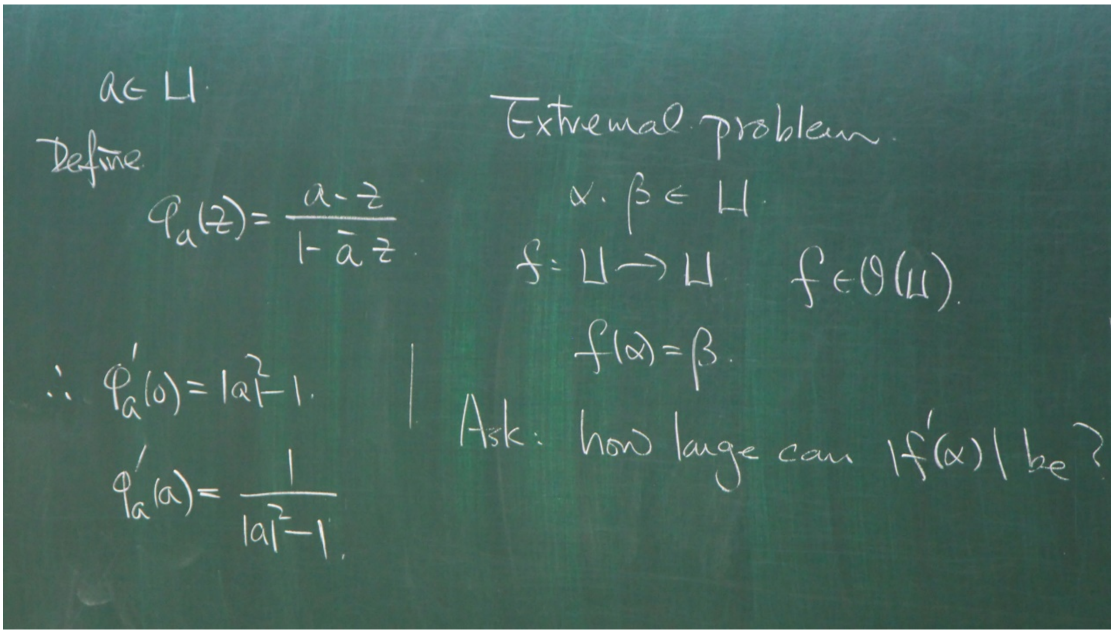
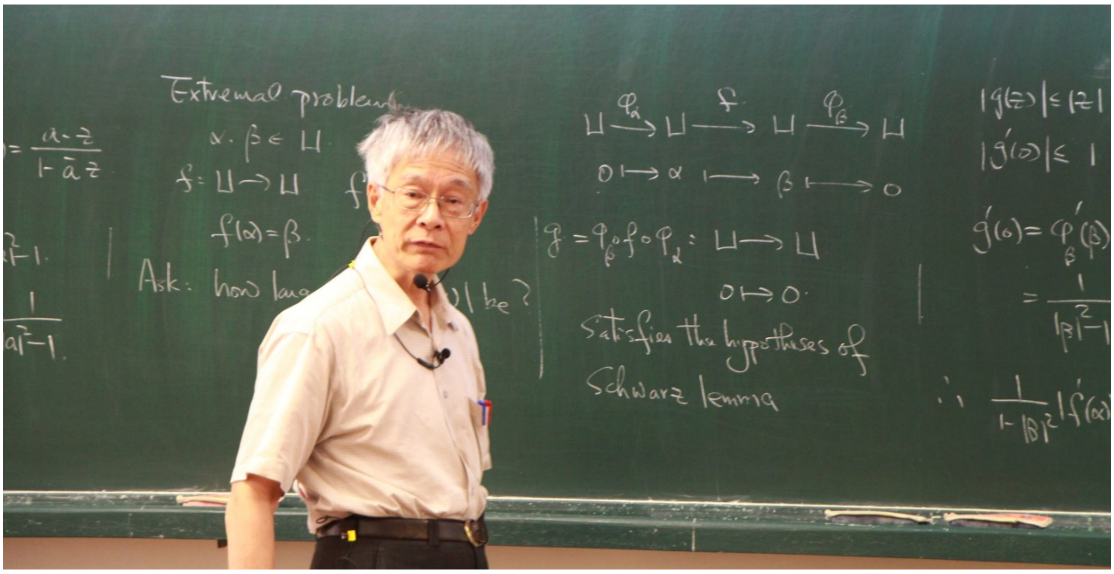
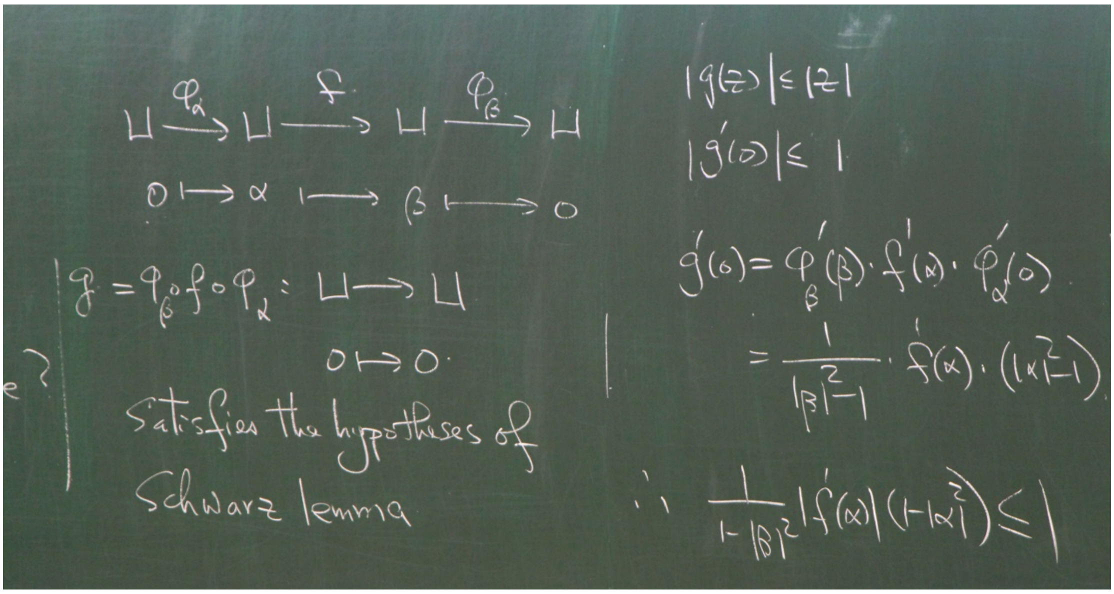
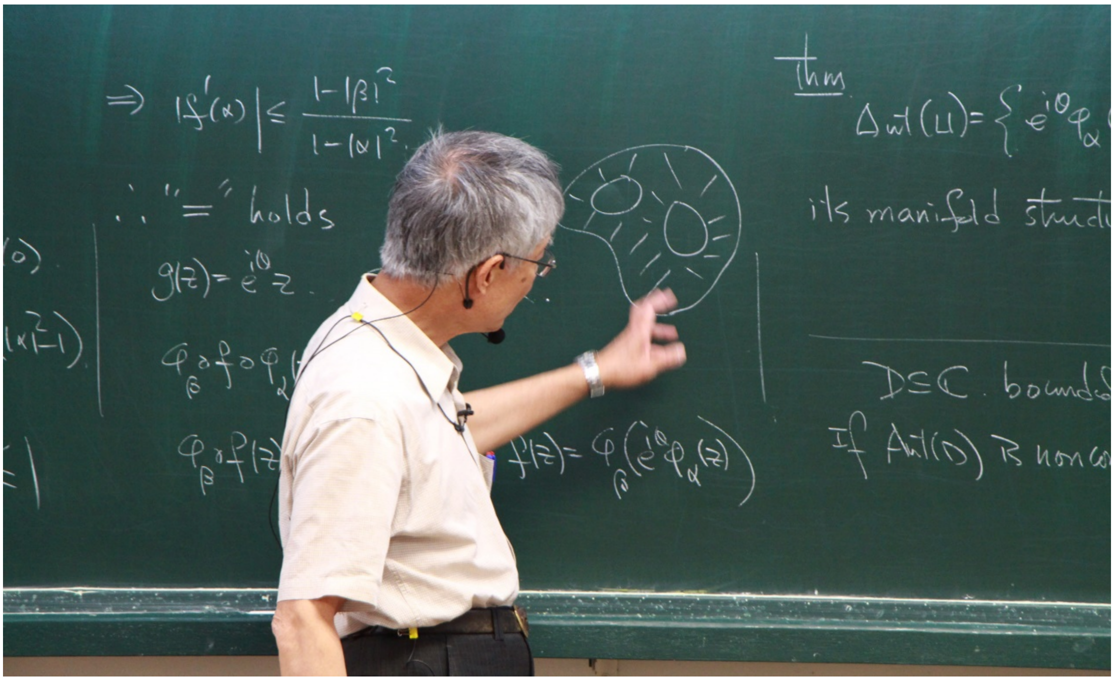
【10920 程守慶教授複變數函數論 / 第 19 堂版書】

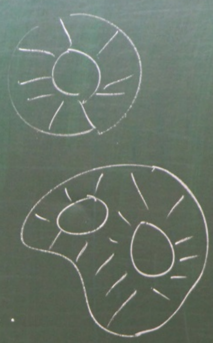





$\mathbb{U} \xrightarrow{\varphi_\alpha} \mathbb{U} \xrightarrow{f} \mathbb{U} \xrightarrow{\varphi_\beta} \mathbb{U}$
$0 \longmapsto \alpha \longmapsto \beta \longmapsto 0$
$g = \varphi_\beta \circ f \circ \varphi_\alpha : \mathbb{U} \longrightarrow \mathbb{U}$
$0 \mapsto 0$.
Satisfies the hypotheses of
Schwarz lemma
$|g(z)| \leq |z|$
$|g'(0)| \leq 1$
$g'(0) = \varphi_\beta'(\beta) \cdot f'(\alpha) \cdot \varphi_\alpha'(0)$
$= \frac{1}{|\beta|^2 - 1} \cdot f'(\alpha) \cdot (|\alpha|^2 - 1)$
$\therefore \frac{1}{1-|\beta|^2} |f'(\alpha)| (1-|\alpha|^2) \leq 1$


$\Rightarrow |f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}$
$\therefore$ "=" holds
$g(z) = e^{i\theta} z$
$f(z) = \varphi_\beta(e^{i\theta} \varphi_\alpha(z))$
thm
its manifold
$D \subseteq \mathbb{C}$ bound

Annulus 環帶



$\Rightarrow |f'(\alpha)| \le \dfrac{1-|\beta|^2}{1-|\alpha|^2}.$

$\therefore$ "=" holds

$g(z) = e^{i\theta} z$. some $\theta \in \mathbb{R}$.

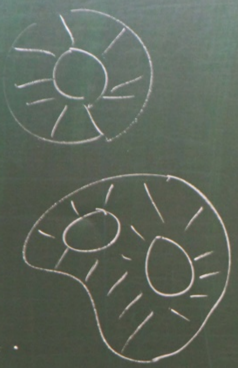
$\varphi_\beta \circ f \circ \varphi_\alpha(z) = e^{i\theta} z$

$\varphi_\beta \circ f(z) = e^{i\theta}\varphi_\alpha(z) \Rightarrow f(z) = \varphi_\beta\left(e^{i\theta}\varphi_\alpha(z)\right)$

$)\cdot f'(\alpha)\cdot \varphi_\alpha'(0)$

$\cdot f'(\alpha)\cdot(|\alpha|^2-1)$

$)|(1-|\alpha|^2) \le 1$



thm

$\mathrm{Aut}(\mathbb{U}) = \left\{ e^{i\theta}\varphi_\alpha(z) \;\middle|\; \theta \in [0,2\pi],\ \alpha \in \mathbb{U} \right\}$

its manifold structure is $S^1 \times \mathbb{U}$.

$\uparrow$ noncompact

$D \subseteq \mathbb{C}$. bounded. $\partial D: C^1$

If $\mathrm{Aut}(D)$ is noncompact $\Rightarrow D \simeq \mathbb{U}$

Pf. $f\in \mathrm{Aut}(\mathbb{U})$ and $f(\alpha)=0$ $\alpha\in\mathbb{U}$

$\therefore\ \exists\ g\in \mathrm{Aut}(\mathbb{U})$ s.t. $g=f^{-1}$

$g\circ f(z)\equiv z$

$\therefore\ g(0)=\alpha$

$g'(0)f'(\alpha)=1$

$|f'(\alpha)|\le \dfrac{1}{1-|\alpha|^2}$ "="

$|g'(0)|\le 1-|\alpha|^2$

"=" holds

$\beta=0$

$f(z)=\varphi_\beta\left(e^{i\theta}\varphi_\alpha(z)\right)$

$= -e^{i\theta}\varphi_\alpha(z)$

$= e^{i(\theta+\pi)}\varphi_\alpha(z)$

$|z|$

≤ 1

$\varphi_\beta'(\beta)\cdot f'(\alpha)\cdot \varphi_\alpha'(0)$

$\dfrac{1}{|\beta|^2-1}\cdot f'(\alpha)\cdot(|\alpha|^2-1)$

$|f'(\alpha)|(1-|\alpha|^2)\le 1$

$\Rightarrow\ |f'(\alpha)|\le \dfrac{1-|\beta|^2}{1-|\alpha|^2}$

"=" holds

$f(z)=\varphi_\beta\left(e^{i\theta}\varphi_\alpha(z)\right)$

(θ_2,β)

$\parallel$

$e^{i\theta_2}\dfrac{\beta-z}{1-\bar\beta z}$

thm

Aut

its ma

(θ_1,α)

$e^{i\theta_1}\dfrac{\alpha-}{1-}$

$|z|$

1

$\varphi_\beta'(\beta)\cdot f'(\alpha)\cdot \varphi_\alpha'(0)$

$\dfrac{1}{|\beta|^2-1}\cdot f'(\alpha)\cdot(|\alpha|^2-1)$

$|f'(\alpha)|(1-|\alpha|^2)\le 1$

$$\Rightarrow\ |f'(\alpha)|\le\frac{1-|\beta|^2}{1-|\alpha|^2}$$

"=" holds

$$f(z)=\varphi_\beta\left(e^{i\theta}\varphi_\alpha(z)\right)$$

$$(\theta_2,\beta) = e^{i\theta_2}\frac{\beta-z}{1-\bar\beta z}$$

thm

$\Delta ut($

its man

(θ_1,α)

$e^{i\theta_1}\dfrac{\alpha-}{1-}$

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thm

$$\mathrm{Aut}(\mathbb{D})=\left\{ e^{i\theta}\varphi_\alpha(z)\ \middle|\ \theta\in[0,2\pi],\ \alpha\in\mathbb{D}\right\}$$

its manifold structure is $S^1\times\mathbb{D}$.

$\uparrow$ noncompact

$$(\theta_2,\beta) = e^{i\theta_2}\frac{\beta-z}{1-\bar\beta z}$$

$$(\theta_1,\alpha)\cdot(\theta_2,\beta)=(\theta_3,\gamma)\qquad \theta_3=\theta_1+\theta_2,\quad \gamma=$$

$$e^{i\theta_1}\frac{\alpha-e^{i\theta_2}\frac{\beta-z}{1-\bar\beta z}}{1-\bar\alpha\left(e^{i\theta_2}\frac{\beta-z}{1-\bar\beta z}\right)}=e^{i\theta_3}\frac{\gamma-z}{1-\bar\gamma z}$$

pf. f

$\therefore\ \exists$

$\therefore$

Thm.

$$\mathrm{Aut}(\mathbb{C}) = \{ a+bz \mid a, b \in \mathbb{C},\ b \neq 0 \}.$$

pf "$\Leftarrow$" $f(z) = a+bz$. $b \neq 0$

$\forall c \in \mathbb{C}$ $\quad a+bz = c \quad z = \dfrac{c-a}{b}$.

"$\Rightarrow$" $f \in \mathrm{Aut}(\mathbb{C})$. $\quad f$: entire 1-1, onto.

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots$$

(i) $f(z)$ is a

of $\deg f$

$f(z) = \omega$

(ii) $a_j \neq 0$ f

$\therefore \infty$ becomes

By Picard's

(i) $f(z)$ is a polynomial of $\deg f \geq 2$

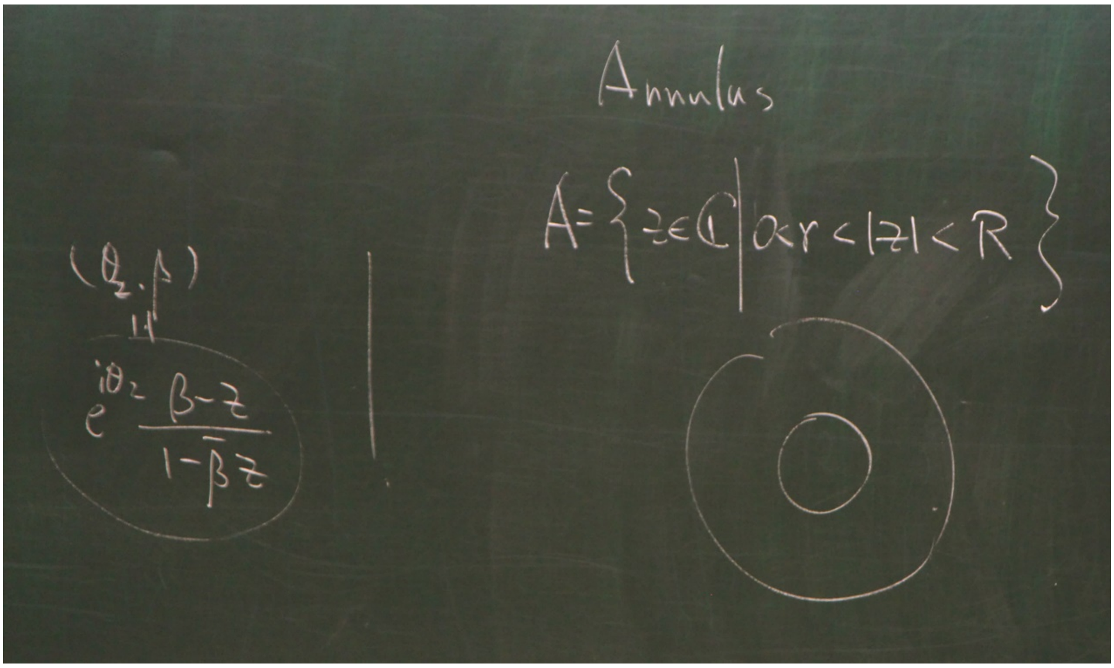
$f(z) = \omega \quad \times$ 1-1

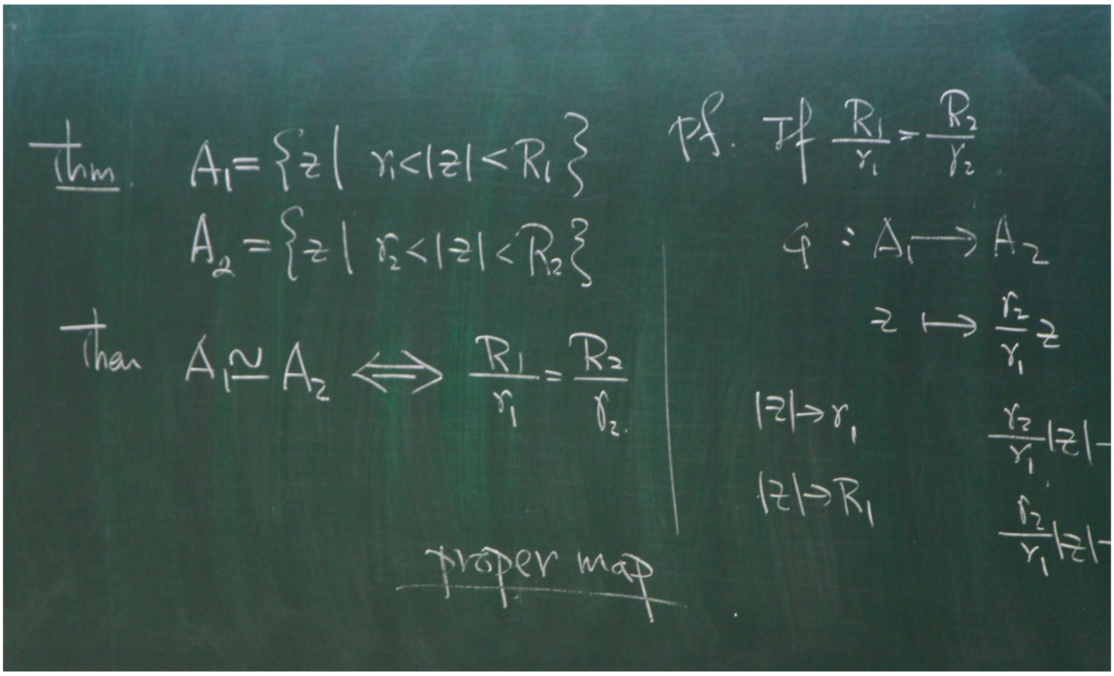
(ii) $a_j \neq 0$ for infinitely many j.

$\therefore \infty$ becomes an essential singularity of f

By Picard's great theorem, $\times$ 1-1

$\therefore f(z) = a+bz$. $b \neq 0$.


Annulus
$A=\{z\in\mathbb{C} \mid 0<r<|z|<R\}$
(θ_2,β)
$e^{i\theta_2}\frac{\beta-z}{1-\bar{\beta}z}$


thm. $A_1=\{z\mid r_1<|z|<R_1\}$
$A_2=\{z\mid r_2<|z|<R_2\}$
Then $A_1\cong A_2 \iff \frac{R_1}{r_1}=\frac{R_2}{r_2}$
Pf. If $\frac{R_1}{r_1}=\frac{R_2}{r_2}$
$\varphi: A_1\to A_2$
$z\mapsto \frac{r_2}{r_1}z$
$|z|\to r_1$
$\frac{r_2}{r_1}|z|\to$
$|z|\to R_1$
$\frac{r_2}{r_1}|z|\to$
~~proper map~~

thm. $A_1 = \{ z \mid r_1 < |z| < R_1 \}$

$A_2 = \{ z \mid r_2 < |z| < R_2 \}$

Then $A_1 \cong A_2 \iff \dfrac{R_1}{r_1} = \dfrac{R_2}{r_2}$.

proper map

pf. If $\dfrac{R_1}{r_1} = \dfrac{R_2}{r_2}$.

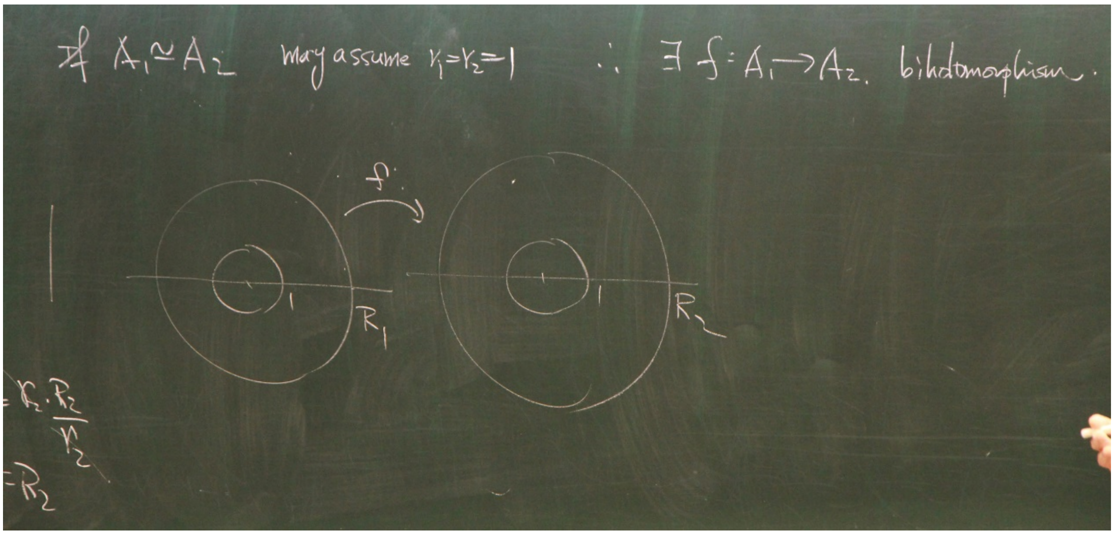
$\varphi : A_1 \longrightarrow A_2$

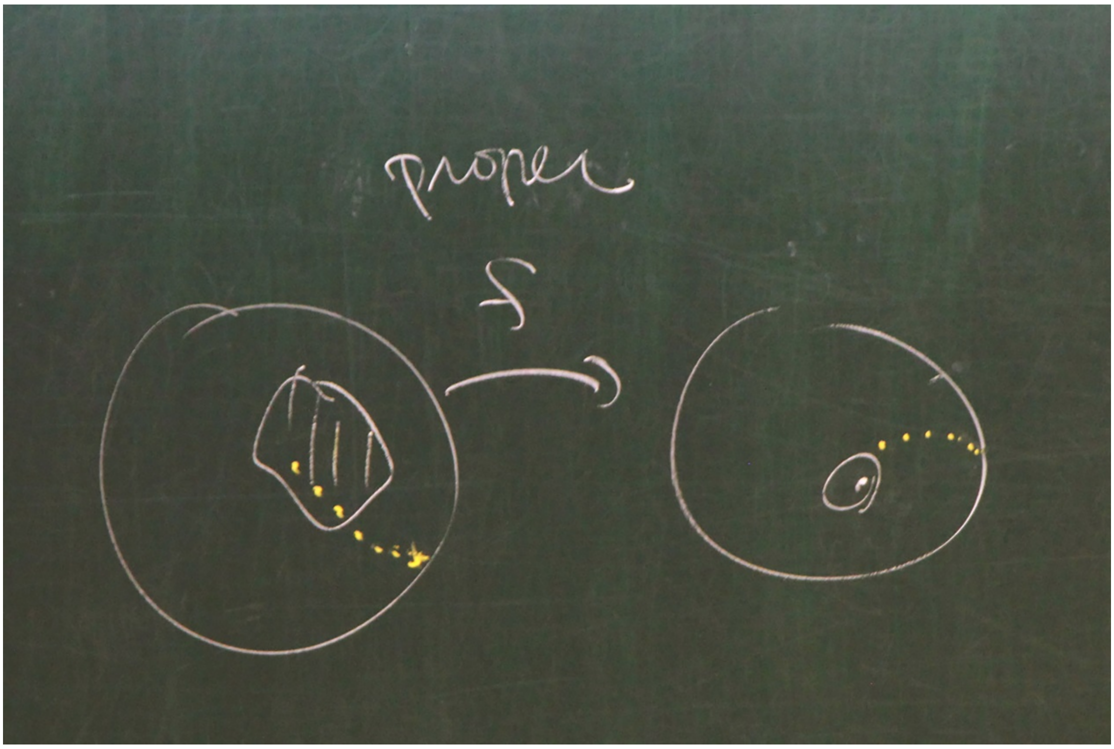
$z \longmapsto \dfrac{r_2}{r_1} z$

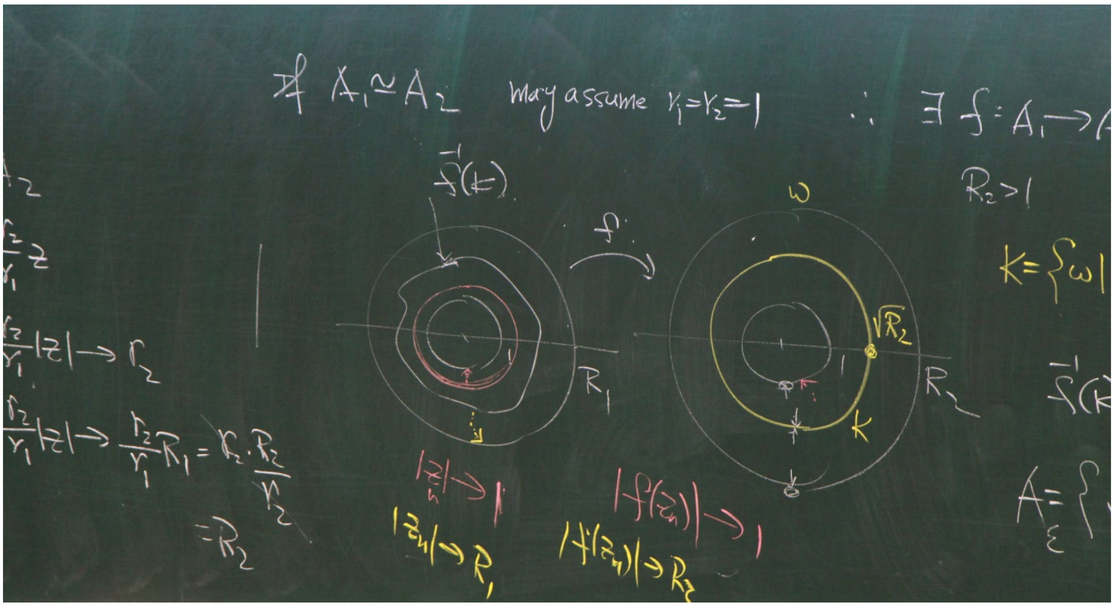
$|z| \to r_1 \qquad \dfrac{r_2}{r_1}|z| \to r_2$

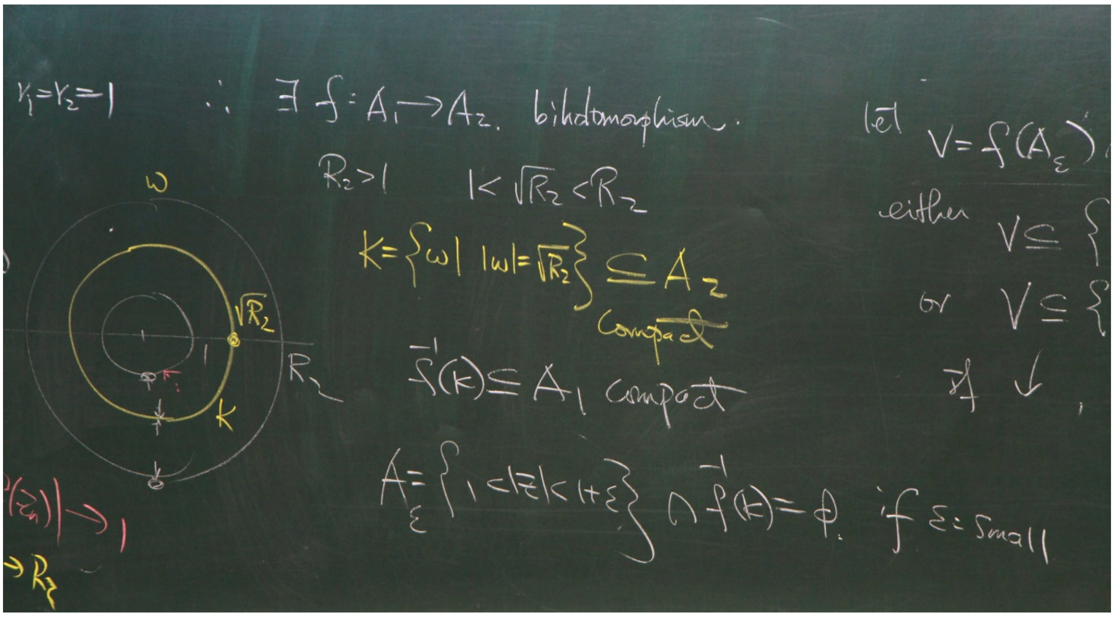
$|z| \to R_1 \qquad \dfrac{r_2}{r_1}|z| \to \dfrac{r_2}{r_1} R_1 = r_2 \cdot \dfrac{R_2}{r_2} = R_2$

If A


If $A_1 \simeq A_2$ may assume $r_1 = r_2 = 1$ ∴ ∃ $f: A_1 \to A_2$ biholomorphism.
f
1
R_1
1
R_2


proper
f


$\nexists\ A_1 \simeq A_2$ may assume $r_1=r_2=1$ $\therefore\ \exists\ f: A_1 \to$
$f^{-1}(K)$
f
ω
$R_2>1$
$K=\{\omega|$
$\sqrt{R_2}$
R_1
R_2
K
$\frac{r_2}{r_1}z$
$\frac{r_2}{r_1}|z| \to r_2$
$\frac{r_2}{r_1}|z| \to \frac{r_2}{r_1}R_1 = r_2 \cdot \frac{R_2}{r_2}$
$= R_2$
$|z_n| \to 1$
$|z_n| \to R_1$
$|f(z_n)| \to 1$
$|f(z_n)| \to R_2$


$r_1=r_2=1$ $\therefore\ \exists\ f: A_1 \to A_2$. biholomorphism.
let $V = f(A_\varepsilon)$
$R_2>1$ $1<\sqrt{R_2}<R_2$
$K=\{\omega \mid |\omega|=\sqrt{R_2}\} \subseteq A_2$
compact
$f^{-1}(K) \subseteq A_1$ compact
$A_\varepsilon=\{1<|z|<1+\varepsilon\} \cap f^{-1}(K)=\emptyset$ if ε small
either $V \subseteq \{$
or $V \subseteq \{$
$\nexists$ $\downarrow$
ω
$\sqrt{R_2}$
R_2
K
$|f(z_n)| \to 1$
$\to R_2$

domorphism.

let $V = f(A_\varepsilon') \cap K' = \phi$.

Assume

$< R_2$

either $V \subseteq \{1 < |w| < \sqrt{R_2}\}$ ← if necessary.

$\} \subseteq A_2$ compact

or $V \subseteq \{\sqrt{R_2} < |w| < R_2\}$

compact

If ↓, Consider $\dfrac{R_2}{f(z)}$ inversion.

$1+\varepsilon\} \cap f^{-1}(K) = \phi$. if ε = small

$\dfrac{R_2}{w}$

Set $m = \dfrac{\log R_2}{\log R_1} > 0$.

$|z| = 1$. $u(z) = 0$.

$|z| = R_1$ $u(z) = 0$

Set

$$u(z) = 2\log|f(z)| - 2m\log|z| \equiv 0 \text{ on } A_1$$

$$= \log|f(z)|^2 - m\log|z|^2.$$

$\dfrac{\partial}{\partial z} u(z) \equiv 0 =$

$0 =$

$$\partial\bar\partial u(z) = \partial\bar\partial\left(\log|f(z)|^2 - m\log|z|^2\right)$$

$$= \partial\left(\frac{f\,\overline{\left(\frac{\partial f}{\partial z}\right)}}{|f|^2} - m\frac{\cancel{z}}{\cancel{z}\bar z}\right) = \partial\left(\frac{\bar f'}{\bar f} - m\frac{1}{\bar z}\right) = 0$$

$\gamma = \sqrt{R_1}e^{it}$

$\Gamma = f(\gamma)$

$$\frac{\ \ \ R_2}{\ \ \ R_1.} > 0.$$

$|z|=1.\quad u(z)=0.$

$|z|=R_1\quad u(z)=0$

$$2\log|f(z)| - 2m\log|z| \equiv 0 \quad \text{on } A_1$$

$$\log|f(z)|^2 - m\log|z|^2.$$

$$\partial\bar\partial\left(\log|f(z)|^2 - m\log|z|^2\right)$$

$$= \partial\left(\frac{f\,\overline{\left(\frac{\partial f}{\partial z}\right)}}{|f|^2} - m\frac{\cancel{z}}{\cancel{z}\bar z}\right) = \partial\left(\frac{\bar f'}{\bar f} - m\frac{1}{\bar z}\right) = 0$$

$$\frac{\partial}{\partial z}u(z) \equiv 0 = \frac{f'\cancel{\bar f}}{f\cancel{\bar f}} - m\frac{\cancel{\bar z}}{z\cancel{\bar z}}$$

$$0 = \frac{f'}{f} - \frac{m}{z}$$

$$\gamma = \sqrt{R_1}e^{it} \subseteq A_1$$

$$\Gamma = f(\gamma) \subseteq A_2$$

Set $m = \dfrac{\log R_2}{\log R_1} > 0.$

$|z|=1.\quad u(z)=0.$

$|z|=R_1\quad u(z)=0$

Set

$$u(z) = 2\log|f(z)| - 2m\log|z| \equiv 0 \quad \text{on } A_1$$

$$= \log|f(z)|^2 - m\log|z|^2.$$

$$\partial\bar\partial u(z) = \partial\bar\partial\left(\log|f(z)|^2 - m\log|z|^2\right)$$

$$= \partial\left(\frac{f\,\overline{\left(\frac{\partial f}{\partial z}\right)}}{|f|^2} - m\frac{\cancel{z}}{\cancel{z}\bar z}\right) = \partial\left(\frac{\bar f'}{\bar f} - m\frac{1}{\bar z}\right) = 0$$

$$\frac{\partial}{\partial z}u(z) \equiv 0 = \frac{f'\cancel{\bar f}}{f\cancel{\bar f}} - m\frac{\cancel{\bar z}}{z\cancel{\bar z}}$$

$$0 = \frac{f'}{f} - \frac{m}{z}$$

$$\gamma = \sqrt{R_1}e^{it} \subseteq A_1$$

$$\Gamma = f(\gamma) \subseteq A_2$$

$$m = \frac{1}{2\pi i}\int_\gamma \frac{m}{z}\,dz$$

set $w = f(z)$.

$$u(z) \equiv 0 = \frac{f'\cancel{\bar z}}{f\cancel{\bar z}} - m\,\frac{\cancel{\bar z}}{z\cancel{\bar z}}$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz$$

$$0 = \frac{f'}{f} - \frac{m}{z}$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{1}{w}\,dw$$

$$= \mathrm{Ind}_\Gamma(0).$$ winding number of Γ.

$$\gamma = \sqrt{R_1}\,e^{it} \subseteq A_1$$

$$\Gamma = f(\gamma) \subseteq A_2$$

$\{z \mid r_1 < |z| < R_1\}$

$\{z \mid r_2 < |z| < R_2\}$

$$A_2 \iff \frac{R_1}{r_1} = \frac{R_2}{r_2}.$$

proper map

$$\frac{d}{dz}\left(\frac{f(z)}{z^m}\right) = \frac{z^m f'(z) - m z^{m-1} f(z)}{z^{2m}}$$

$$= \frac{z f'(z) - m f(z)}{z^{m+1}} \equiv 0.$$

$$\therefore\ f(z) = \lambda z^m \qquad \lambda = \text{const}.$$

$$\Rightarrow m = 1$$

$\nexists\ A_1 \simeq A_2$