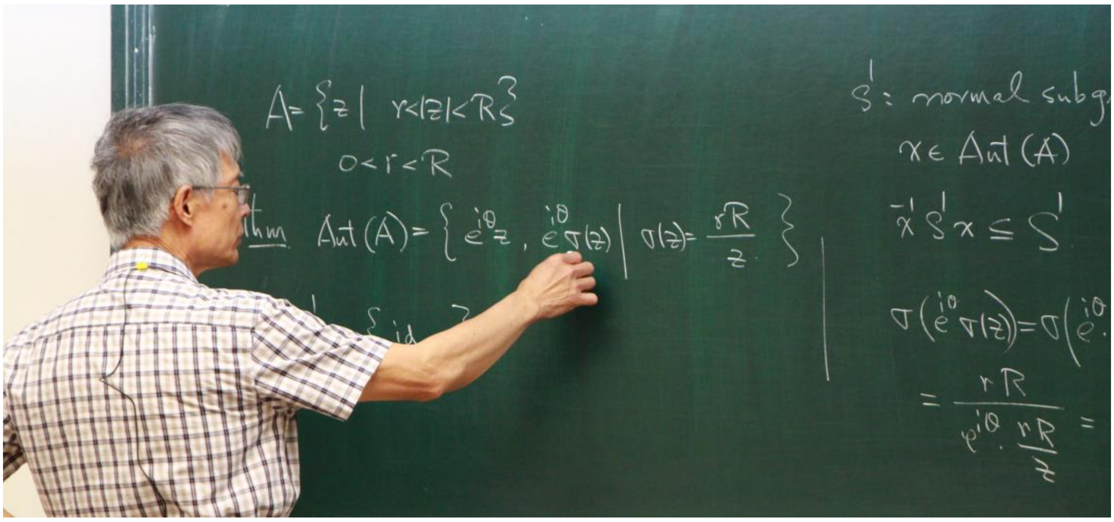
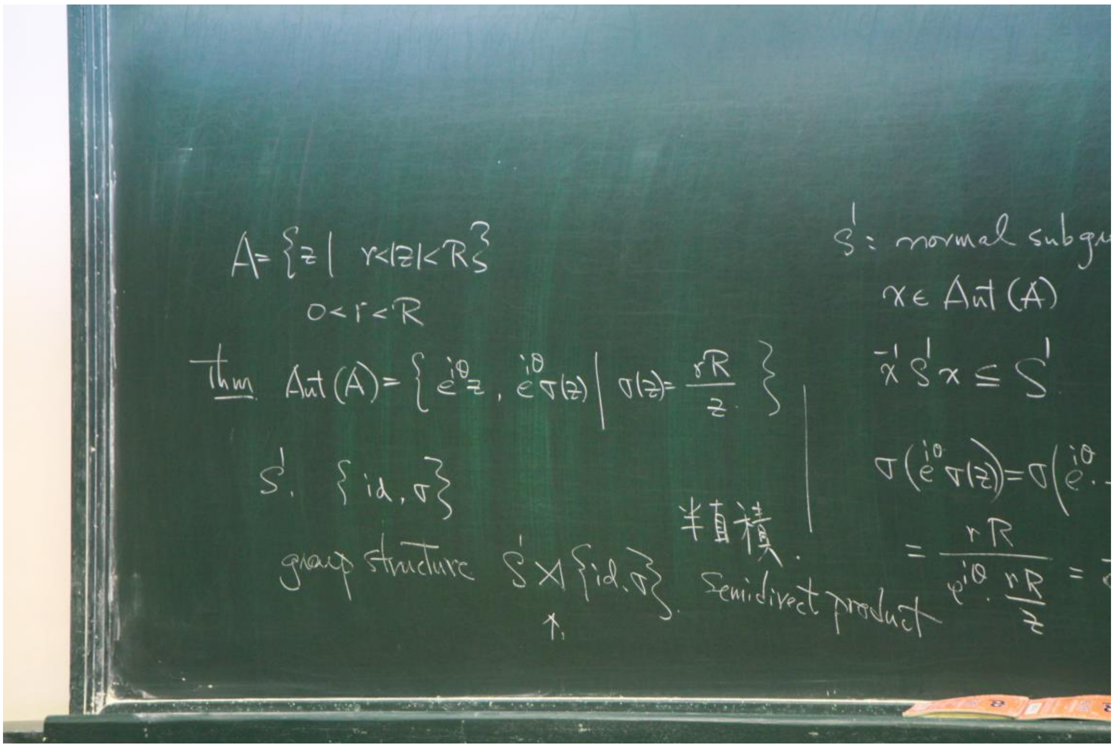
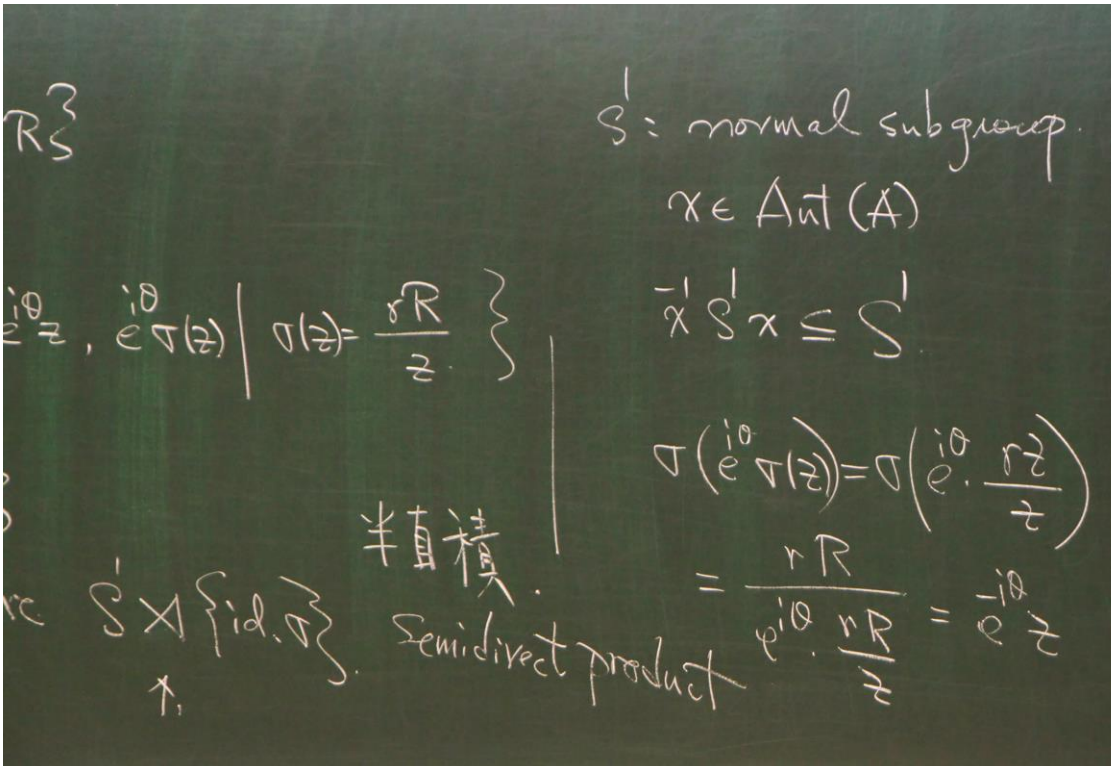
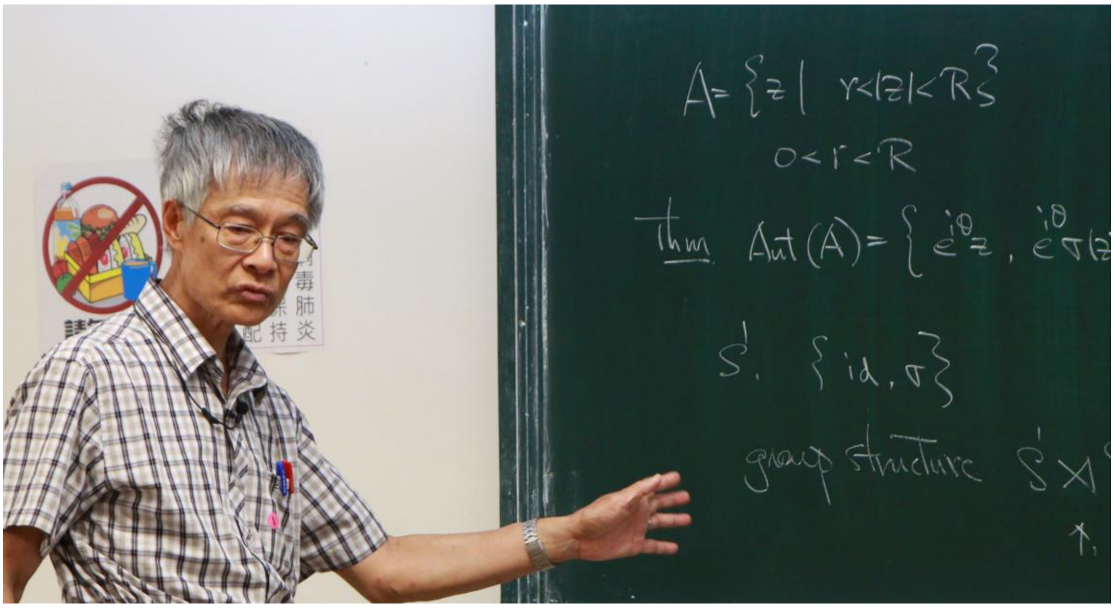
【10920 程守慶教授複變數函數論 / 第 21 堂版書】






S¹: normal subgroup.
x ∈ Aut(A)
x⁻¹ S¹ x ⊆ S¹
σ(e^{iθ} σ(z)) = σ(e^{iθ} · rz/z)
= rR / (e^{iθ} · rR/z) = e^{-iθ} z
σ(z) = rR/z
S¹ ⋊ {id, σ}
半直積
Semidirect product


A = {z | r < |z| < R}
0 < r < R
Thm Aut(A) = {e^{iθ}z, e^{iθ}σ(z)
S¹, {id, σ}
group structure S¹ ⋊

normal subgroup.

$\in \mathrm{Aut}(A)$

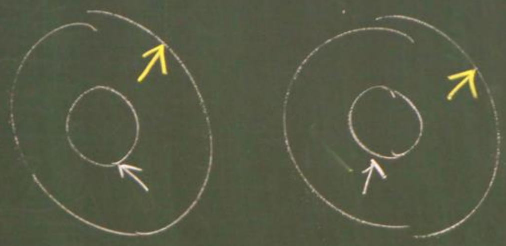
$S^1 x \leq S^1$.

$e^{i\theta}\sigma(z)) = \sigma\left(e^{i\theta} \cdot \frac{rR}{z}\right)$

$= \frac{rR}{e^{i\theta} \cdot \frac{rR}{z}} = e^{-i\theta} z$

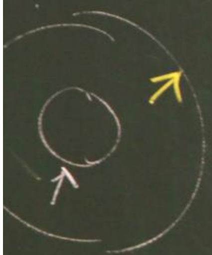
pf. $f : A \to A$.

If necessary, consider $\sigma \circ f$



A.

Consider $\sigma \circ f$



$\log\left|\frac{f(z)}{z}\right|$ harmonic on A.

$|z| \to r \quad |f(z)| \to r$

$|z| \to R \quad |f(z)| \to R$

$\log\left|\frac{f(z)}{z}\right| \equiv 0$ on A

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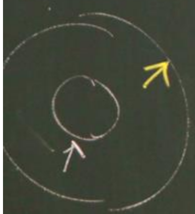

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$\log\left|\frac{f(z)}{z}\right| \equiv 0$ on A

$\lambda = e^{i\theta}$.

i.e., $\left|\frac{f(z)}{z}\right| \equiv 1$ on $A \Rightarrow f(z)=\lambda z.\ |\lambda|=1.$

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$\sigma\circ f(z) = e^{i\theta} z$

$f(z) = \sigma\left(e^{i\theta} z\right)$

$= e^{-i\theta}\sigma(z)$

$\frac{rR}{e^{i\theta} z} = e^{-i\theta}$

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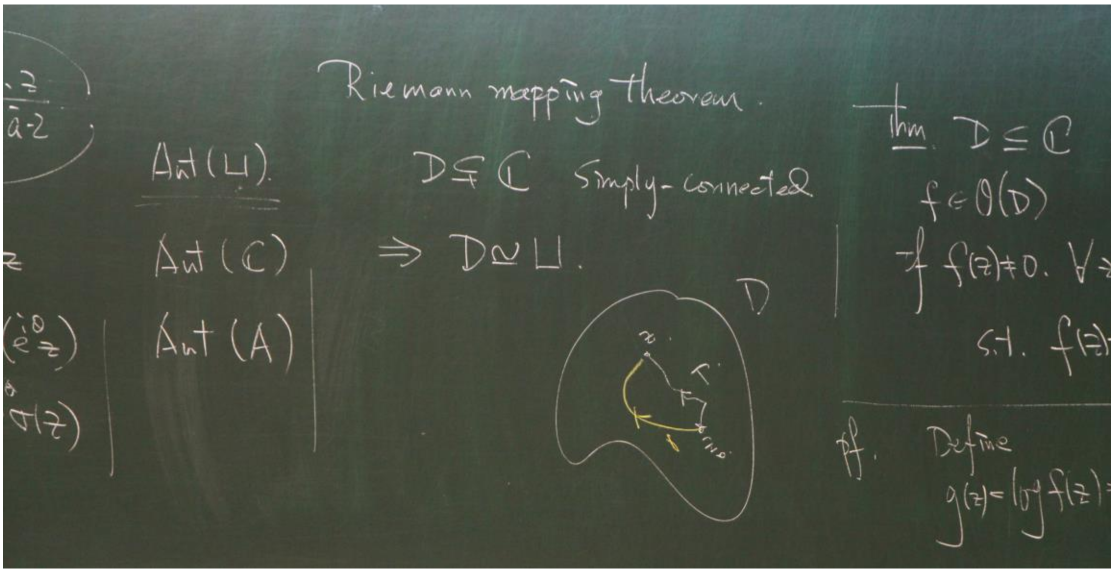
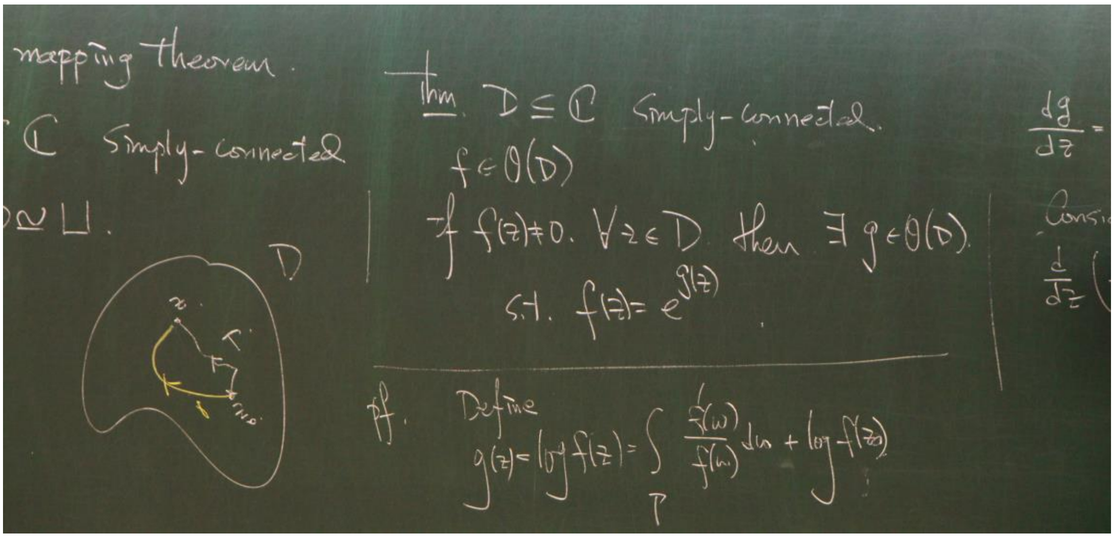
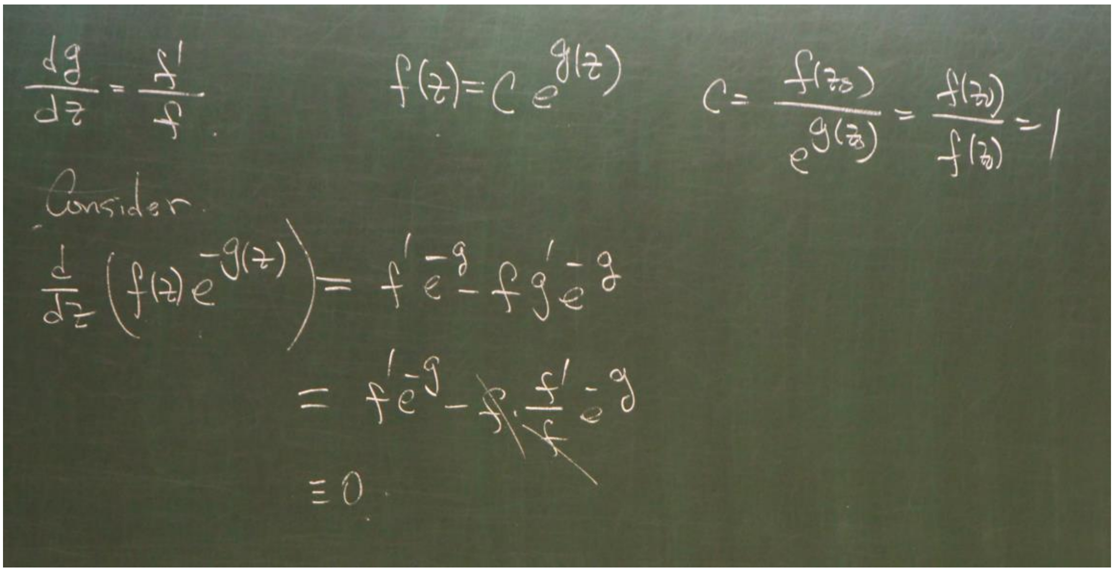
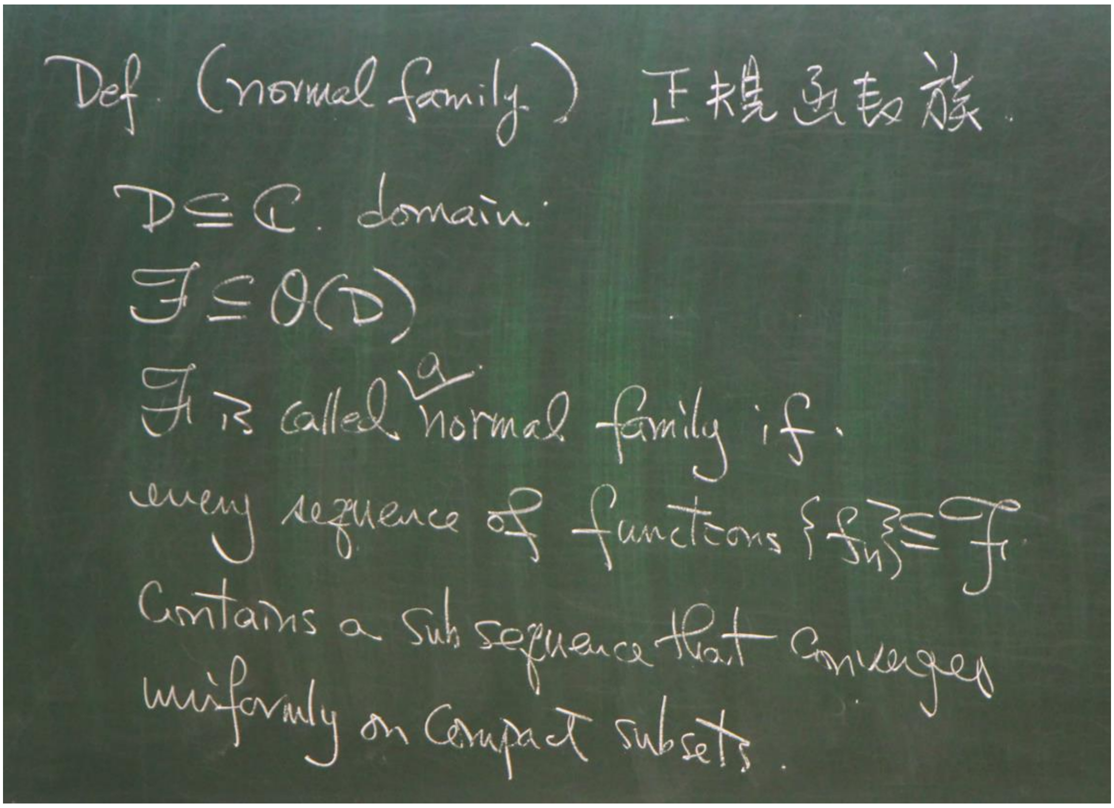

Riemann mapping theorem
$\mathrm{Aut}(\mathbb{U})$
$\mathrm{Aut}(\mathbb{C})$
$\mathrm{Aut}(A)$
$D \subsetneq \mathbb{C}$ Simply-connected
$\Rightarrow D \cong \mathbb{U}$
D
thm. $D \subseteq \mathbb{C}$
$f \in \mathcal{O}(D)$
pf. Define


mapping theorem
thm. $D \subseteq \mathbb{C}$ Simply-connected.
$f \in \mathcal{O}(D)$
If $f(z) \neq 0$. $\forall z \in D$. then $\exists\, g \in \mathcal{O}(D)$.
s.t. $f(z) = e^{g(z)}$
D
pf. Define
$g(z) = \log f(z) = \int_\gamma \frac{f'(w)}{f(w)}\,dw + \log f(z_0)$


$\frac{dg}{dz} = \frac{f'}{f}$
$f(z) = C e^{g(z)}$
$C = \frac{f(z_0)}{e^{g(z_0)}} = \frac{f(z_0)}{f(z_0)} = 1$
Consider
$\frac{d}{dz}\left(f(z) e^{-g(z)}\right) = f' e^{-g} - f g' e^{-g}$
$= f' e^{-g} - \cancel{f} \cdot \frac{f'}{\cancel{f}} e^{-g}$
$\equiv 0$.


Def. (normal family) 正規函數族.
$D \subseteq \mathbb{C}$. domain.
$\mathcal{F} \subseteq \mathcal{O}(D)$.
$\mathcal{F}$ is called a normal family if.
every sequence of functions $\{f_n\} \subseteq \mathcal{F}$
contains a subsequence that converges
uniformly on compact subsets.

the limit function is not required to be in $\mathcal{F}$.

Def. (equicontinuity)

$D \subseteq \mathbb{C}$ domain.

$\mathcal{F} \subseteq C(D)$ 等度連續

$\mathcal{F}$ is said to be equicontinuous

if. given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t

$|f(x) - f(y)| < \varepsilon$ if $|x-y| < \delta$. $f \in \mathcal{F}$.

等度連續

uicontinuous

>0 s.t

$|x-y| < \delta$. $f \in \mathcal{F}$.

Thm (Arzelá - Ascoli).

$\mathcal{F} \subseteq C(D)$

Suppose $\mathcal{F}$ is a normal family.

and equicontinuous.

Then $\mathcal{F}$ is normal.

pf.

pf. let $\{f_n\} \subseteq \mathcal{F}$

$\{z_k\}_{k=1}^{\infty} \subseteq D$

rewrite $f_n = f_{0n}$

$|f_{0n}(z_1)| \le M$.

there exists a subseq. $\{f_{1n}\} \subseteq \{f_{0n}\}$

s.t. ① $f_{1n}(z_1)$ converges

② the order of f_{1n} is preserved $\quad f_{1n} \quad f_{1,n+1}$

(I).

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$|f_{1n}(z_2)| \le M$.

$\exists$ subseq. $\{f_{2n}\} \subseteq \{f_{1n}\}$

s.t ① $f_{2n}(z_2)$ converg

② the order is pre

$|f_{2n}(z_3)| \le M$,

$\vdots$

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$|f_{2n}(z_3)| \leq M$,

$\vdots$

f_{0n}

$$\begin{array}{ccccc} f_{11} & f_{12} & f_{13} & f_{14} & \cdots \\ f_{21} & f_{22} & f_{23} & f_{24} & \cdots \\ f_{31} & f_{32} & f_{33} & f_{34} & \cdots \\ f_{41} & f_{42} & f_{43} & f_{44} & \cdots \\ & \vdots & & & \end{array}$$

$\{f_{kn}\}$ $\quad z_k$

$f_{11} \; f_{22} \; f_{33} \; f_{44} \cdots f_{kk} \cdots$

f_{0n}

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$\{f_{kn}\}$ $\quad z_k$

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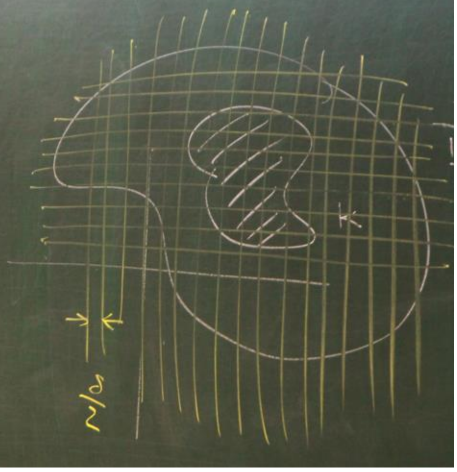
(II) let $K \subseteq D$ compact subset.

Given $\{f_n\} \subseteq \mathcal{F}$.

Given $\varepsilon > 0$, $\exists\, \delta > 0$, s.t.

$|f(x) - f(y)| < \varepsilon$. if $|x-y| < \delta$. $f \in \mathcal{F}$



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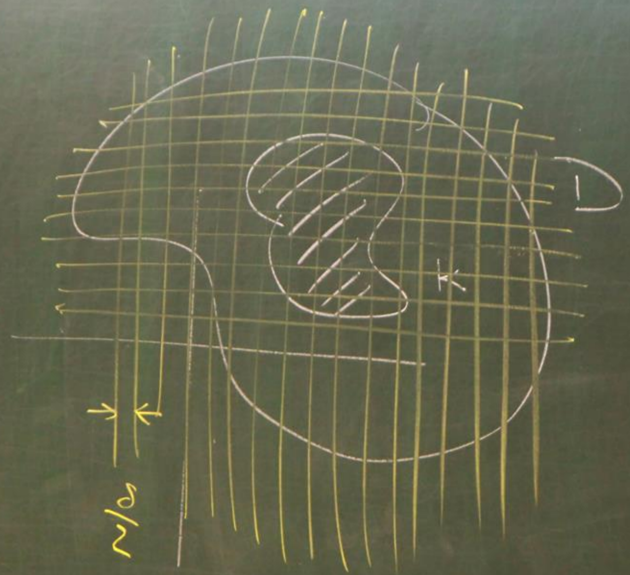

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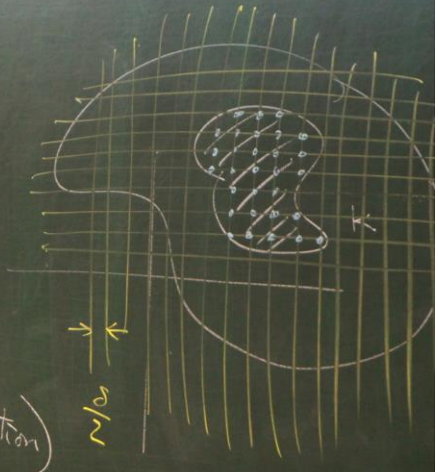
finite lattice points $\in K$.

$z_1, \dots z_m$

By (I), there exists a subseq $\{f_n\}$ (same notation)

s.t. $f_n(z_j)$ converges. $1 \le j \le m$



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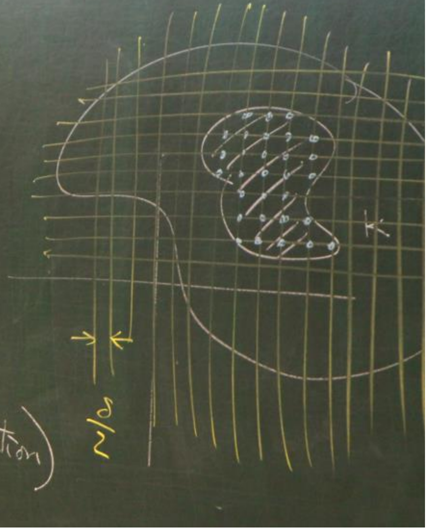

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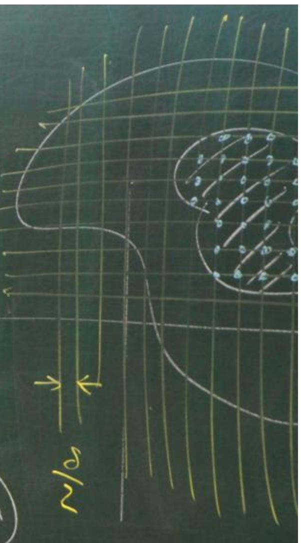


If $z \in K$, then $\exists\, z_j$, $1 \le j \le m$,

s.t. $|z - z_j| \le \sqrt{2} \cdot \frac{\delta}{2} = \frac{\delta}{\sqrt{2}} < \delta$.

$\exists\, n_0$ s.t.

$|f_n(z_k) - f_j(z_k)| < \varepsilon$

for $n, j \ge n_0$, $1 \le k \le m$

for $n, l \ge n_0$

$$|f_n(z) - f_l(z)| \le |f_n(z) - f_n(z_j)| + |f_n(z_j) - f_l(z_j)| + |f_l(z_j) - f_l(z)| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

Thm (Arzelá-Ascoli).

$\mathcal{F} \subseteq C(D)$

Suppose $\mathcal{F}$ is uniformly bounded and equicontinuous.

Then $\mathcal{F}$ is normal.

Pf.

(III). Find a seq. of compact subsets

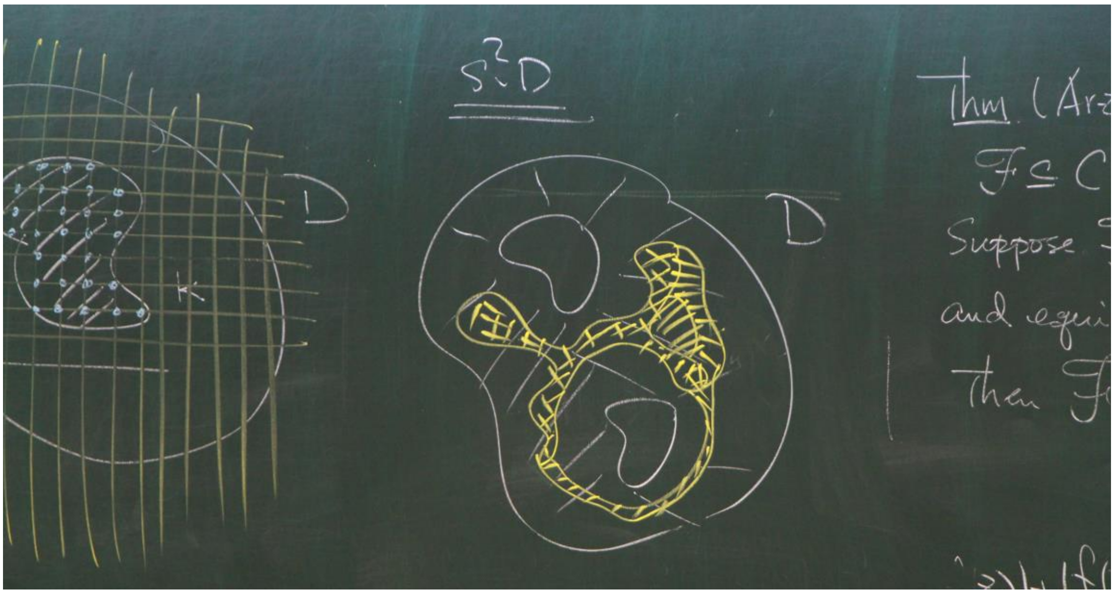
pf. K_n of D s.t.

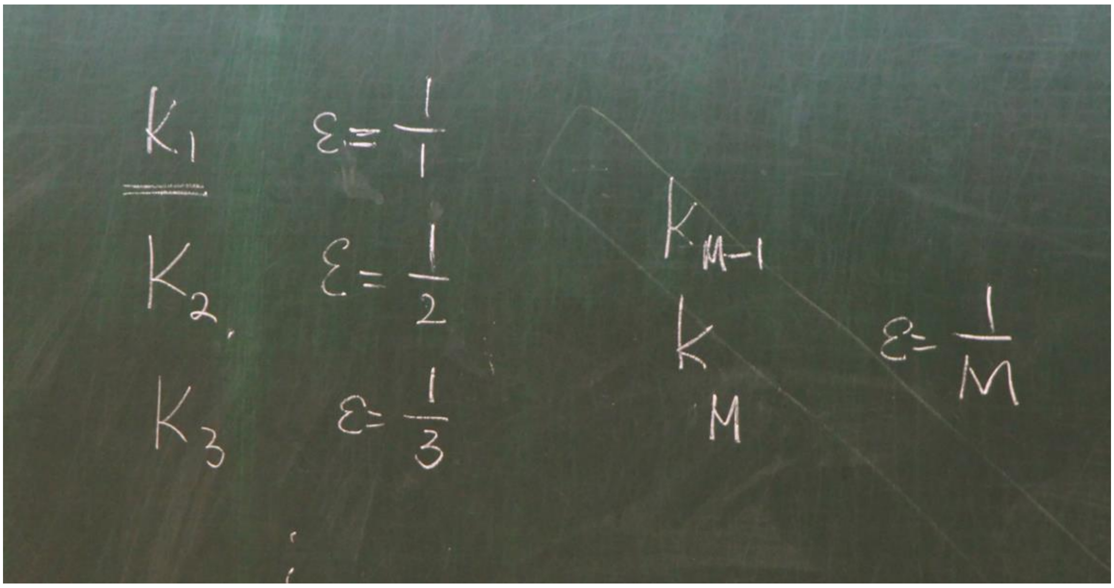
(i) $\bigcup_{n=1}^{\infty} K_n = D$

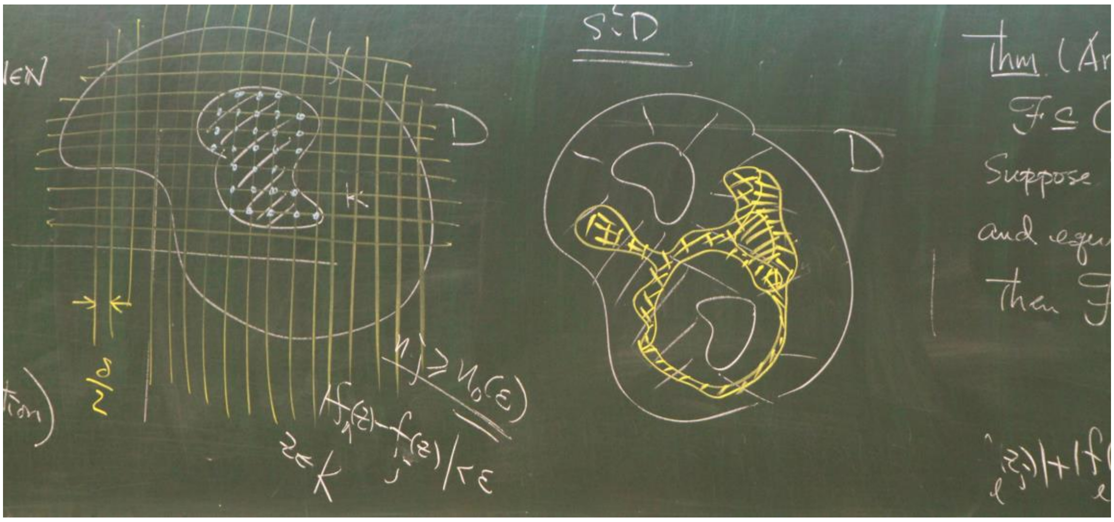
(ii) $K_n \subseteq \text{int } K_{n+1}$

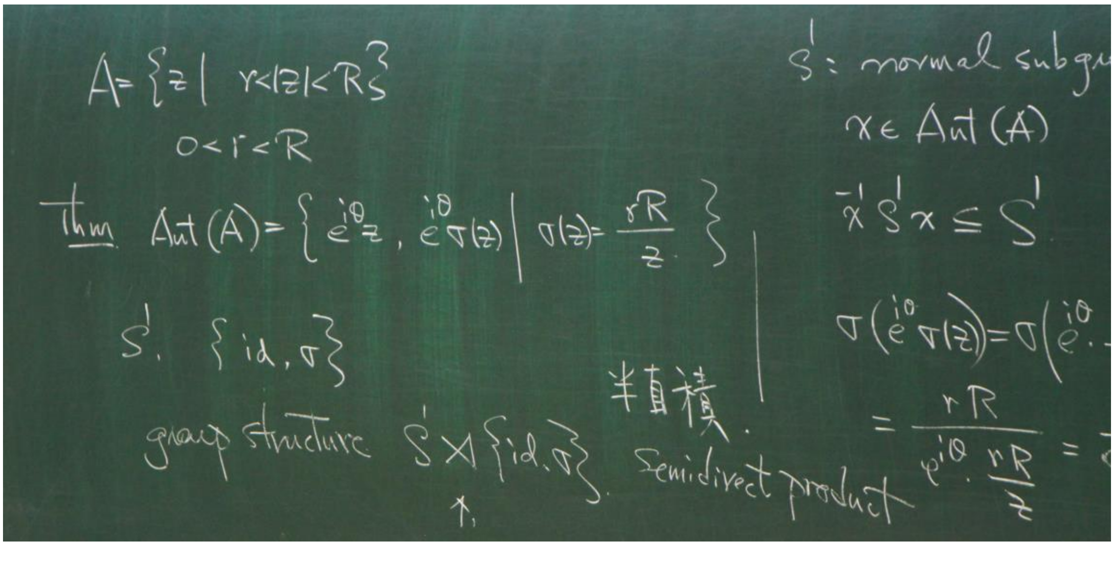
(iii) Any compact subset is contained in one of K_n's.

(iv) Any component of $S^2 \setminus K_n$ contains a component of $S^2 \setminus D$.




K1 ε = 1/1
K2 ε = 1/2
K3 ε = 1/3
K M-1
K M ε = 1/M


D
K
n ≥ n0(ε)
|fn(z) − fj(z)| < ε
z ∈ K
S:D
D
Thm. (Ar
Suppose
and equ
Then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