【10920 程守慶教授複變數函數論 / 第 12 堂版書】

Rouché

Thm. (Hurwitz)

$D \subseteq \mathbb{C}$ domain.

$f_n \in \mathcal{O}(D)$, $n \in \mathbb{N}$

$f_n \longrightarrow f$ u.c.c

$f_n(z) \neq 0$ $\forall z \in D$, $n \in \mathbb{N}$

$\Rightarrow$ Either $f(z) \neq 0$ $\forall z \in D$

or $f(z) \equiv 0$ on D.

$\forall z \in D$

on D.

pf. $f \in \mathcal{O}(D)$

If f is not a constant function.

and $Z_f \neq \emptyset$. say $z_0 \in Z_f$.

$\therefore \exists\ \delta > 0$ s.t. $\overline{B}(z_0;\delta) \subseteq D$.

and $f(z) \neq 0$ on $\overline{B}(z_0;\delta)$ except $z = z_0$.

$$\therefore\ 1 \leq N_f = \frac{1}{2\pi i}\int_{\partial B} \frac{f'(z)}{f(z)}\,dz = \lim_{n\to\infty} \underbrace{\frac{1}{2\pi i}\int_{\partial B} \frac{f_n'(z)}{f_n(z)}\,dz}_{=N_{f_n}} \geq 1$$

Thm $D \subseteq \mathbb{C}$ domain

$f_n \in \mathcal{O}(D)$ and f_n is 1-1 $\forall n$

$f_n \longrightarrow f$ u.c.c.

$\Rightarrow$ Either f is also 1-1

or $f(z)$ is a constant function.

pf. Assume f

$N_{f_n} = 0$

$\frac{1}{2\pi i}\int_{\partial B} \frac{f_n'(z)}{f_n(z)}\,dz \geq 1$

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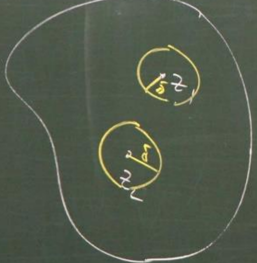
pf. Assume f is not a constant function. ; $1 \leq N_{f-\alpha} = \frac{1}{2\pi i}$

and $\exists\ z_1 \neq z_2 \in D$ s.t. $f(z_1) = f(z_2) = \alpha$.

$\exists\ \delta > 0$ s.t.

$\overline{B}(z_1;\delta) \cap \overline{B}(z_2;\delta) = \phi$.



$f(z) \neq \alpha$. $z \in \overline{B}(z_1;\delta) \setminus \{z_1\}$.

$\overline{B}(z_2;\delta) \setminus \{z_2\}$

$$1 \leq N_{f-\alpha} = \frac{1}{2\pi i}\int_{\partial B(z_1;\delta)} \frac{f'(z)}{f(z)-\alpha}\,dz = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial B(z_1;\delta)} \frac{f_n'(z)}{f_n(z)-\alpha}\,dz$$

$$= \lim_{n\to\infty} N_{f_n-\alpha}$$

$\therefore\ N_{f_n-\alpha} \geq 1$. if n is suff. large

$\Downarrow$

$f_n(z_n) = \alpha$. some $z_n \in B(z_1;\delta)$

then $f_n(z_n') = \alpha$. some z_n'

$\Rightarrow f_n$ is not 1-

※

$$\ldots_{;\delta)} \frac{f'(z)}{f(z)-\alpha}\,dz = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{\partial B(z_1;\delta)}\frac{f_n'(z)}{f_n(z)-\alpha}\,dz$$

$$= \lim_{n\to\infty} N_{f_n-\alpha}$$

$\therefore N_{f_n-\alpha} \geq 1$. if n is suff. large

$\Downarrow$

$f_n(z_n)=\alpha$. some $z_n\in B(z_1;\delta)$

$\Nearrow$ then $f_n(z_n')=\alpha$. some $z_n'\in B(z_2;\delta)$

$\Rightarrow$ f_n is not 1−1 n: large.

$\times$

$$;\quad 1\leq N_{f-\alpha} = \frac{1}{2\pi i}\int_{\partial B(z_1;\delta)}\frac{f'(z)}{f(z)-\alpha}\,dz = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{\partial B(z_1;\delta)}\frac{f_n'(z)}{f_n(z)-\alpha}\,dz$$

$$= \lim_{n\to\infty} N_{f_n-\alpha}$$

$= \phi$.

$\delta)\setminus\{z_1\}$

$\delta)\setminus\{z_2\}$

$\therefore N_{f_n-\alpha} \geq 1$. if n is suff. large

$\Downarrow$

$f_n(z_n)=\alpha$. some $z_n\in B(z_1;\delta)$

$\Nearrow$ then $f_n(z_n')=\alpha$. some $z_n'\in B(z_2;\delta)$

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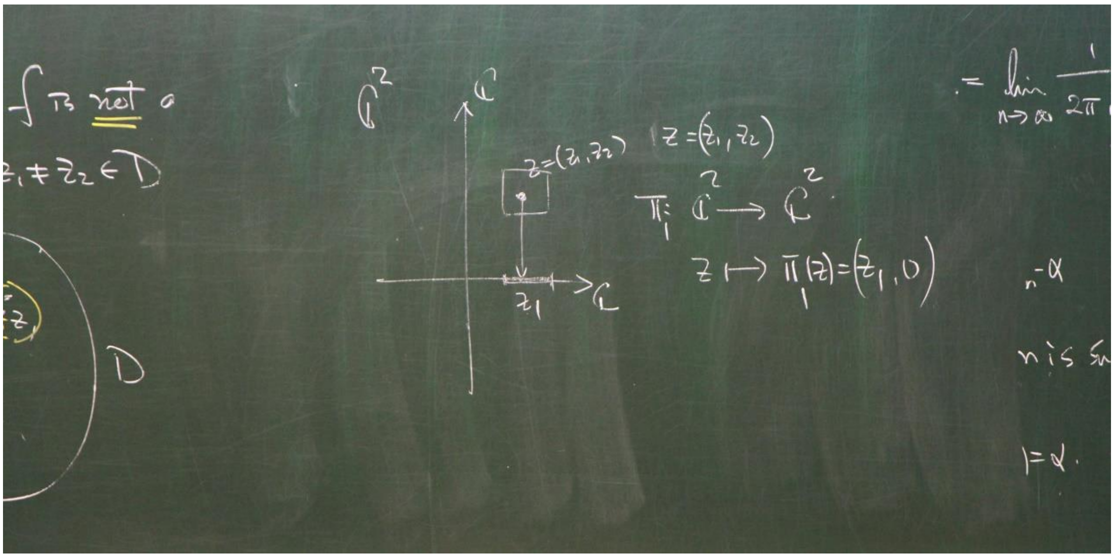
$\times$

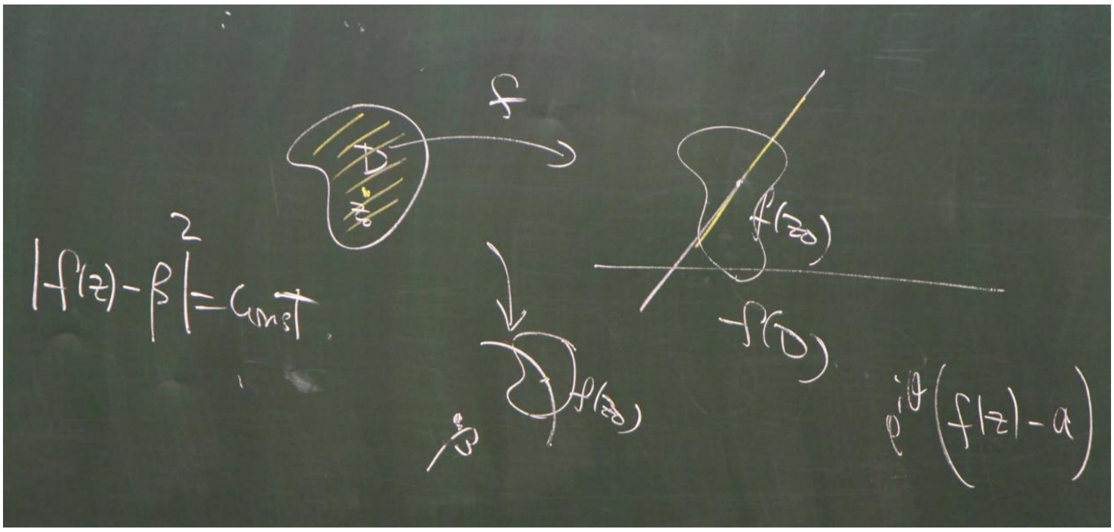
Thm (Open mapping theorem).

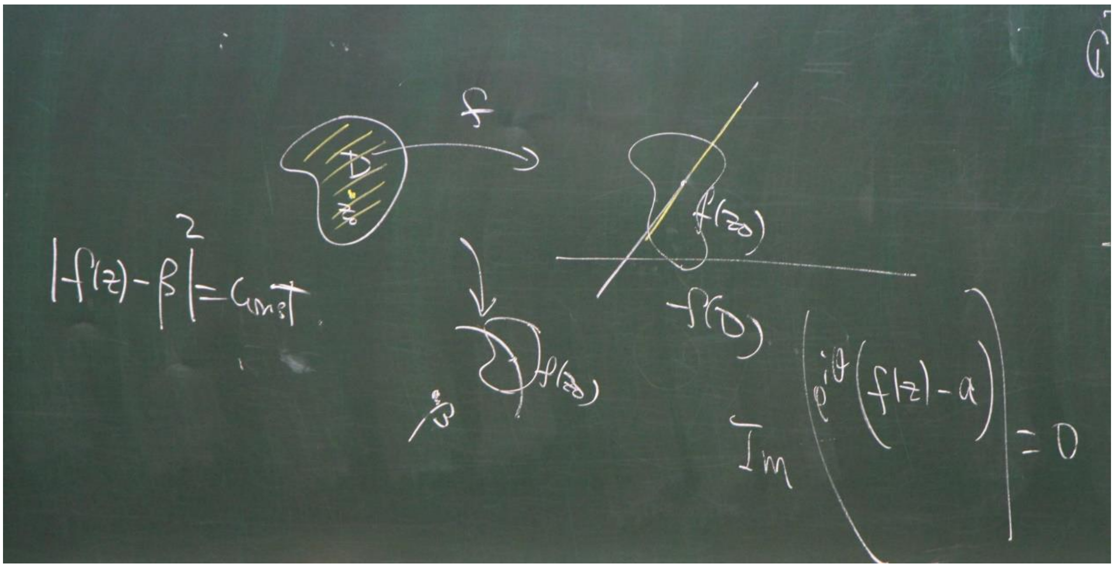
$D\subseteq\mathbb{C}$ domain.

$f\in\mathcal{O}(D)$. not a constant function.

$\Rightarrow$ f is an open mapping.


is not a
$z_1 \neq z_2 \in D$
z_1
D
$\mathbb{C}^2$
$\mathbb{C}$
$z=(z_1,z_2)$
$z=(z_1,z_2)$
$\pi_1: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$
$z \longmapsto \pi_1(z)=(z_1,0)$
z_1
$\mathbb{C}$
$= \lim_{n\to\infty} \frac{1}{2\pi}$
n is sm
$=\alpha$


f
D
z_0
$|f(z)-\beta|^2 = \text{const}$
$f(z_0)$
$f(D)$
$f(z_0)$
β
$e^{i\theta}(f(z)-a)$


f
D
z_0
$|f(z)-\beta|^2 = \text{const}$
$f(z_0)$
$f(D)$
$f(z_0)$
β
$\mathrm{Im}\left(e^{i\theta}(f(z)-a)\right) = 0$

Thm (Open mapping theorem).

$D \subseteq \mathbb{C}$ domain.

$f \in \mathcal{O}(D)$. not a constant function.

$\Rightarrow$ f is an open mapping.

i.e. $V \subseteq D$ open.

show $f(V)$ is open.

pf. let $z_0 \in D$

$g(z) = f(z) - f(z_0)$

$\therefore \exists\, \eta > 0$.

$\bar{B}(z_0;$

$f(D)$

w-space.

$(z) \neq 0$. on $\bar{B}(z_0;\eta)$

ept $z = z_0$

Assume.

$$\sup_{|z-z_0|=\eta} |g(z)| = 2\delta > 0$$

<u>claim:</u> $B(f(z_0);\delta) \subseteq f(V)$

$|w - f(z_0)| < \delta$

show $\exists\, z \in B(z_0;\eta)$ s.t. $f(z) = w$

$f(z) - w = 0$

$f(z) - w$

...orem)

...tant function

...g.

pf. let $z_0 \in V$. $f(z_0) \in f(V)$

$g(z)=f(z)-f(z_0)$ $\quad g(z_0)=0$

$\therefore\ \exists\ \eta>0$. s.t.

$\bar{B}(z_0;\eta)\subseteq V$ and. $g(z)\neq 0$. on $\bar{B}(z_0;\eta)$ except $z=z_0$

Assume.

$$\inf_{|z-z_0|=\eta} |g(z)| = 2\delta > 0$$

claim: $B(f(z_0);\delta)\subseteq f(V)$

$|w-f(z_0)|<\delta$

show $\exists\ z\in B(z_0;\eta)$ s.t. $f(z)=w$

$f(z)-w=0$

$g(z)\neq 0$. on $\bar{B}(z_0;\eta)$ except $z=z_0$

$$f(z)-w = \underbrace{f(z)-f(z_0)}-\underbrace{(w-f(z_0))}$$

$$|w-f(z_0)|<\delta<2\delta\le \underset{|z-z_0|=\eta}{|f(z)-f(z_0)|}$$

$$\therefore\ N_{f(z)-w} = N_{f(z)-f(z_0)} \ge 1$$

O.K.

$f(z_0);\delta)\subseteq f(V)$

s.t. $f(z)=w$

$f(z)-w=0$

Ex.

$f(x)=x$...

$$f(z)-f(z_0)-\big(w-f(z_0)\big)$$

$$\ldots)| < \delta < 2\delta \le \min_{|z-z_0|=\eta} |f(z)-f(z_0)|$$

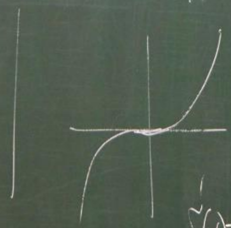
$$= N \ge 1$$

$$f(z)-f(z_0)$$

O.K.

Ex.

$f(x)=x^3$. $x\in\mathbb{R}$



1-1.

$f'(x)=3x^2 \Rightarrow f'(0)=0$

Thm: $D\subseteq\mathbb{C}$

$f\in\mathcal{O}(D)$

$\Rightarrow f'(z)\neq 0$.

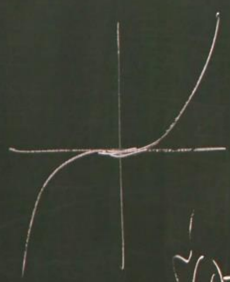
Thm: $D\subseteq\mathbb{C}$ domain.

$f\in\mathcal{O}(D)$ and 1-1.

$\Rightarrow f'(z)\neq 0$. $\forall\, z\in D$

Ex.

$f(x)=x^3$. $x\in\mathbb{R}$



1-1.

$f'(x)=3x^2 \Rightarrow f'(0)=0$

$$f: D \longrightarrow \overset{\text{domain}}{f(D)} \quad \text{open connected}$$

$$\longleftarrow \quad ; f^{-1}$$

$\therefore$ D, $f(D)$ biholomorphic

f. biholomorphism.

$$\lim_{w\to w_0}\frac{f^{-1}(w)-f^{-1}(w_0)}{w-w_0}=\lim_{z\to z_0}\frac{z-z_0}{f(z)-f(z_0)}=\lim_{z\to z_0}\frac{1}{\frac{f(z)-f(z_0)}{z-z_0}}=\frac{1}{f'(z_0)}$$

domain

$f : D \longrightarrow f(D)$ open connected

$\longleftarrow$; f^{-1}

$\therefore$ D, $f(D)$ biholomorphic

f. biholomorphism.

$$\lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \lim_{z \to z_0} \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}} = \frac{1}{f'(z_0)}$$

$) - f(z_0))$

$|f(z) - f(z_0)|$

$|z - z_0| = \eta$

$|$

O.K,

locally.

holomorphic function

is a m-to-1 covering map

$g(z) = z^m$, near 0

Thm: $D \subseteq \mathbb{C}$ domain

$f \in \mathcal{O}(D)$ and 1-1.

$\Rightarrow f'(z) \neq 0$, $\forall z \in D$

Pf. let $z_0 \in D$.

Assume $z_0 = 0$. $\quad f(z_0) = 0$

Write near 0:

$$f(z) = a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots$$

$a_1 = f'(0)$.

$f(z_0) = 0$

$a_2 z^2 + \cdots + a_k z^k + \cdots$

Assume. $a_1 = f'(0) = 0$

let $m \geq 2$ be the multiplicity of 0.

$\therefore f(z) = z^m \varphi(z)$. near 0

and $\varphi(0) \neq 0$

$\therefore \exists\, \eta > 0$. s.t.

$$\begin{cases} f(z) \neq 0 & \text{on } \overline{B}(0;\eta). \text{ except } z = 0 \\ f'(z) \neq 0 & \text{〃} \quad \text{〃} \\ \varphi(z) \neq 0 & \text{on } \overline{B}(0;\eta) \end{cases}$$

let

$\inf\limits_{|z|=\eta} |f(z$

If $|\omega| < \delta$

$f(z)$

On $|z| = \eta$.

$|\omega| < \delta <$

let

$$\inf_{|z|=\eta} |f(z)| = 2\delta > 0$$

If $|\omega| < \delta$. Consider

$$f(z) - \omega$$

On $|z| = \eta$,

$$|\omega| < \delta < 2\delta \leq \underset{|z|=\eta}{|f(z)|}$$

By Rouché

$$N_{f(z)} = N_{f(z)-\omega} = m \geq 2$$

If $\omega \neq 0$ $\quad f(z) = \omega$

$\uparrow$

$z \neq 0$ Simple root

$\because f'(z) \neq 0 \quad f(z) - \omega = 0$

$\therefore \exists$ m distinct points, mapped to ω

let

$$\inf_{|z|=\eta} |f(z)| = 2\delta > 0$$

If $|\omega| < \delta$. Consider

$$f(z) - \omega$$

On $|z| = \eta$,

$$|\omega| < \delta < 2\delta \leq \underset{|z|=\eta}{|f(z)|}$$

By Rouché

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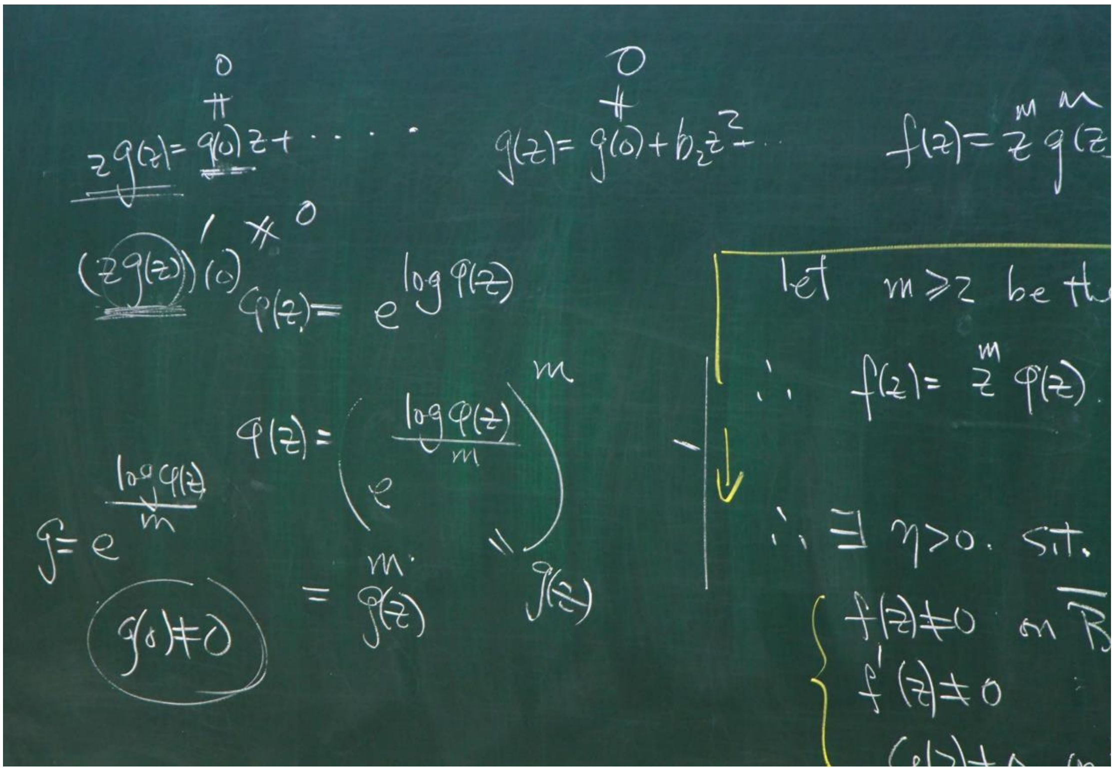
If $\omega \neq 0$ $\quad f(z) = \omega$

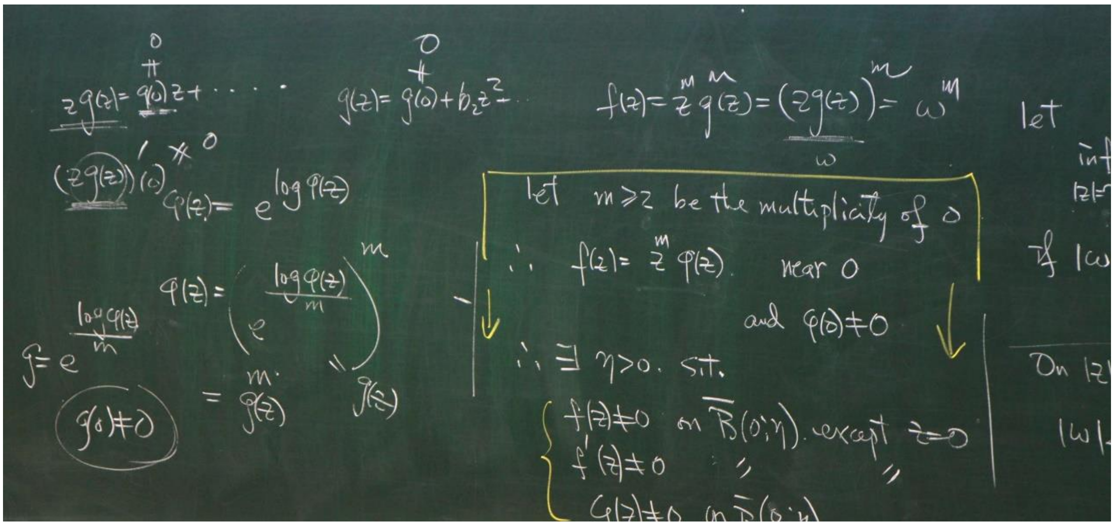
$\uparrow$

$z \neq 0$ Simple root

$\because f'(z) \neq 0 \quad f(z) - \omega = 0$

$\therefore \exists$ m distinct points, mapped to ω


zg(z) = g(0) z + · · · ·
g(z) = g(0) + b₂z² + · · ·
f(z) = z^m g(z)
(zg(z))′(0) ≠ 0
φ(z) = e^{log φ(z)}
φ(z) = (e^{log φ(z) / m})^m = g(z)^m
g = e^{log φ(z) / m}
g(0) ≠ 0
let m ≥ 2 be the
∴ f(z) = z^m φ(z)
∴ ∃ η > 0. s.t.
f(z) ≠ 0 on B̄
f′(z) ≠ 0


zg(z) = g(0) z + · · · ·
g(z) = g(0) + b₂z² + · · ·
f(z) = z^m g(z)^m = (zg(z))^m = ω^m
ω
(zg(z))′(0) ≠ 0
φ(z) = e^{log φ(z)}
φ(z) = (e^{log φ(z) / m})^m = g(z)^m
g = e^{log φ(z) / m}
g(0) ≠ 0
let m ≥ 2 be the multiplicity of 0
∴ f(z) = z^m φ(z) near 0
and φ(0) ≠ 0
∴ ∃ η > 0. s.t.
f(z) ≠ 0 on B̄(0; η) except z = 0
f′(z) ≠ 0 " "
let
If
On

$zg(z) = \overset{\neq 0}{g(0)}\, z + \cdots$

$g(z) = \overset{\neq 0}{g(0)} + b_2 z^2 \cdots$

$f(z) = z^m g^m(z) = (\underbrace{zg(z)}_{\omega})^m = \omega^m$

$(zg(z))'(0) \neq 0$

$\varphi(z) = e^{\log \varphi(z)}$

$\varphi(z) = \left(e^{\frac{\log \varphi(z)}{m}} \right)^m = g^m(z)$

$g = e^{\frac{\log \varphi(z)}{m}}$

$g(0) \neq 0$

let $m \geq 2$ be the multiplicity of 0

∴ $f(z) = z^m \varphi(z)$ near 0

and $\varphi(0) \neq 0$

∴ $\exists\ \eta > 0$ s.t.

$f(z) \neq 0$ on $\overline{B}(0;\eta)$ except $z=0$

$f'(z) \neq 0$ 〃 〃

$\varphi(z) \neq 0$ on $\overline{B}(0;\eta)$

Darboux-Picard.

≥ 2

ω

Simple root

$f(z) - \omega = 0$

its, mapped to ω

$\mathbb{C}$,

$\mathbb{C}^n$, $n \geq 2$

proper mapping

locally.

holomorphic function

is a m-to-1 cove

$g(z) = z^m$, ne