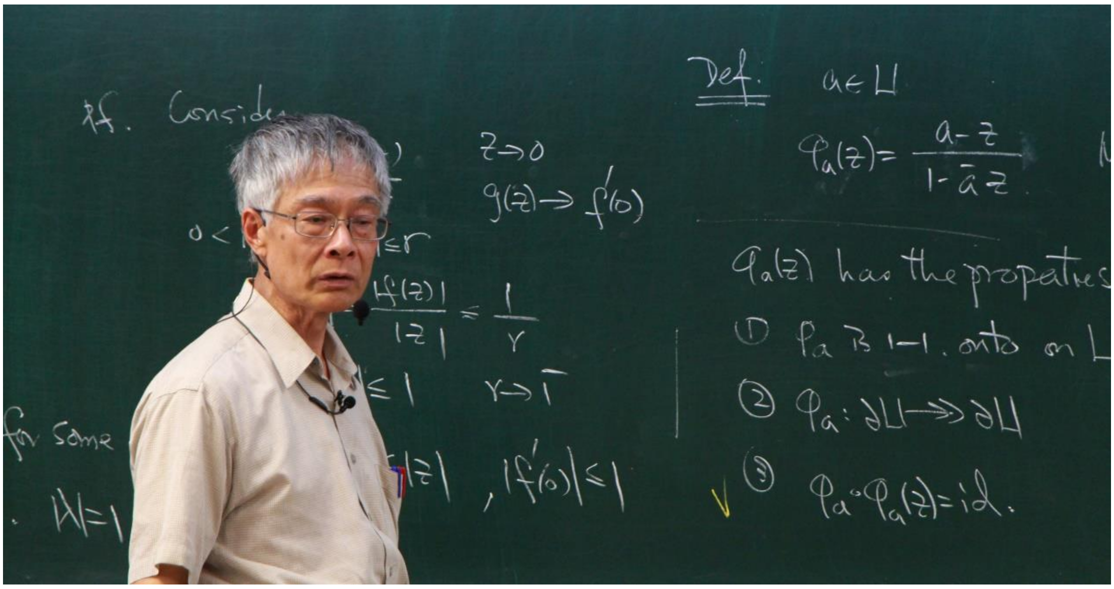
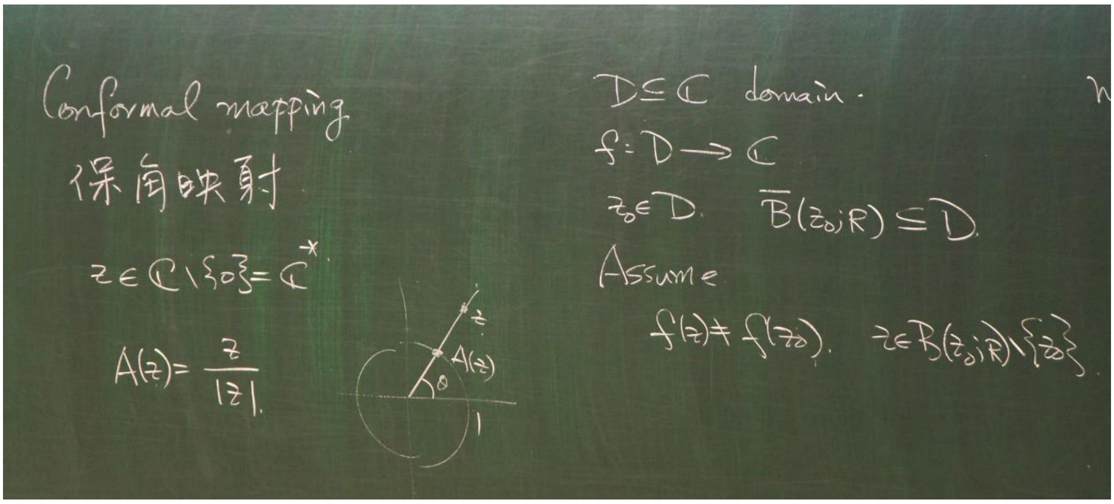
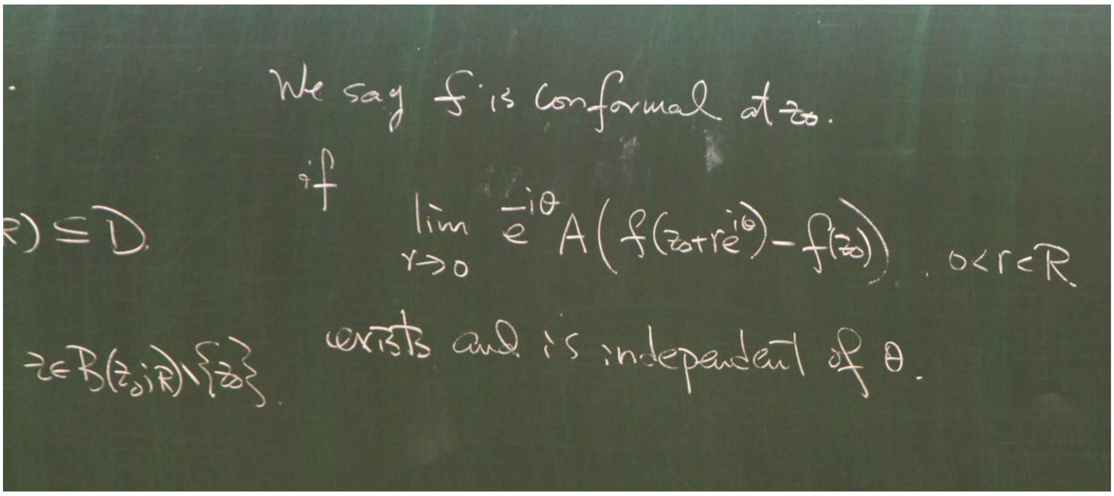
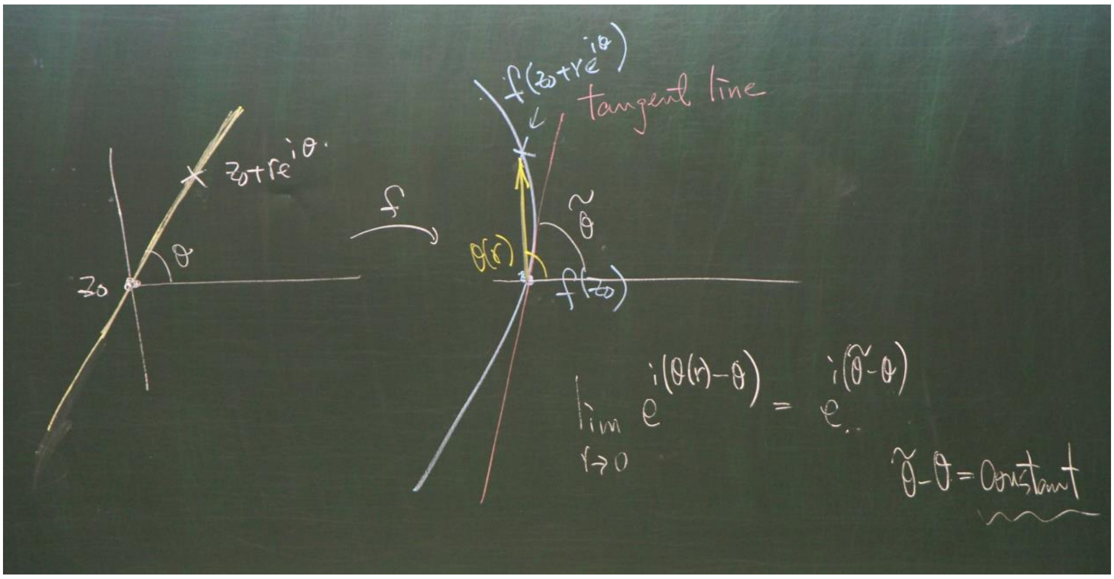
【10920 程守慶教授複變數函數論 / 第 18 堂版書】

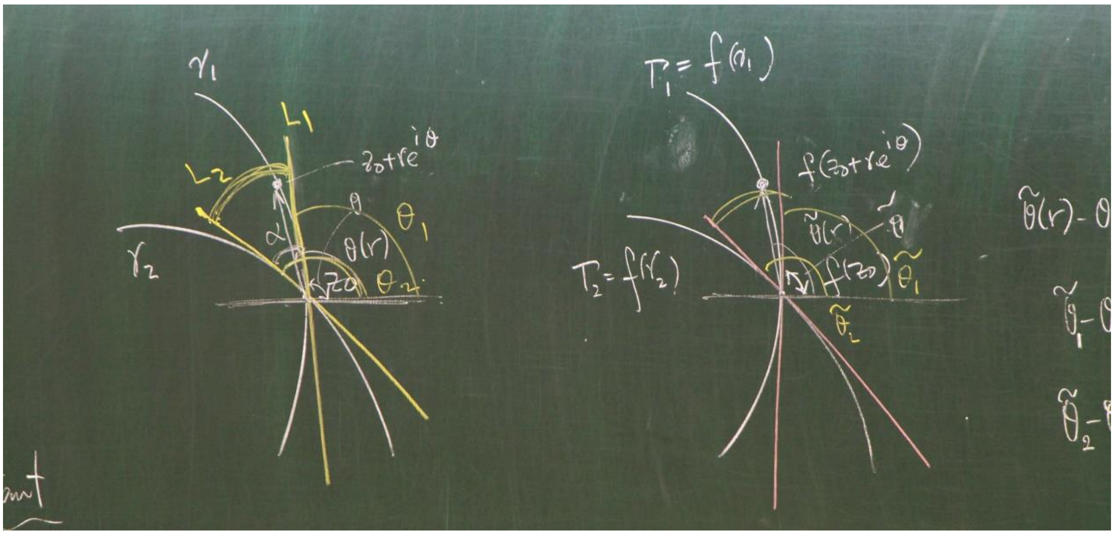




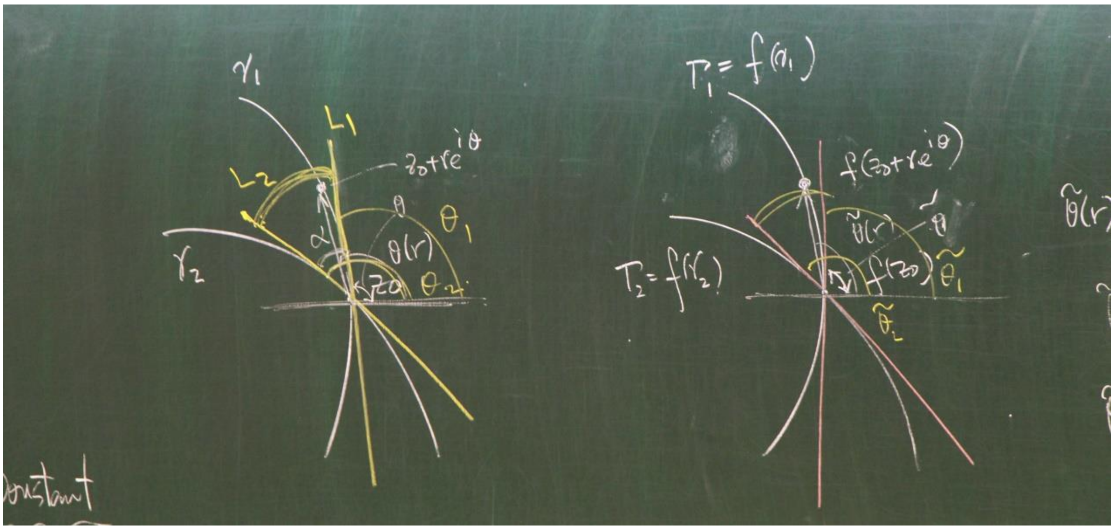




$f(z_0+re^{i\theta})$
tangent line
$z_0+re^{i\theta}$
f
θ
z_0
$\theta(r)$
$\tilde{\theta}$
$f(z_0)$
$\lim_{r\to 0} e^{i(\theta(r)-\theta)} = e^{i(\tilde{\theta}-\theta)}$
$\tilde{\theta}-\theta = \text{constant}$



γ_1
L_1
$z_0+re^{i\theta}$
L_2
θ
θ_1
α
$\theta(r)$
γ_2
z_0
θ_2
$T_1 = f(\gamma_1)$
$f(z_0+re^{i\theta})$
$\tilde{\theta}(r)$
$\tilde{\theta}$
$T_2 = f(\gamma_2)$
$f(z_0)$
$\tilde{\theta}_1$
$\tilde{\theta}_2$



γ_1
L_1
$z_0+re^{i\theta}$
L_2
θ
θ_1
α
$\theta(r)$
γ_2
z_0
θ_2
$T_1 = f(\gamma_1)$
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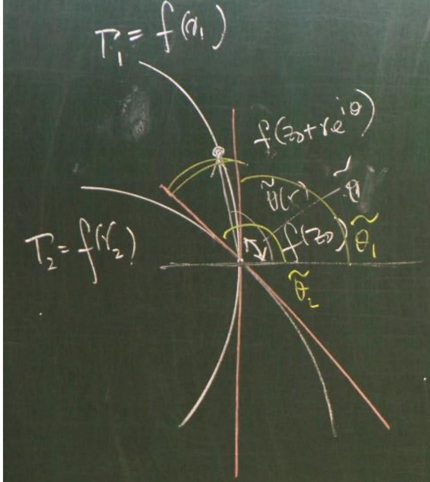
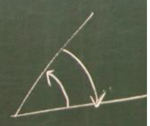


$r \to 0$

$\tilde\theta(r) - \theta(r) \longrightarrow \beta$



$\tilde\theta_1 - \theta_1 = \beta$ $\qquad$ $\tilde\theta_1 - \theta_1 = \tilde\theta_2 - \theta_2$

$\tilde\theta_2 - \theta_2 = \beta$ $\qquad$ $\boxed{\theta_2 - \theta_1 = \tilde\theta_2 - \tilde\theta_1}$ 保角

thm. $D \subseteq \mathbb{C}$ domain.

$f : D \to \mathbb{C}$. $z_0 \in D$.

then. ① $f \in \mathcal{O}(D)$ and $f'(z_0) \neq 0$.

$\Rightarrow$ f is conformal at z_0.

② f has a differential at z_0.

(differential is nonzero)

and f is conformal at z_0

$\Rightarrow$ f is diffentiable at z_0 and $f'(z_0) \neq 0$.

$D \subseteq \mathbb{C}$

$f : D \longrightarrow$

$z_0 \in D$.

Assume

$f(z) \neq$

Pf ① Assume $z_0=0$. $f(z_0)=0$

$\because$ $f'(z_0)=a_1\neq 0$

$\therefore$ $f(z)=a_1z+a_2z^2+\cdots$

$$\lim_{r\to 0} e^{-i\theta}A\left(f(re^{i\theta})\right)=\lim_{r\to 0} e^{-i\theta}\left(\frac{a_1re^{i\theta}+a_2(re^{i\theta})^2+\cdots}{\left|a_1re^{i\theta}+a_2(re^{i\theta})^2+\cdots\right|}\right)$$

$$=\lim_{r\to 0}\frac{a_1+a_2re^{i\theta}+a_3r^2e^{i2\theta}+\cdots}{\left|a_1+a_2re^{i\theta}+a_3r^2e^{i2\theta}+\cdots\right|}=\frac{a_1}{|a_1|}$$

② near $z_0=0$

$f(z)=\alpha z+\beta\bar{z}+|z|\eta(z)$ $\quad \lim_{|z|\to 0}\eta(z)=0$

not both α and β are zero.

f is conformal at 0.

$\therefore$ $\lim_{r\to 0} e^{-i\theta}A\left(f(re^{i\theta})\right)$ exists indep. of θ.

$$=\lim_{r\to 0} e^{-i\theta}\frac{f(re^{i\theta})}{|f(re^{i\theta})|}=\lim_{r\to 0} e^{-i\theta}\frac{\alpha re^{i\theta}+\beta re^{-i\theta}+r\eta(z)}{\left|\alpha re^{i\theta}+\beta re^{-i\theta}+r\eta(z)\right|}$$

$$\lim_{r\to 0} \frac{\alpha+\beta e^{-i2\theta}+\eta(z)}{|\alpha+\beta e^{-i2\theta}+e^{-i\theta}\eta(z)|} = \frac{\alpha+\beta e^{-i2\theta}}{|\alpha+\beta e^{-i2\theta}|}$$ indep. of θ.

$\Rightarrow \beta=0 . \ \alpha\neq 0$

$$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}$$

$$=\lim_{z\to 0}\frac{f(z)}{z}=\lim_{z\to 0}\frac{\alpha z+|z|\eta(z)}{z}=\lim_{z\to 0}\left(\alpha+\frac{|z|}{z}\eta(z)\right)=\alpha\neq 0.$$

$$\lim_{r\to 0} \frac{\alpha+\beta e^{-i2\theta}+\eta(z)}{|\alpha+\beta e^{-i2\theta}+e^{-i\theta}\eta(z)|} = \frac{\alpha+\beta e^{-i2\theta}}{|\alpha+\beta e^{-i2\theta}|}$$ indep. of θ.

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$D_1, D_2 \subseteq \mathbb{C}$

$D_1 \underset{f}{\overset{f}{\rightleftarrows}} D_2$ 1-1 onto holo.

biholomorphic

biholomorphism

$D \subseteq \mathbb{C}^n$ domain $n \geq 1$.

$f: D\to D$ biholomorphism

automorphism. 自同構.

$D \xrightarrow{f} D \xrightarrow{g} D$, $g\circ f$

$\mathrm{Aut}(D)=\{$ all of … of …

forms a group.

$D \subseteq \mathbb{C}^n$ domain $n \geq 1$.

$f: D \to D$ biholomorphism

automorphism 自同構.

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & D \arrow[r, "g"] & D \end{tikzcd}$$

$\mathrm{Aut}(D) = \{$ all of the automorphisms of $D \}$.

forms a group.

Thm (Cartan) $D \subseteq \mathbb{C}^n$ $n \geq 1$. bounded domain. ✓

Then $\mathrm{Aut}(D)$ is a C^ω Lie group.

and $\pi: \mathrm{Aut}(D) \times D \longrightarrow D$

$(\varphi, z) \longmapsto \varphi(z)$

is also C^ω.

–2008)

1980. Wolf medal

$a \in \sqcup$

$$\mathrm{Aut}(\sqcup) = \{ e^{i\theta} \varphi_a(z) \} \qquad \varphi_a(z) = \frac{a - z}{1 - \bar{a} z}$$

$S^1 \times \sqcup$

bounded $D \subseteq \mathbb{C}$. $\partial D : C^1$

$\mathrm{Aut}(D)$: noncompact

$\Rightarrow D \simeq \sqcup$

Schwarz lemma

$f \in \mathcal{O}(\mathbb{U})$ and $f : \mathbb{U} \to \mathbb{U}$.

$f(0) = 0$.

Then ① $|f(z)| \le |z|$. $\forall z \in \mathbb{U}$.

② $|f'(0)| \le 1$

In particular, if "=" holds in ① for some $z \neq 0$ or "=" holds in ② $\Rightarrow$ $f(z) = \lambda z$. $|\lambda| = 1$

pf. Consider

$$g(z) = \frac{f(z)}{z} \qquad z \to 0,\quad g(z) \to f'(0)$$

$0 < r < 1$ $\quad |z| \le r$

$$|g(z)| = \frac{|f(z)|}{|z|} \le \frac{1}{r}$$

$\therefore$ $|g(z)| \le 1$ $\quad r \to 1$

$\Rightarrow$ $|f(z)| \le |z|$, $|f'(0)| \le 1$

De

$z \to 0$

$g(z) \to f'(0)$

$r \to 1$

, $|f'(0)| \le 1$

Def. $a \in \mathbb{U}$

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a} z} \qquad \text{Möbius transformation.}$$

$\varphi_a(z)$ has the properties.

① φ_a is 1-1. onto on $\mathbb{U}$.

② $\varphi_a : \partial\mathbb{U} \to \partial\mathbb{U}$

③ $\varphi_a \circ \varphi_a(z) = id$.

pf.

$$\frac{a - \dfrac{a-z}{1-\bar{a}z}}{1 - \bar{a}\left(\dfrac{a-z}{1-\bar{a}z}\right)} = \frac{\cancel{a} - |a|^2 z - \cancel{a} + z}{1 - \cancel{\bar{a}z} - |a|^2 + \cancel{\bar{a}z}} = \frac{(1-|a|^2)z}{1-|a|^2} = z$$

$$|1-\bar{a}z|^2 - |a-z|^2 = (1-\bar{a}z)(1-a\bar{z}) - (a-z)(\bar{a}-\bar{z})$$

$$= 1 - \cancel{\bar{a}z} - \cancel{a\bar{z}} + |a|^2|z|^2 - \left(|a|^2 + |z|^2 - \cancel{a\bar{z}} - \cancel{\bar{a}z}\right)$$

$$= 1 - |a|^2 - |z|^2 + |a|^2|z|^2$$

$$= (1-|a|^2)(1-|z|^2) > 0$$

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z} \qquad -1 + \cancel{\bar{a}z} + |a|^2 - \cancel{\bar{a}z}$$

$$\varphi_a'(z) = \frac{-(1-\bar{a}z) + \bar{a}(a-z)}{(1-\bar{a}z)^2} = \frac{-1+|a|^2}{(1-\bar{a}z)^2} \Bigg|_{z=a} = \frac{-1+|a|^2}{(1-|a|^2)^2} = \frac{-1}{1-|a|^2}$$

$$\varphi_a'(0) = -1+|a|^2 \quad \sim$$

$$\varphi_a'(a) = \frac{-1}{1-|a|^2} = \frac{1}{-1+|a|^2}$$

$$\ldots z = \frac{(1-|a|^2)z}{1-|a|^2} = z$$

$$(a-z)(\bar{a}-\bar{z})$$

$$-\left(|a|^2 + |z|^2 - \cancel{\bar{a}z} - \cancel{a\bar{z}}\right)$$

$$> 0$$

$f : \mathbb{D} \longrightarrow \mathbb{D}$ holo.

$f(\alpha) = \beta \qquad \alpha \in \mathbb{D}$

How large can $|f'(\alpha)|$ be?

$$\mathrm{Aut}(\mathbb{D}) = \left\{ e^{i\theta} \varphi_a(z) \right\}$$

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$$

$$\varphi_a'(z) = \frac{-(1-\bar{a}z) \ldots}{(1-\bar{a}z \ldots}$$

$$\varphi_a'(0) = -1+|a \ldots$$

$$\varphi_a'(a) = \frac{-1}{1-|a \ldots}$$