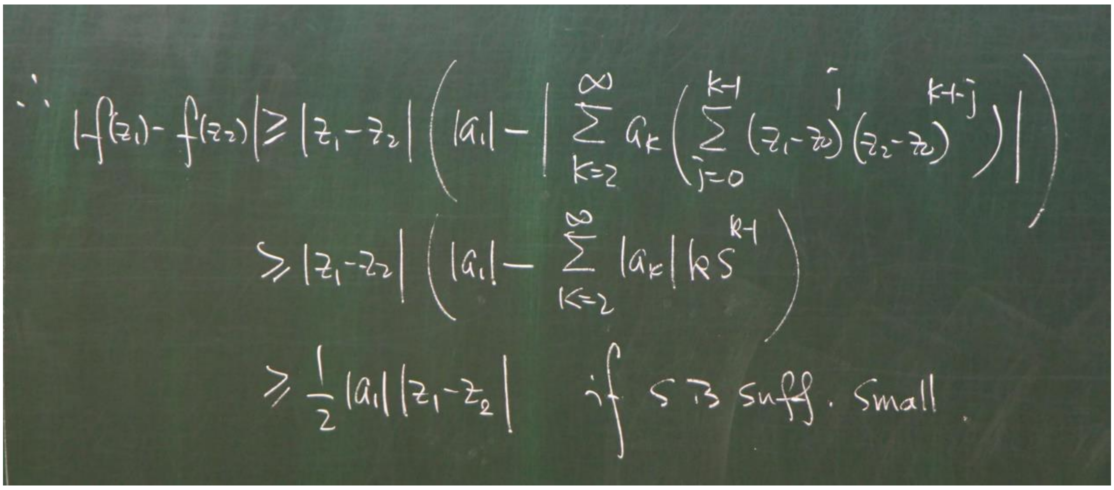


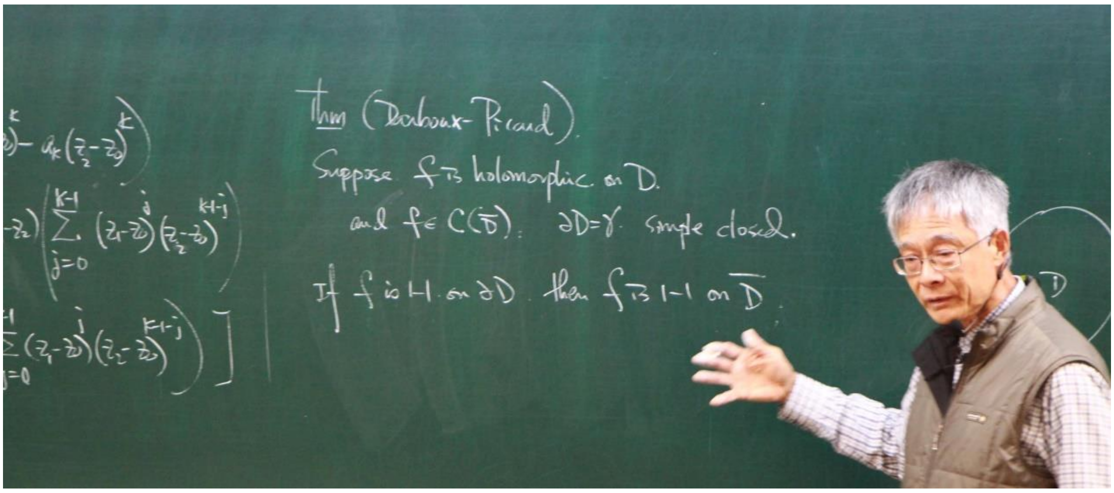
Consider $0 < s < R$

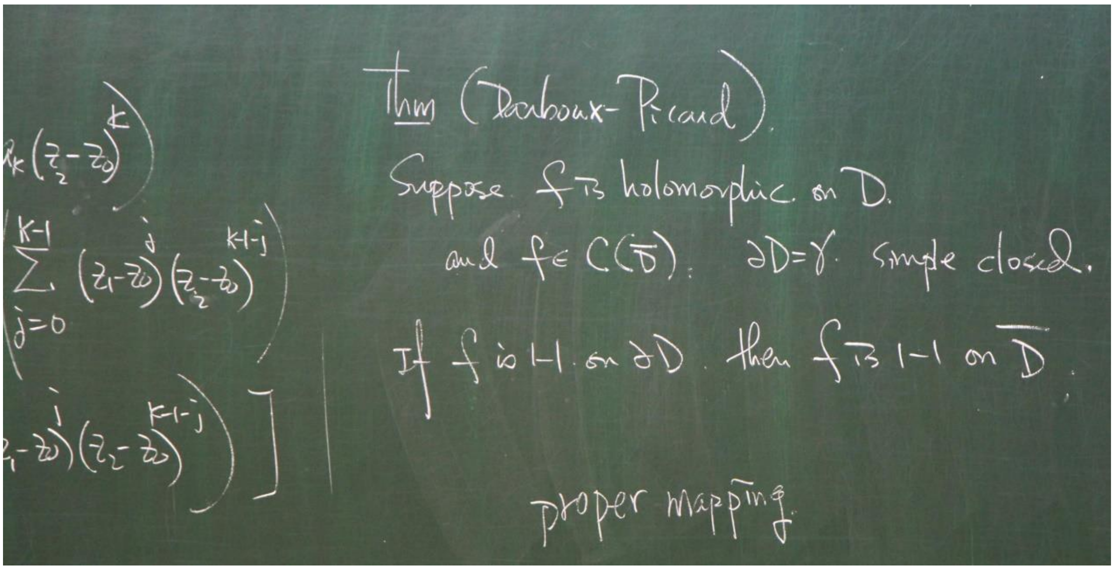
$z_1, z_2 \in B(z_0; s)$

$a_1 = f'(z_0) \neq 0 \quad \Rightarrow \quad |z_1 - z_0| < s$

$|z_2 - z_0| < s$


$\therefore\ |f(z_1)-f(z_2)| \geq |z_1-z_2| \left(|a_1| - \left| \sum_{k=2}^{\infty} a_k \left(\sum_{j=0}^{k-1} (z_1-z_0)^j (z_2-z_0)^{k-1-j} \right) \right| \right)$
$\geq |z_1-z_2| \left(|a_1| - \sum_{k=2}^{\infty} |a_k| k \delta^{k-1} \right)$
$\geq \frac{1}{2}|a_1||z_1-z_2|$ if δ is suff. small.


Thm (Darboux-Picard).
Suppose f is holomorphic on D.
and $f \in C(\overline{D})$. $\partial D = \gamma$ simple closed.
If f is 1-1 on ∂D then f is 1-1 on $\overline{D}$.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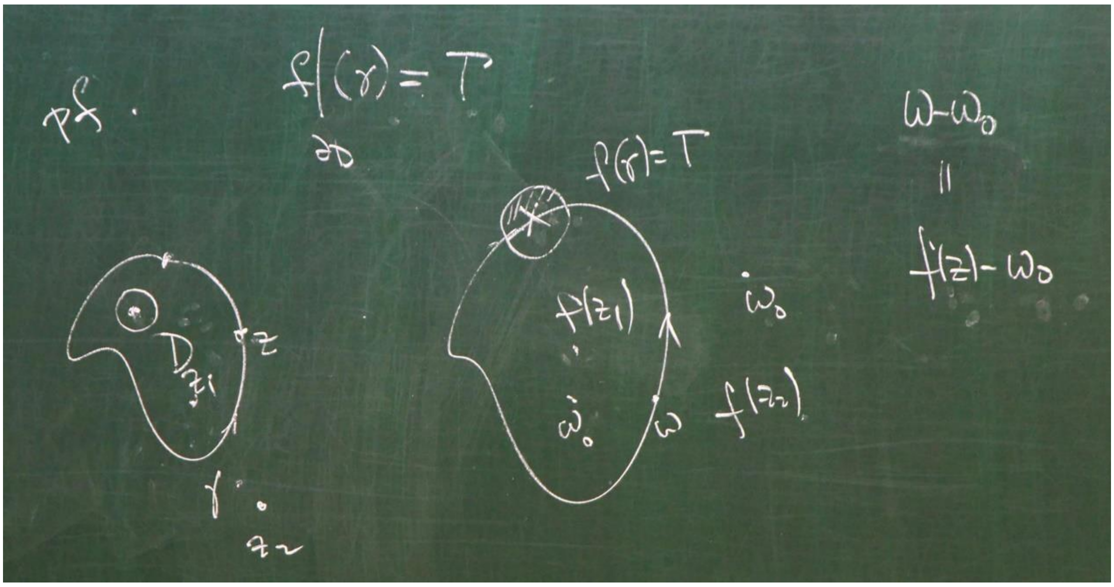
proper mapping



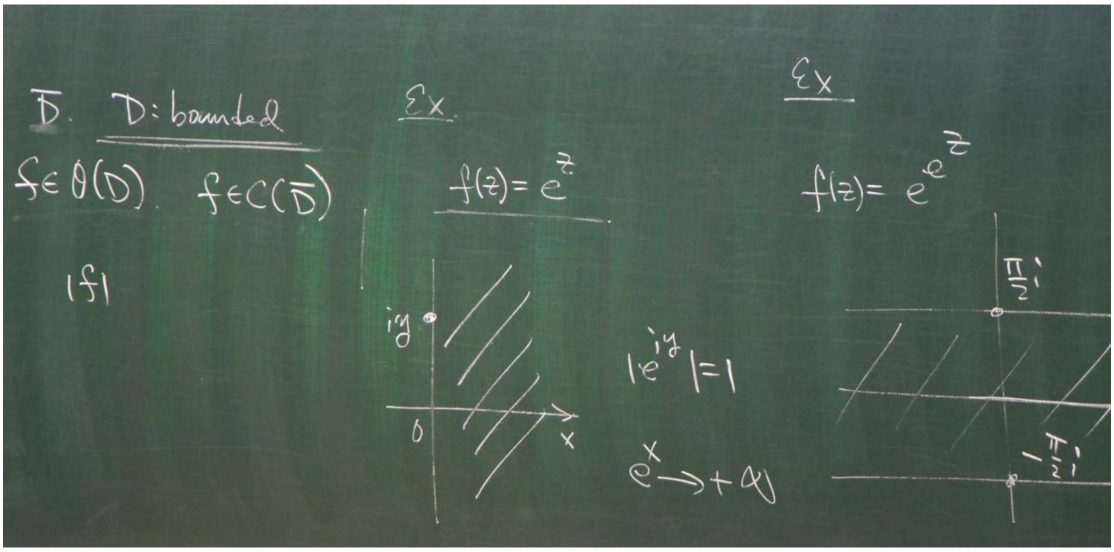
Δ: open unit disc

$f \in \mathcal{O}(\Delta)$ $f(0)=0$. $f'(0)=1$ univalent

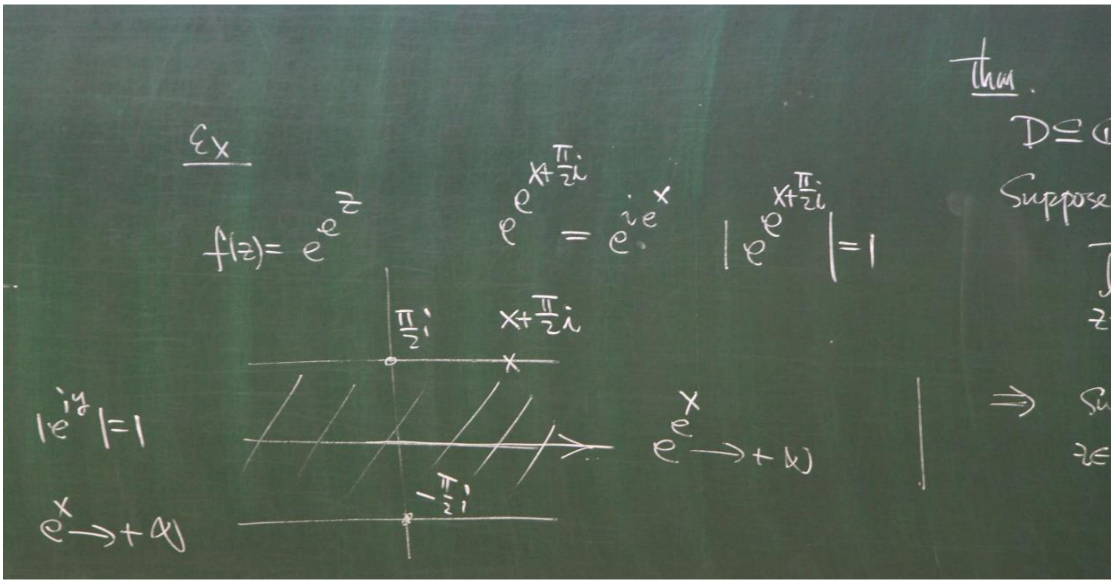
$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots$$

1916 Bieberbach conjecture. then $|a_n| \le n$.

1984 L. de Branges operator theory



D̄. D: bounded
f ∈ 𝒪(D). f ∈ C(D̄)
|f|
Ex.
f(z) = e^z
iy
0
x
|e^{iy}| = 1
e^x → +∞
Ex.
f(z) = e^{e^z}
π/2 i
−π/2 i



Ex
f(z) = e^{e^z}
e^{e^{x+π/2 i}} = e^{ie^x}
|e^{e^{x+π/2 i}}| = 1
π/2 i
x+π/2 i
−π/2 i
|e^{iy}| = 1
e^x → +∞
e^{e^x} → +∞
thm.
Suppose
⇒

$$\left| e^{e^{x+\frac{\pi}{2}i}} \right| = 1$$

$$e^{e^{x}} \to +\infty$$

Thm.

$D \subseteq \mathbb{C}$ <u>bounded domain</u>

Suppose $\exists\, M > 0$ s.t.

$$\overline{\lim_{z \to z_0 \in \partial D}} |f(z)| \le M.$$

$$\Rightarrow \sup_{z \in D} |f(z)| \le M.$$

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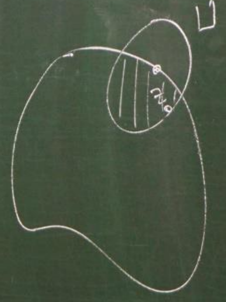


$\forall \varepsilon > 0$, $\exists$ open nbh. U of z_0

s.t.

$$|f(z)| \le M + \varepsilon.$$

$$z \in U \cap D$$

Claim

$\sup_{z \in D} |f(z)|$ is bounded.

If not, $\exists \{z_n\} \subseteq D$ s.t.

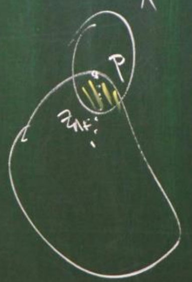
$|f(z_n)| > n$.

$\therefore \exists$ subseq. $\{z_{n_k}\}$ s.t.

$z_{n_k} \to P \in \overline{D}$.

$p \in D$ ✗

$\therefore p \in \partial D$. $\forall \varepsilon > 0$, $\exists U$



s.t. $|f(z)| \le M + \varepsilon$.

$z \in U \cap D$.

$z_{n_k} \in U \cap D$ $\quad |f(z_{n_k})| \le M$

k: large.

$\sup_{z \in D} |f(z)|$ is bounded. $\quad p \in D$ ✗

$\exists \{z_n\} \subseteq D$ s.t. $\quad \therefore p \in \partial D$. $\forall \varepsilon > 0$, $\exists U$

$|f(z_n)| > n$.

subseq. $\{z_{n_k}\}$ s.t.

$z_{n_k} \to P \in \overline{D}$.

s.t. $|f(z)| \le M + \varepsilon$.

$z \in U \cap D$.

$z_{n_k} \in U \cap D$ $\quad |f(z_{n_k})| \le M + \varepsilon$

k: large.

Set

$\sup_{z \in D} |f(z)| = C < \infty$

$\therefore \exists \{z_n\} \subseteq D$, s.t.

$|f(z_n)| \to C$.

$\exists$ subseq $\{z_{n_k}\}$ s.t.

$z_{n_k} \to p \in \overline{D}$

① $p \in D$. $\therefore |f(p)| = C$.

$|f(z)| \equiv C \le M$.

② $p \in \partial D$. $\forall \varepsilon$. $\exists U$.

$|f(z)| \le M + \varepsilon$. $\quad z \in U \cap D$

$\therefore C \le M + \varepsilon$,

$\therefore C \le M$

$|f(z_{n_k})| \le M + \varepsilon$

...ounded domain

$M>0$. s.t.

$|f(z)| \le M$.

...D

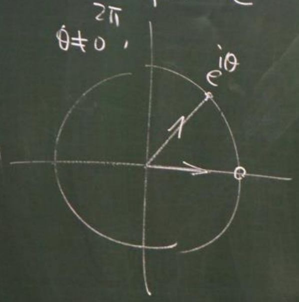
...D)

...z)$| \le M$.

Ex.

$$f(z) = e^{\frac{1+z}{1-z}} \quad \text{on } \mathbb{D}$$

$\theta \neq 0, 2\pi$.



$$f(x) = e^{\frac{1+x}{1-x}}$$

$f(x) \to +\infty$ as $x \to 1^-$

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} \xrightarrow{r \to 1^-}$$

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} \xrightarrow{r \to 1^-} \frac{1+e^{i\theta}}{1-e^{i\theta}} = \frac{(1+e^{i\theta})(1-e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}$$

$\sin\theta = \sin 2($

$$\left| e^{i\cot\frac{\theta}{2}} \right| = 1$$

$$= \frac{\cancel{1}+e^{i\theta}-e^{-i\theta}-\cancel{1}}{|1-e^{i\theta}|^2} = \frac{e^{i\theta}-e^{-i\theta}}{4\left|\frac{e^{-\frac{1}{2}\theta}-e^{\frac{1}{2}\theta}}{2i}\right|^2} = \frac{2i\sin\theta}{4\sin^2\frac{\theta}{2}} = \frac{\cancel{4}}{\cancel{4}}$$

$\theta \neq 0, 2\pi$

$$\frac{1+e^{i\theta}}{1-e^{i\theta}} = \frac{(1+e^{i\theta})(1-e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}$$

$$\sin\theta = \sin 2\left(\frac{\theta}{2}\right)$$

$$= \frac{\cancel{1}+e^{i\theta}-e^{-i\theta}-\cancel{1}}{|1-e^{i\theta}|^2} = \frac{e^{i\theta}-e^{-i\theta}}{4\left|\frac{e^{-\frac{1}{2}\theta}-e^{\frac{1}{2}\theta}}{2i}\right|^2} = \frac{2i\sin\theta}{4\sin^2\frac{\theta}{2}} = \frac{\cancel{4}\, i\, \cancel{\sin\frac{\theta}{2}}\cos\frac{\theta}{2}}{\cancel{4}\, \sin^{\cancel{2}}\frac{\theta}{2}} = i\cot\frac{\theta}{2}$$

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} \xrightarrow{r\to 1} \frac{1+e^{i\theta}}{1-e^{i\theta}} = \frac{(1+e^{i\theta})(1-e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}$$

$\sin\theta = \sin 2\left(\frac{\theta}{2}\right)$

$$= \frac{\cancel{1}+e^{i\theta}-e^{-i\theta}-\cancel{1}}{|1-e^{i\theta}|^2} = \frac{e^{i\theta}-e^{-i\theta}}{4\left|\frac{e^{-\frac{1}{2}i\theta}-e^{\frac{1}{2}i\theta}}{2i}\right|^2} = \frac{2i\sin\theta}{4\sin^2\frac{\theta}{2}} = \frac{\cancel{4}\,i\,\cancel{\sin\frac{\theta}{2}}\cos\frac{\theta}{2}}{\cancel{4}\sin^{\cancel{2}}\frac{\theta}{2}} = i\cot\frac{\theta}{2}$$

$$\left|e^{i\cot\frac{\theta}{2}}\right| = 1$$

$\theta \neq 0, 2\pi$

$f(z) = ic \quad c\in\mathbb{R} \quad |f(z)| = |c|$

Ex.

$f(z) = i\log(z+1)$ on $\mathbb{H}$

$$= i\left(\ln|z+1| + i\arg(z+1)\right)$$

$$= -\arg(z+1) + i\ln|z+1|$$

$z+1 = z-(-1)$